Power Mean Graphs

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Abstract: *A graph $G = (V, E)$ is called a Power mean graph with p vertices and q edges, if it is possible to label the vertices $x \in V$ with distinct elements $f(x)$ from $1, 2, 3, \cdots, q+1$ in such way that when each edge $e = uv$ is labeled with*

$$f(e=uv) = \left\lceil \left(f(u)^{f(v)} f(v)^{f(u)} \right)^{\frac{1}{f(u)+f(v)}} \right\rceil$$

or

$$f(e=uv) = \left\lfloor \left(f(u)^{f(v)} f(v)^{f(u)} \right)^{\frac{1}{f(u)+f(v)}} \right\rfloor$$

then the resulting edge labels are distinct. Here f is called a Power mean labeling of G. We investigate Power mean labeling for some standard graphs.

Key Words: *Graphs, Power mean labeling, Power Mean graph, Path, Cycle, Comb, Ladder, K_n, $K_{1,n}$.*

I. Introduction

The graphs considered here are finite and undirected graphs. For a detailed survey of graph labeling one may refer to Gallian[2] and also [1]. For all other standard terminology and notations we follow Harary[3]. In [4], Somasundaram and Ponraj introduced and studied mean labeling for some standard graphs. Somasundaram et al. [5], [7] introduced Harmonic mean labeling of graphs. Somasundaram et al. [6] introduced the concept of Geometric mean labeling of graphs and studied their behaviour. Somasundaram et al. [6] studied harmonic mean labeling technique. In this paper we define Power mean labeling and investigate some standard graphs like Path, Cycle, Complete graph, Star, Comb , Ladder, K_n, $K_{1,n}$ for power mean labeling.

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II. Definition and Results

Now we introduce the main concept and its related results in this paper.

Definition 2.1. A graph $G = (V, E)$ with p vertices and q edges is said to be a Power Mean Graph if it is possible to label the vertices $x \in V$ with distinct labels $f(x)$ from $1, 2, 3, ..., q+1$ in such a way that when each edge $e = uv$ is labeled with

$$f(e=uv) = \left\lceil \left(f(u)^{f(v)} f(v)^{f(u)} \right)^{\frac{1}{f(u)+f(v)}} \right\rceil$$

or

$$f(e=uv) = \left\lfloor \left(f(u)^{f(v)} f(v)^{f(u)} \right)^{\frac{1}{f(u)+f(v)}} \right\rfloor$$

then the resulting edge labels are distinct. In this case, f is called a Power mean labeling of G.

Remark 2.1. If G is a Power mean labeling graph, then 1 must be a label of one of the vertices of G, since an edge should get label 1.

Remark 2.2. If $p > q+1$, then the graph $G = (p, q)$ is not a Power mean graph, since it doesn't have sufficient labels from $\{1, 2, 3, ..., q+1\}$ for the vertices of G.

The following Proposition will be used in the edge labelings of some standard graphs to get Power mean labeling.

Proposition 2.1. *Let a, b and i be positive integers with $a < b$. Then*

(i) $a < (a^b b^a)^{\frac{1}{a+b}} < b$,

(ii) $i < (i^{1+2}(i+2)^i)^{\frac{1}{2i+2}} < (i+1)$,

(iii) $i < (i^{i+3}(i+3)^i)^{\frac{1}{2i+3}} < (i+2)$,

(iv) $i < (i^{i+4}(i+4)^i)^{\frac{1}{2i+4}} < (i+2)$, and

(v) $(1^i i^1)^{\frac{1}{i+1}} = i^{\frac{1}{i+1}} < 2$.

Proof. **(i)** Since $a^{a+b} = a^a a^b < b^a a^b < b^a b^b = b^{a+b}$, we get the inequality in Proposition 2.1.(i). That is, the Power mean of two numbers lies between the numbers a and b. This leads to infer that if vertices u, v have labels $i, i+1$ respectively, then the edge uv may be labeled i or $i+1$ for Power mean labeling.

(ii) As a proof of this inequality, we see

$$\begin{aligned} i^{i+2}(i+2)^i &< i^2[i(i+2)]^i, \\ &< i^2(i+1)^{2i}, \\ &\quad \text{since } i(i+2) < (i+1)^2, \\ &< (i+1)^2(i+1)^{2i}, \\ &= (i+1)^{2i+2}. \end{aligned}$$

This leads to $[(i^{i+2}(i+2)^i)^{\frac{1}{2i+2}}] < i+1$.

