Alternative Methods to Prove Theorem of Intersection of Two Subspace of a Vector Space

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Abstract - *In this paper, we study about alternative methods by which we can proof the theorem, Intersection of any two subspaces of a vector space V(F) is again a subspace of V(F). We all are familiar with the methods of proving the given theorems mentioned in books as reference books but there are also other methods by which we can prove the theorem using some theorems directly as statements.*

Keywords - *Vector Space, Vector-Subspace, Necessary and Sufficient Condition of Vector Subspace to be a subspace, Linear Sum of Two Subspace of a Vector space.*

1. INTRODUCTION

We have already studied out the two important basic algebraic structures 'groups' and 'rings'. In groups, we have studied the algebraic system with one binary operation and in the rings, integral domain and fields we have studied the algebraic system with two binary operations. Now we shall consider another important algebraic system known as linear vector space or simply vector space upon which the whole of linear algebra is based.

1.1. Vector Space :-

If $(F,+,\cdot)$ is a field and V is a non-empty set of vectors, then an algebraic structure V(F) is called a vector space if it satisfy the following conditions :

1.1.1.V is an abelian additive group of vectors.
1.1.2.$\forall$ a ε F and α ε V(F) => aα ε V(F) i.e. V is closed for the scalar multiplication.
1.1.3.$\forall$ a ε F and α, β ε V(F), the following four laws of scalar multiplication are satisfied :
1.1.3.1 $a\cdot(\alpha+\beta) = a\cdot\alpha + a\cdot\beta$

1.1.3.2. $(a+b)\cdot\alpha = a\cdot\alpha + b\cdot\alpha$

1.1.3.3. $a\cdot(b\cdot\alpha) = (a\cdot b)\cdot\alpha$

1.1.3.4. $1\cdot\alpha = \alpha$, 1 is the unit element of F.

1.2. Vector-Subspace :-

Let V(F) be a vector space and W is a non-empty subset of V(F), then W is called a sub-space of V(F) if w is itself a vector space under the same operations that defined for V(F).

1.3. Necessary and Sufficient Condition of Vector Subspace to be a subspace :-

1.3.1.W is closed under vector addition in V(F).
1.3.2. W is closed under scalar multiplication in V(F).

1.4.Linear Sum of Two Subspace of a Vector space :-

Let V(F) is a vector space and W_1 ,W_2 are two sub-space of V(F), then the linear sum of W_1 and W_2 is denoted by $W_1 + W_2$ and defined as :

$$W_1 + W_2 = \{ (w_1 + w_2) : w_1 \varepsilon W_1 \text{ and } w_2 \varepsilon W_2 \}.$$

2. ALTERNATIVE METHODS

2.1 METHOD 1

Let W_1 and W_2 be two subspaces of a vector space V(F), then we have to show that $W_1 \cap W_2$ is also a subspace of V(F).

Since $0\varepsilon\, W_1$ and $0\varepsilon\, W_2 \Rightarrow 0\varepsilon\, W_1 \cap W_2$

Therefore, $W_1 \cap W_2$ is not an empty i.e. $W_1 \cap W_2 \neq \Phi$.

Let a, b ε F and $\alpha, \beta\ \varepsilon\ W_1 \cap W_2$.

Now, $\alpha\ \varepsilon\ W_1 \cap W_2 \Rightarrow \alpha\ \varepsilon\ W_1$ and $\alpha\ \varepsilon\ W_2$

$\beta\ \varepsilon\ W_1 \cap W_2 \Rightarrow \beta\ \varepsilon\ W_1$ and $\beta\ \varepsilon\ W_2$.

Again , $\alpha, \beta\ \varepsilon\ W_1$ and a, b ε F $\Rightarrow a\alpha+b\beta\ \varepsilon\ W_1$, (W_1 is a subspace)

Also, $\alpha, \beta\ \varepsilon\ W_2$ and a, b ε F $\Rightarrow a\alpha+b\beta\ \varepsilon\ W_2$. (W_2 is a subspace)

Hence, we see that $a\alpha+b\beta$ is common element of W_1 and W_2.

Therefore, $a\alpha+b\beta\ \varepsilon\ W_1 \cap W_2$.

Thus, $\alpha, \beta\ \varepsilon\ W_1 \cap W_2$ and a, b ε F $\Rightarrow a\alpha+b\beta\ \varepsilon\ W_1 \cap W_2$.

By using theorem,

The non-empty subset W of vector space V(F) is a subspace of V(F) iff

I. $0\varepsilon\, W$ ($V \neq \Phi$)
II. a, b ε F and $\alpha, \beta\ \varepsilon\ W \Rightarrow a\alpha+b\beta\ \varepsilon\ W$.

