On $S_{2\frac{1}{2}}$ mod $\mathcal{I}$ spaces and θ^I closed sets

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Abstract

In this paper we will introduce $S_{2\frac{1}{2}}$ mod $\mathcal{I}$ spaces and discuss their properties. We also introduce θ^I closed sets using the local closure function and obtain the sufficient conditions for a set to be θ^I closed.

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1 Introduction and Preliminaries

The subject of ideals in topological spaces has been studied by Kuratowski[3] and Vaidyanathaswamy[5]. An ideal $\mathcal{I}$ on a topological space (X, τ) is a collection of subsets of X which satisfies that (i) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X known as ideal topological space and $(.)^* : \wp(X) \to \wp(X)$, called a local function[3] of A with respect to $\mathcal{I}$ and τ, is defined as follows: for $A \subseteq X$, $A^*(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I}$ for every open nhd. U of x in $X\}$. A Kuratowski closure operator $cl^*(.)$ for a topology $\tau^*(\mathcal{I}, \tau)$, called the $*$-topology, finer than τ, is defined by $cl^*(A) = A \cup A^*(\mathcal{I}, \tau)$[4]. A topological (X, τ) is said to be $S_{2\frac{1}{2}}$ if for any two distinct points x, y of X, whenever one of them has open set not containing the other then there exist open sets U and V such that $x \in U, y \in V$ and $\overline{U} \cap \overline{V} = \emptyset$. When there is no chance of confusion, we will simply write A^* for $A^*(\mathcal{I}, \tau)$ and $\tau^*(\mathcal{I})$ for $\tau^*(\mathcal{I}, \tau)$.

Throughout this paper (X, τ) will denote topological space on which no separation axioms are assumed. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal space. For a subset A of X, $cl(A)$ and $int(A)$ will denote the closure of A, interior of A in (X, τ), respectively, $cl^*(A)$ and $int^*(A)$ will denote the closure of A, interior of A in (X, τ^*), respectively, and A^C will denote the complement of A in X.

Lemma 1.1. *[1] Let $(X, \tau, \mathcal{I})$ be an ideal space. Then for any subset A of X the following holds:*

(a) $A^* \subset \Gamma(A)(\mathcal{I}, \tau) \subset cl_\theta(A)$.

(b) $\Gamma(A)(\mathcal{I}, \tau) = cl(\Gamma(A)(\mathcal{I}, \tau))$.

2 Results

We begin by defining $S_{2\frac{1}{2}}$ mod $\mathcal{I}$ spaces.

Definition 2.1. An ideal space $(X, \tau, \mathcal{I})$ is said to be $S_{2\frac{1}{2}}$ mod $\mathcal{I}$ if for any two distinct points x, y of X, whenever one of them has open set not containing the other then there exist open sets U and V such that $x \in U, y \in V$ and $\overline{U} \cap \overline{V} \in \mathcal{I}$.

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Since $\emptyset \in \mathcal{I}$. Therefore, $S_{2\frac{1}{2}}$ space is $S_{2\frac{1}{2}}$ mod $\mathcal{I}$, but the following Example 2.1 shows that the converse need not be true.

Example 2.1. Let $X = \{x, y, z\}$, $\tau = \{\emptyset, \{x\}, \{z\}, \{x, z\}, \{y, z\}, X\}$, $\mathcal{I} = \{\emptyset, \{y\}, \{z\}, \{y, z\}\}$. Then X is $S_{2\frac{1}{2}}$ mod $\mathcal{I}$ but not $S_{2\frac{1}{2}}$.

Theorem 2.1. *If an ideal space $(X, \tau, \mathcal{I})$ is $S_{2\frac{1}{2}}$ mod $\mathcal{I}$ and $\mathcal{I} \subset \mathcal{J}$ then $(X, \tau, \mathcal{J})$ is $S_{2\frac{1}{2}}$ mod $\mathcal{J}$.*

Proof. Proof is obvious and hence is omitted. □

The following Example 2.2 shows that if (X, τ^*) is $S_{2\frac{1}{2}}$, then X need not be $S_{2\frac{1}{2}}$ mod $\mathcal{I}$.