Therefore, if u, v have labels $i, i+2$ respectively, then the edge uv may be labeled i or $i+1$.

(iii) Next we have

$$\begin{aligned} i^{i+3}(i+3)^i &= i^3[i(i+3)]^i, \\ &< i^3(i+2)^{2i}, \text{ since } i(i+3) < (i+2)^2, \\ &< (i+2)^3(i+2)^{2i}, \\ &= (i+2)^{2i+3}. \end{aligned}$$

This leads to $[i^{i+3}(i+3)^i]^{\frac{1}{2i+3}} < (i+2)$. Hence, if u, v have labels $i, i+3$ respectively, then the edge uv may be labeled $i+1$ without ambiguity.

(iv) Now

$$\begin{aligned} i^{i+4}(i+4)^i &= i^4[i(i+4)]^i, \\ &< i^4(i+2)^{2i}, \text{ since } i(i+4) < (i+2)^2 \\ &< (i+2)^4(i+2)^{2i}, \\ &= (i+2)^{2i+4}. \end{aligned}$$

Therefore

$$[i^{i+4}(i+4)^i]^{\frac{1}{2i+4}} < i+2.$$

Hence if u, v have labels $i, i+4$ respectively, then the edge uv may be labeled $i+1$.

(v) Now

$$\begin{aligned} 2^{i+1} &= (i+1)^{i+1}, \\ &= 1 + {}^{(i+1)}C_1 + \cdots + {}^{(i+1)}C_{i+1}, \\ &\geq 1 + 1 + \cdots + (i+2) \text{ terms}, \\ &\geq i+2 > i. \end{aligned}$$

Therefore $(1^i i^1)^{\frac{1}{i+1}} = i^{\frac{1}{i+1}} < 2$. Thus we observe that if u, v are labeled $1, i$ respectively, then the edge uv may be labeled 1 or 2. ■

2.1 *Power Mean labeling for Path* P_n

Path: Path is a finite or infinite sequence of edges which connect a sequence of vertices and all are distinct from one another.

We examine the possibility of Power mean labeling to a path with an example.

Theorem 2.1. *Any path is a Power mean graph.*

Proof. Let P_n be a path on n vertices namely $u_1, u_2, u_3, ..., u_n$ with $n-1$ edges. Define a function

$$f : V(P_n) \longrightarrow \{1, 2, 3, \ldots, q+1 = n\} \text{ by}$$

$$f(u_i) = i \;;\; 1 \leq i \leq n.$$

Then we get edge labels as $f(e_i) = i; 1 \leq i \leq n-1$, by Proposition 2.1.($i$). As the edge labels are distinct and the graph, P_n is a Power mean graph. ■

Example 2.1. *A Power mean labeling of P_6 is given by Figure 2.1.*

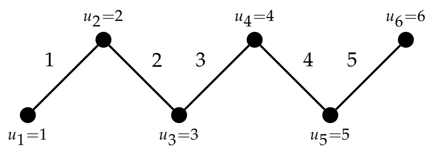


Figure 2.1: Power mean labeling of Path P_6

2.2 *Power Mean Labeling for Cycle Graph* $C_n, n \geq 3$

Cycle C_n **:** Cycle is a connected closed path.

We investigate the assignment of Power mean labeling to a Cycle with an example.

Theorem 2.2. *Any cycle is a Power mean graph*

Proof. Let C_n be the cycle $u_1, u_2, u_3, \ldots, u_n, u_1$ of length n.
Define a function

$$f : V(G) \longrightarrow \{1, 2, 3, \ldots, q+1 = n+1\}$$

by

(i) $f(u_1) = 1,$

(ii) $f(u_{i-1}) = i.$; $3 \leq i \leq n+1.$

We get edge labels as

(i) $E(u_i u_{i+1}) = i+1.$; $2 \leq i \leq n-1,$

(ii) $E(u_1 u_2) = 1,$

(iii) $E(u_n u_1) = 2.$

by Proposition 2.1.(i) , (ii) and (iv). Hence any cycle is a Power mean graph. ■

Example 2.2. *A Power mean labeling of* C_8 *is given below in Figure 2.2.*

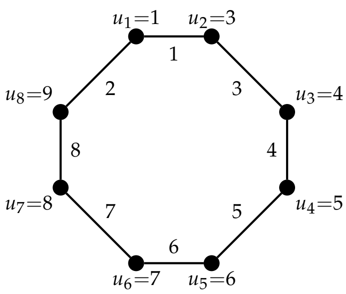


Figure 2.2: Power mean labeling of C_8

2.3 *Power Mean labeling for Comb*

Comb: Comb is a graph obtained by joining a single pendant edge to each vertex of a path.

