Greek Labelings of graphs

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ABSTRACT: *Graph parameters related to Roman military defense strategy were studied by many persons [1, 2, 3, 4, 5, 6]. Satheesh and Sureshkumar introduced Roman labeling and Roman number of graphs [7]. In this paper, we extend this labeling to a new type of labeling called Greek labeling and a related graph parameter, called Greek Number, for a graph. The motivation for Roman and Greek labelings were discussed. Also, its properties are studied and its values for special types of graphs are explored.*

Keywords: *Graph, labeling, Roman labeling, Roman number*

1. *Introduction*

The basic idea behind it is that if we consider the edges and the vertices of a graph as some streets and the junctions, which are the meeting points of the streets, then the label of a vertex is the number of soldiers deployed at that junction and we require that every street (edge) should be guarded by at least 1 soldier. That is, in case of any street having no soldiers, then there should be an adjacent junction with two soldiers so that one of them can be deployed to the former junction in case of emergency.

2. Applications in Military and Defense Strategic Planning

Motivation behind Roman labeling is the "defense in depth" strategy of Emperor Constantine, adopted in the beginning of 4^{th} century AD. Due to the huge expense of setting up military camps at a strategically important place, some places are selected to set up camps. Selection of such places are made after considering the total cost of maintaining the troops at the stations and the cost of defending the places which could be targeted by enemies. Defense centers should be selected so that troops could defend all other places effectively from the places where they are based at. We can minimize the cost of defense by minimizing the number of centers. Thus the problem is equivalent the problem to find the minimal Roman labeling of the underlying graph. It is based on the oversimplified assumption that total cost of maintaining defense center at a vertex is same.

We can restate the problem in a more realistic way. Let the cost of setting up a defense station at the vertex v_i be s_i and the running cost be r_i. The total cost of v_i is $t_i = s_i + r_i$. It is assumed that cost of a military operation from a vertex to another vertex is same, between any pairs of vertices.

The best strategy of the problem is the Roman labeling with minimum cost. This minimum cost corresponds to a minimal Roman labeling.

The problem can be more generalized assuming that the cost of maintaining two units of army at a place is double the cost of maintaining one unit. Thus we have two choices of cost at each vertex. At v_i if there is only one unit, $t_i = s_i + r_i$ and $t_i = s_i + 2r_i$ otherwise. Minimum cost of this problem may not correspond to a minimal Roman labeling. Thus we have to consider all Roman labelings in this case, to obtain the solution.

3. Main Results

The Roman labeling and Roman number of graphs was introduced by Satheesh and Sureshkumar[9]. In this paper, a variation of Roman labeling and Roman number is introduced and studied.

Roman labeling of a connected graph G is a function f: V(G) → {0, 1, 2} such that any vertex with label 0 is adjacent to a vertex with label 2. The function value f(v) of a vertex v of the graph G is called the label of v.

It can be easily seen that if G has a Roman labeling, then for any edge e = {u, v}, either both u and v are adjacent to vertices with labels at least 1 or the edge e is incident with a vertex with label 2.

Clearly, the function, f, partitions the vertex set, V(G) into 3 vertex subsets, V_0, V_1 and V_2 .which are the subsets of V(G) with labels 0, 1,2 respectively.

Weight of a Roman labeling, f is defined as the sum of all vertex labels.That is, $w(f) = \sum_{v \in V(G)} f(v)$. Roman number of a graph G is defined as the minimum weight of a Roman labeling on G and is denoted by S(G). A Roman labeling with minimum weight is called a minimal Roman labeling.

For terms and definitions not explicitly defined here, reader can refer Harary [8].

In this section, we extend the concept of Roman labeling and Roman number to Greek labeling and Greek Number. We also determine Greek number of some classes of graphs.

Definition 3.1. A function f : E → {0, 1, 2} is called a Greek labeling if every vertex v of G is either incident with an edge with label, 1 or incident with an edge which is adjacent to an edge with label, 2.

A Greek labeling is a minimal Greek labeling if we cannot obtain another Greek labeling, reducing the label at any of the vertices. The Greek Number of a graph G is, T(G)=min{f(E):f is an Greek labeling of G}.

Weight of a Greek labeling, f is defined as the sum of all vertex labels. That is, $w(f) = \sum_{v\in V(G)} f(v)$. Greek number of a graph G is defined as the minimum weight of a Greek labeling on G and is denoted by T(G). A Greek labeling with minimum weight is called a minimal Greek labeling.

We can define three related edge sub-sets, E_1, E_2, E_3, which are the subsets of E(G) with labels 0, 1, 2 respectively.