Example 2.2. Let $X = \{x, y, z\}$, $\tau = \{\emptyset, \{y\}, \{z\}, \{y, z\}, X\}$, $\mathcal{I} = \{\emptyset, \{y\}, \{z\}, \{y, z\}\}$. So $\tau^* = \wp(X)$ and hence (X, τ^*) is obviously $S_{2\frac{1}{2}}$, but X is not $S_{2\frac{1}{2}}$ mod $\mathcal{I}$. Since $\{y\}$ has a open set not containing $\{z\}$, but $\overline{\{y\}} \cap \overline{\{z\}} = \{x, y\} \cap \{x, z\} = \{x\} \notin \mathcal{I}$.

Even though we have seen that if (X, τ^*) is $S_{2\frac{1}{2}}$, then X need not be $S_{2\frac{1}{2}}$ mod $\mathcal{I}$. but the following Theorem 2.2 shows that for codense ideals (X, τ^*) is $S_{2\frac{1}{2}}$ implies X is $S_{2\frac{1}{2}}$ mod $\mathcal{I}$.

Theorem 2.2. *Let $(X, \tau, \mathcal{I})$ be an ideal space where $\mathcal{I}$ is codense and (X, τ^*) is $S_{2\frac{1}{2}}$ then X is $S_{2\frac{1}{2}}$ mod $\mathcal{I}$.*

Proof. Let $x, y \in X$ be any two distinct points such that one of them has τ-open and hence τ^*-open subset not containing the other. Then (X, τ^*) is $S_{2\frac{1}{2}}$ implies there exist basic open set $G - I, H - J$ where G, H are open in x and $I, J \in \mathcal{I}$ such that $x \in G - I, y \in H - J$ and $cl^*(G - I) \cap cl^*(H - J) = \emptyset$ and so by $[cl^*(G) - I] \cap [cl^*(H) - J] = \emptyset$. This implies that $(cl^*(G) \cap cl^*(H)) - (I \cup J) = \emptyset$. Therefore, $(cl^*(G) \cap cl^*(H)) \subset (I \cup J) \in \mathcal{I}$. Now $\mathcal{I}$ is codense implies that $cl^*(G) = cl(G)$ for every open subset G of X. Hence $cl(G) \cap cl(H) \in \mathcal{I}$ implies that X is $S_{2\frac{1}{2}}$ mod $\mathcal{I}$. □

Definition 2.2. An ideal space $(X, \tau, \mathcal{I})$ is said to be ${}^*S_{2\frac{1}{2}}$ mod $\mathcal{I}$ if for any two distinct points x, y of X, whenever one of them has τ^*-open set not containing the other then there exist open sets U and V such that $x \in U, y \in V$ and $\overline{U} \cap \overline{V} \in \mathcal{I}$.

It can be seen easily that $(X, \tau, \mathcal{I})$ is ${}^*S_{2\frac{1}{2}}$ mod $\mathcal{I}$ implies $S_{2\frac{1}{2}}$ mod $\mathcal{I}$ but the following Example 2.3 shows that the converse is not true.

Example 2.3. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X\}$, $\mathcal{I} = \{\emptyset, \{a\}\}$. So $\tau^* = \{\emptyset, \{b, c\}, X\}$ and hence $(X, \tau, \mathcal{I})$ is obviously $S_{2\frac{1}{2}}$ mod $\mathcal{I}$, but $\{c\}$ has a τ^*-open set not containing $\{a\}$ and X is the only open subset containing $\{a\}$ and $\{c\}$ implies that $(X, \tau, \mathcal{I})$ is not ${}^*S_{2\frac{1}{2}}$ mod $\mathcal{I}$.

In [2], Gupta and Noiri introduced QHC spaces with respect to an ideal written $\mathcal{I}$-QHC(where An ideal space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-QHC if for every open cover $\{G_\alpha : \alpha \in \Delta\}$ of X, there exists a finite subset Δ_0 of Δ such that $X - \bigcup\{cl(G_\alpha) : \alpha \in \Delta_0\} \in \mathcal{I}$). We now discussed some properties of $\mathcal{I}$-QHC spaces.