We do here the Power mean labeling for a Comb with an example.

Theorem 2.3. *Comb is a Power mean graph.*

Proof. Let G be a comb obtained from a path $P_n = u_1, u_2, u_3, ...u_n$ by joining a vertex v_i to u_i ; $1 \leq i \leq n$
Define a function $f : V(G) \longrightarrow \{1, 2, 3, ..., q+1\}$ by

(i) $f(v_i) = 2i$, $1 \leq i \leq n,$

(ii) $f(u_i) = 2i - 1$, $1 \leq i \leq n.$

We get edge labels as

(i) $E(u_i v_i) = 2i - 1$, $1 \leq i \leq n,$

(ii) $E(v_i v_{i+1}) = 2i$, $1 \leq i \leq n-1.$

by Proposition 2.1.(i) and (ii). Thus f is a Power mean labeling of G. ■

Example 2.3. *The Comb obtained from* P_4 *is given in Figure 2.3.*

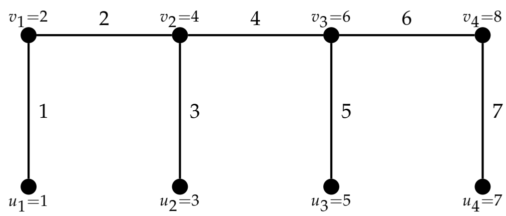


Figure 2.3: *Comb*

2.4 *Power Mean Labeling for Ladder Graph* L_n, $n \geq 2$

Ladder graph L_n : Ladder is an undirected graph consisting of two paths $u_1, u_2, u_3, \ldots, u_n$ and $v_1, v_2, v_3, \ldots, v_n$.

Herein, we establish the Power mean Labeling to a Ladder with an example.

Theorem 2.4. *Any Ladder is a Power mean graph.*

Proof. Let L_n denote the ladder graph. L_n has $2n$ vertices and $3n-2$ edges
Define a function

$$f : V(L_n) \longrightarrow \{1,2,3,...,q+1 = 3n-1\} \text{ by}$$

(i) $f(u_1) = 1,$

(ii) $f(u_2) = 4,$

(iii) $f(u_i) = 3i-3 \ , \ 3 \leq i \leq n,$

(iv) $f(v_i) = 2i \ , \ 1 \leq i \leq 2,$

(v) $f(v_i) = 3i-1 \ , \ 3 \leq i \leq n.$

We get edge labels as

(i) $E(u_i v_i) = 3i-2 \ , \ 1 \leq i \leq n,$

(ii) $E(u_i u_{i+1}) = 3i-1 \ , \ 1 \leq i \leq n-1,$

(iii) $E(v_i v_{i+1}) = 3i \quad 1 \leq i \leq n-1.$

by Proposition 2.1.(i), (ii) and (iii). Thus the Ladder L_n is a Power mean graph. ■

Example 2.4. *Power mean labeling of L_5 with 10 vertices and 13 edges is given in Figure 2.4.*

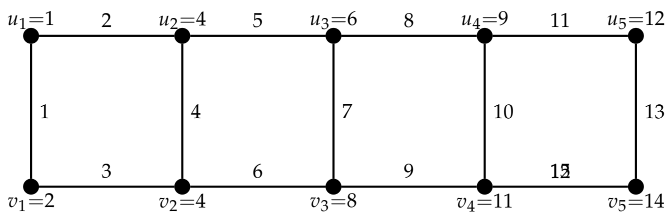


Figure 2.4: Power mean labeling of L_5

2.5 *Power Mean labeling for Complete Graph $K_n, n = 1,2,3$*

Complete graph K_n : K_n is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge.

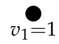
We analyse the possibility of assigning Power mean labeling to Complete graph $K_n, n = 1,2,3,\ldots$ with illustrative examples.