Thus, $W_1 \cap W_2$ is a subspace of a vector space V(F).

1.2. METHOD 2

Let $\alpha, \beta\ \varepsilon\ W_1 \cap W_2 \Rightarrow \alpha, \beta\ \varepsilon\ W_1$ and $\alpha, \beta\ \varepsilon\ W_2$.

⇨ $\alpha+(-\beta)\ \varepsilon\ W_1$ and $\alpha+(-\beta)\ \varepsilon\ W_2$, (if $\beta\ \varepsilon\ W_1$ and $W_2 \Rightarrow -\beta\ \varepsilon\ W_1$ and W_2)
⇨ $\alpha+(-\beta)$ or $\alpha-\beta\ \varepsilon\ W_1 \cap W_2$.

Again , let a ε F and $\alpha\ \varepsilon\ W_1 \cap W_2 \Rightarrow$ a ε F, $\alpha\ \varepsilon\ W_1$ and $\alpha\ \varepsilon\ W_2$.

⇨ $a\alpha\ \varepsilon\ W_1$ and $a\alpha\ \varepsilon\ W_2$, (W_1 and W_2 are subspaces of a vector space V(F))
⇨ $a\alpha\ \varepsilon\ W_1 \cap W_2$.

Thus, $W_1 \cap W_2$ is closed under scalar multiplication.

By using theorem ,

A non – empty subset W of a vector space V(F) is a subspace of V(F) iff

I. $\alpha, \beta \varepsilon W \Rightarrow \alpha-\beta \varepsilon W$.
II. $a \varepsilon F, \alpha \varepsilon W \Rightarrow a\alpha \varepsilon W$.

Hence, $W_1 \cap W_2$ is a subspace of a vector space V(F).

1.3. METHOD 3

Firstly, let $\alpha, \beta \varepsilon W_1 \cap W_2 \Rightarrow \alpha, \beta \varepsilon W_1$ and $\alpha, \beta \varepsilon W_2$.

⇨ $\alpha+\beta \varepsilon W_1$ and $\alpha+\beta \varepsilon W_2$
⇨ $\alpha+\beta \varepsilon W_1 \cap W_2$.

Since, we know that W_1 and W_2 are subspaces of a vector space V(F) then their linear sum be also a subspace of V(F)

i.e. $W_1 + W_2$ is also a subspace.

⇨ $\alpha+\beta \varepsilon W_1 + W_2$.

Thus, $W_1 \cap W_2$ is closed under vector addition.

Secondly, let $a \varepsilon F$ and $\alpha \varepsilon W_1 \cap W_2 \Rightarrow a \varepsilon F, \alpha \varepsilon W_1$ and $\alpha \varepsilon W_2$.

⇨ $a\alpha \varepsilon W_1$ and $a\alpha \varepsilon W_{2,}$
⇨ $a\alpha \varepsilon W_1 \cap W_{2.}$.

Since, we know that W_1 and W_2 are subspaces of a vector space V(F) then their linear sum be also a subspace of V(F).

i.e. $W_1 + W_2$ is also a subspace.

⇨ $a\alpha \varepsilon W_1 + W_2$.

Thus, $W_1 \cap W_2$ is closed under scalar multiplication.

By using theorem,

A non – empty subset W of a vector space V(F) is a subspace of V(F) iff

I. W is non- empty
II. W is closed under vector addition i.e. $\alpha, \beta \varepsilon W \Rightarrow \alpha+\beta \varepsilon W$.
III. W is closed under scalar multiplication i.e. $a\alpha \varepsilon W \ \forall \ a \varepsilon F$ and $\alpha \varepsilon W$.

Thus, $W_1 \cap W_2$ is a subspace of a vector space V(F)

3. REFERENCES

1. Linear Algebra 4th Edition by Seymour Lipschutz (Temple University) and Marc Lars Lipson (University of Virginia), ISBN : 978-0-07-154353-8.
2. Halmos, Paul Richard,1916, Finite- Dimensional Vector Spaces,reprint of 2nd ed. Published by D. Van Nostrand Co., Princeton,1958. (Springer)
3. Linear Algebra 2nd Edition by Kenneth Hoffman (Massachusetts Institute of Technology) and Ray Kunze(University of California,Irvine).
4. Linear Algebra And Matrices by S.J. Publications (A unit of Kedar Nath Ram Nath), Edition-2015.
5. Linear Algebra And Matrices (Textbook) by Vigyan Bodh Prakashan, ISBN : 978-81-924048-3-7.