Theorem 2.3. *Let $(X, \tau, \mathcal{I})$ be $S_{2\frac{1}{2}}$ mod $\mathcal{I}$ space and F be $\mathcal{I}$-QHC subset of X such that $x \notin \overline{F}$ then there exist open subsets U and V such that $x \in U$ and $F - \overline{V} \in \mathcal{I}$ and $\overline{U} \cap \overline{V} \in \mathcal{I}$*

Proof. Let F be any $\mathcal{I}$-QHC subset of X and $x \in X$ be any element such that $x \notin \overline{F}$ then $x \in X - \overline{F}$. Therefore, for all $y \in F$, x has a open set $X - \overline{F}$ not containing the elements of F and so X is $S_{2\frac{1}{2}}$ mod $\mathcal{I}$ implies that there exist open subsets U_y, V_y containing x, y respectively such that $\overline{U_y} \cap \overline{V_y} \in \mathcal{I}$ and $F \subseteq \bigcup_{y \in F} V_y$. Further, F is $\mathcal{I}$-QHC subset of X implies that there exist finite subset F_0 of F such that $F - \bigcup_{y \in F_0} \overline{V_y} \in \mathcal{I}$ and so $F - \overline{\bigcup_{y \in F_0} V_y} \in \mathcal{I}$. Consider $U = \bigcap_{y \in F_0} U_y$ and $V = \bigcup_{y \in F_0} V_y$ then U is the open subset containing x and $F - \overline{V} \in \mathcal{I}$ and $\overline{U} \cap \overline{V} \in \mathcal{I}$. □

Theorem 2.4. *Let $(X, \tau, \mathcal{I})$ be an ideal space and K be $\mathcal{I}$-QHC subset of X then $cl^*(K)$ is also $\mathcal{I}$-QHC.*

Proof. Let $G_{\alpha\alpha}$ be open open cover of $cl^*(K)$ so that $cl^*(K) \subseteq \bigcup_\alpha G_\alpha$ and so $K \subseteq cl^*(K) \subseteq \bigcup_\alpha G_\alpha$. But K is $\mathcal{I}$-QHC subset of X implies that $K - \bigcup_{i=1}^n \overline{G_{\alpha_i}} \in \mathcal{I}$. Let $G = \bigcup_{i=1}^n G_{\alpha_i}$ so that $K - \overline{G} \in \mathcal{I}$. Now we will prove that $cl^*(K) - \overline{G} \in \mathcal{I}$. For this we will prove that $cl^*(K) - \overline{G} \subseteq K - \overline{G}$.

Let $x \notin K - \overline{G}$. Then there can be two possibilities: case(i) $x \notin K$ case(ii) $x \in \overline{G}$. Now if $x \in \overline{G}$ then obviously $x \notin cl^*(K) - \overline{G}$ and if $x \notin K$ but $x \notin \overline{G}$. Then $x \in (\overline{G})^C$. This implies that $(\overline{G})^C$ is open set containing x and $(\overline{G})^C \cap K \in \mathcal{I}$ implies that $x \notin K^*$ and so $x \notin K \cup K^*$ and so $x \notin cl^*(K)$. Thus, $x \notin cl^*(K) - \overline{G}$. Therefore, $cl^*(K) - \overline{G} \subseteq K - \overline{G}$. Hence $cl^*(K) - \overline{G} \subseteq K - \overline{G} \in \mathcal{I}$ and so $cl^*(K) - \overline{G} \in \mathcal{I}$ implies that $cl^*(K)$ is $\mathcal{I}$-QHC. □

In [1], Al-Omari and Noiri defined the local closure function in ideal topological spaces (where in an ideal topological sapce $(X, \tau, \mathcal{I})$ for a subset A of X, the local closure function of A denoted by $\Gamma(A)(\mathcal{I}, \tau)$ is defined as $\Gamma(A)(\mathcal{I}, \tau) = \{x \in X : \overline{U} \cap A \notin \mathcal{I}$ for every τ-nhd. U of x in X $\})$. Before our further results firstly, we will define θ^I closed sets using the local closure function.

Definition 2.3. Let $(X, \tau, \mathcal{I})$ be an ideal space and A be any subset of X. Then A is said to be θ^I closed if $\Gamma(A)(\mathcal{I}, \tau) \subseteq A$.