Theorem 2.5. *If $n > 3$, K_n is not a Power mean graph.*

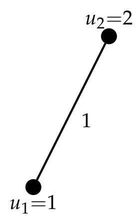
Proof. **Case(i):** Clearly K_1, K_2 and K_3 are Power mean graphs as shown in figure 2.5.

Case(ii): Suppose $K_n, n > 3$ is a Power mean graph. By Remark 2.1, a vertex, say u should get label 1. By Proposition 2.1. (ii), an edge with vertex labels 1 and i should get label 1 or 2. There are atleast 3 vertices adjacent to the vertex u and hence atleast 3 edges incident at u. Distinct labels of these edges with 1 or 2 are not possible. Hence $K_n, n > 3$ is not a Power mean graph. ■

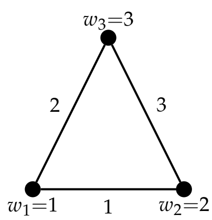
Example 2.5. *Power mean labeling of K_1 , K_2 and K_3 are given by Figure 2.5.*



(a) K_1



(b) K_2



(c) K_3

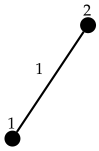
Figure 2.5: Power Mean labeling for Complete Graph $K_n, n = 1,2,3$

2.6 *Power Mean Labeling for Star* $K_{1,n}, n \leq 8$

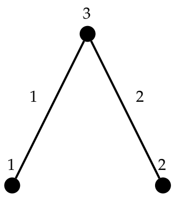
$K_{1,n}$: $K_{1,n}$ is a graph with a central vertex u and n pendant vertices adjacent to u. We investigate the star graph $K_{1,n}$ for Power mean labeling.

Theorem 2.6. *$K_{1,n}$ is a Power mean graph if and only if $n \leq 8$.*

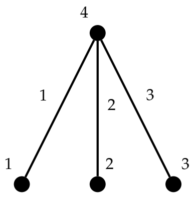
Proof. **Case(i):** The Power mean labelings for $K_{1,n}$, $n \leq 8$ are given below Figure 2.6.



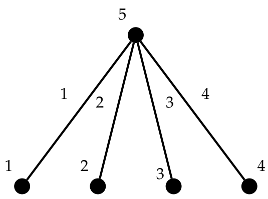
$K_{1,1}$



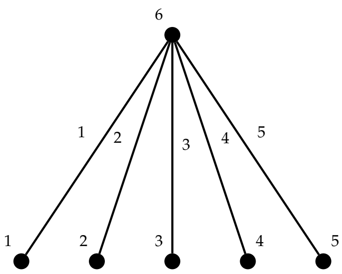
$K_{1,2}$



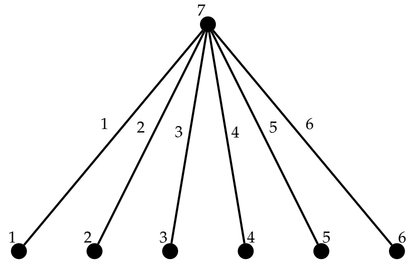
$K_{1,3}$



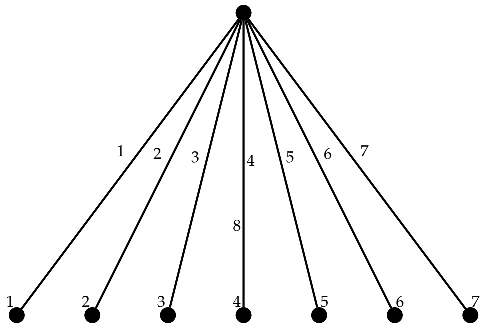
$K_{1,4}$



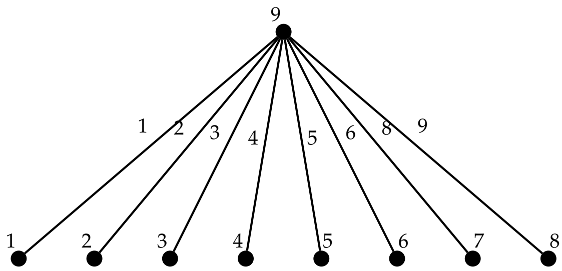
$K_{1,5}$



$K_{1,6}$



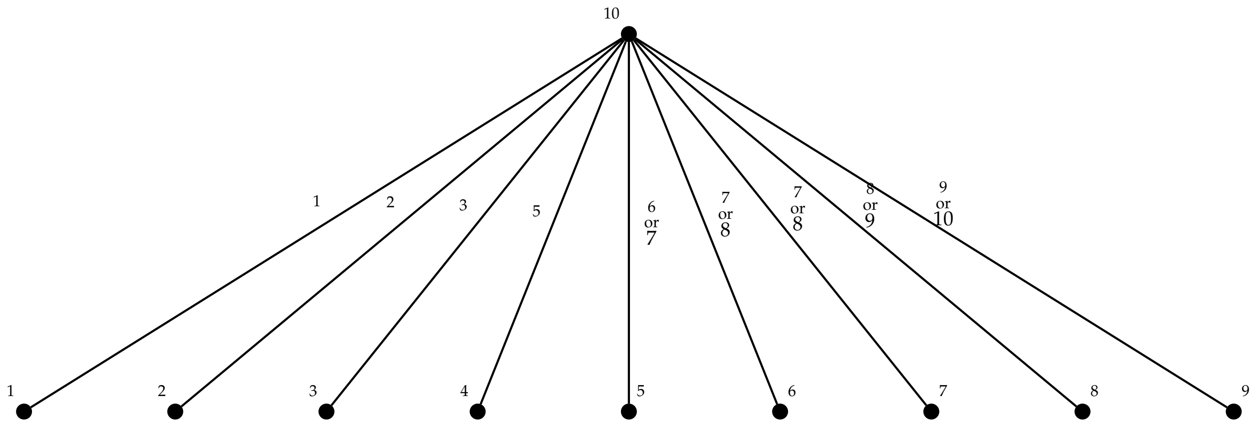
$K_{1,7}$



$K_{1,8}$

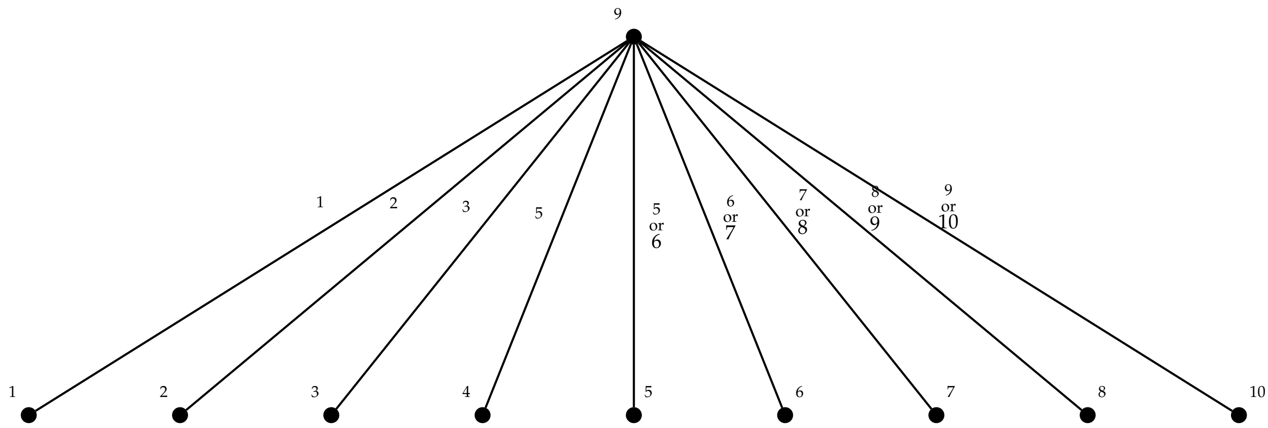
Figure 2.6: Power Mean Labeling for Star $K_{1,n}, n \leq 8$

Case(ii): We discuss $K_{1,9}$ with different central vertex labels (Figure 2.7).



$K_{1,9}(a)$

In this case, no edge will get label 4.



$K_{1,9}(b)$

Figure 2.7: Star Graph

In this case also no edge will get label 4. Similarly, the argument extends for cases $n \geq 9$. Thus $K_{1,n}$ is a Power mean graph only for $n \leq 8$. ∎

III. Conclusion

In this paper we defined Power mean labeling concept and showed how to label graphs like Path, Cycle, Comb, Ladder, Complete graph K_n and Star $K_{1,n}$. We also provided illustrative examples for possible implementation of power mean labeling technique.

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