Theorem 2.5. *Let $(X, \tau, \mathcal{I})$ be ${}^*S_{2\frac{1}{2}}$ mod $\mathcal{I}$ space and K be any $\mathcal{I}$-QHC subset of X. Then K is θ^I closed if and only if K or K^C is union of τ^*-closed subsets of X.*

Proof. Firstly, let K is θ^I closed and so τ^*-closed. This implies that K is union of τ^*-closed sets. Conversely, let $K = \bigcup_\alpha F_\alpha$, where F_α are τ^*-closed subsets of X. Then we will prove that $\Gamma(K)(\mathcal{I}, \tau) \subset K$. Let $x \in \Gamma(K)(\mathcal{I}, \tau)$ be any element then for every open subset G containing x, $\overline{G} \cap K \notin \mathcal{I}$. Consider the filter $\mathcal{F}$ generated by the filterbase $\mathcal{F}(\mathcal{B}) = \{\overline{G} \cap A : G$ is open subset of X containing x $\}$. Then it can be easily seen that $\mathcal{F}$ is the filter containing the closure of every open set containing x and $\mathcal{F} \cap \mathcal{I} = \emptyset$. Further, K is $\mathcal{I}$-QHC subset of X implies that there exists $y \in K$ such that $y \in \bigcap_{F \in \mathcal{F}} \Gamma(F)(\mathcal{I}, \tau)$. Therefore, there exists α such that $y \in F_\alpha$. Now, let $x \notin K$, then $x \notin F_\alpha$. So $x \in F_\alpha^C$. This implies that F_α^C is τ^*-open nhd. of X containing x but not y. Therefore, X is ${}^*S_{2\frac{1}{2}}$ mod $\mathcal{I}$ implies that there exist open sets U and V of X containing x and y respectively such that $\overline{U} \cap \overline{V} \in \mathcal{I}$ and so $y \notin \Gamma(\overline{U})(\mathcal{I}, \tau)$. Also U is open subset of X containing x implies that $\overline{U} \in \mathcal{F}$. Therefore, $y \in \Gamma(\overline{U})(\mathcal{I}, \tau)$ which means that $\overline{U} \cap \overline{V} \notin \mathcal{I}$, which is a contradiction. Therefore, $x \in K$ and so $\Gamma(K)(\mathcal{I}, \tau) \subseteq K$. Hence K is θ^I closed. □

The following Examples show that we can not replace ${}^*S_{2\frac{1}{2}}$ mod $\mathcal{I}$ space by $S_{2\frac{1}{2}}$ mod $\mathcal{I}$ space or by (X, τ^*) is $S_{2\frac{1}{2}}$.

Example 2.4. Let X be any infinite set with indiscrete topology and $\mathcal{I} = \mathcal{I}_f$ = ideal of finite subsets of X. Then $\tau^* = \{G \subseteq X | X - G$ is finite$\}$ i.e. τ^* is cofinite topology. Now, it can be easily seen that X is $S_{2\frac{1}{2}}$ mod $\mathcal{I}$ space since no point of X has a neighbourhood not containing the other. Further, X is the only open subset of X so every subset of X is $\mathcal{I}$-QHC. Let K = any infinite subset of X so $K = \bigcup_{x \in K}\{x\}$ where each $x \in K$ is τ^*-closed i.e. K is union of τ^*-closed subsets of X. But K is not θ^I closed. Since X is the only open subset of X and $\overline{X} \cap K = X \cap K = K \notin \mathcal{I}$. Therefore, $\Gamma(K)(\mathcal{I}, \tau) = X$ and so $\Gamma(K)(\mathcal{I}, \tau) \nsubseteq K$. Hence K is not θ^I closed.

Example 2.5. Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. So $\tau^* = \wp(X)$. So it can be easily seen that (X, τ^*) is $S_{2\frac{1}{2}}$ but X is not ${}^*S_{2\frac{1}{2}}$ mod $\mathcal{I}$. Since a has τ^*-open subset $\{a\}$ not containing b but $\overline{a} \cap \overline{b} = \{a, c\} \cap \{b, c\} = \{c\} \notin \mathcal{I}$. Now, $\{c\}$ is τ^*-closed but $\Gamma(\{c\})(\mathcal{I}, \tau) = \{a, b, c\}$ and so $\Gamma(\{c\})(\mathcal{I}, \tau) \nsubseteq \{c\}$. Hence $\{c\}$ is not θ^I closed.

Even though we cannot replace ${}^*S_{2\frac{1}{2}}$ mod $\mathcal{I}$ space by $S_{2\frac{1}{2}}$ mod $\mathcal{I}$ space. But the following Theorem 2.6 holds.

Theorem 2.6. *Let $(X, \tau, \mathcal{I})$ be $S_{2\frac{1}{2}}$ mod $\mathcal{I}$ space and K be any $\mathcal{I}$-QHC subset of X. Then K or K^C is union of closed subsets of X implies that K is θ^I closed.*

Proof. Proof is similar to Theorem 2.5 and hence is omitted. □

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