Let f be a Greek labeling of a graph. All vertices in G are either adjacent to a vertex, which is incident with an edge in E_2 or incident with an edge in E_1. Thus all edges in G are either a member of $E_1 \cup E_2$ or adjacent to at least one edge in $E_1 \cup E_2$. Also, all edges in E_0 are adjacent to at least one edge in E_2.

Lemma 3.2. Let f be a minimal Greek labeling of graph G. An edge a in E_2 and an edge b in E_1 cannot be adjacent.

Proof. Let b={u, v}. Since f(a) = 2 and a is incident with either u or v, the second guard at a can take care of the other vertex. The function g defined by g(a) = 2, g(c) = 0, where c is any other edge of G is a Greek labeling such that w(g) < w(f). It contradicts the fact that f is a minimal Greek labeling.

Lemma 3.3. If $E_2 = \Phi$ for all minimal Greek labeling, f of G, then G is the union of a set of K_2's.

Proof. We assume that the graph contains at least two edges. So the graph has minimum one minimal Greek labeling. Also, $|E_1^f| = |E_1^g|$ for any twominimal Greek labelings, f, gof G. Each end vertex of an edge with label 1 should be incident with an edge of label 1. So, G contains only $2|E_1|$ edges. Thus G can contain more edges,which links two edges present in E_1^f. In other words, there exists a path $P_3 = (v_1e_1v_2e_2v_3e_3v_4)$ such that e_1 and e_3 are in E_1. Now we can easily define a new minimal Greek labeling h where $h(e_2) = 2$, $h(e_1) = 0$, $h(e_3) = 0$ and h(e) = f(e) for all other edges. Then E_2^h is non-empty, which is a contradiction.

We observe that if f is a Greek labeling of G and e is a leaf, then either e belongs to E^f_1 or there is an edge in E^f_2, which is adjacent to e. That is, the edge e must have label 1 or it must be adjacent to another edge with label 2.

A spider graph is denoted by S_n has the vertex set $V(S_n) = \{u, v_1, v_2, ..., v_n, w_1, w_2, ..., w_n\}$, where d(u) = n, $d(v_i) = 2$ and $d(w_i) = 1$ for all i = 1, 2, ..., n.

Proposition 3.4. Let G be a spider graph containing vertices $\{u, v_1, v_2, ..., v_n, w_1, w_2, ..., w_n\}$. Then $T(S_n) = n + 1$.

Proof. We can assign labels to the edges of the spider byassigning 1 to each edge (vi,wi) for all i and to one of the edges (u, vi). Then, T(G) = n+1.

Another function g can be defined by assigning two to (u, vi), for exactly one value of i. Assume that the value of i is 1, for simplicity. Then we must assign g(vi,wi) = 1 for i = 2, 3, ..., n. So g(E) = (n−1)+2 = n+1.We cannot define any other function having smaller function value.

A spider with one or more of its legs missing is called a wounded spider. A leg of a spider is the edge (vi,wi) for one i. If all legs are missing the spiderbecomes a star graph. A wounded spider with r legs can de denoted by Sn,r.

V(Sn,r) = {u, v1, v2, ..., vn,w1, w2, ...,wr} such that d(u) = n, d(vi) = 2 fori = 1, 2, ..., r, d(vi) = 1 for i = r + 1, r + 2, ..., n and d(wi) = 1 for alli = 1, 2, ..., r.

Proposition 3.5. Let G be a wounded spider graph containing a vertex uhaving degree n and r legs, where r<n. T(G) = n−r+1.

Proof. A labeling, g can be defined by assigning 2to (u, vi), for exactly one value of i. Assume that the value of i is 1,for simplicity. Then we must assign g(vi,wi) = 1 for i = 2, 3, ..., n. So T(G) = (n−r−1)+2 = n−r+1.

Lemma 3.6. If f is a minimal Greek labeling of G, then the induced subgraph $<E_1^f>$does not contain a path of length 3.

Proof. Suppose E_1^f contains a path $P_3= (v_1e_1v_2e_2v_3e_3v_4)$. Now we can definea new function g by $g(e_2) = 2$, $g(e_1) = 0$, $g(e_3) = 0$ and g(e) = f(e) for all other edges. Then, w(g) <w(f), contradicting the fact that f is a minimal Greek labeling of G.

Proposition 3.7.Let f be a Greek labeling of a graph G and $|E_1^f| = a$ and $|E_2^f| = b$. Then, $n - 2a - 2b \le \sum_x d(x)$, where x is adjacent to an edge in E_2^f

Proof. Since, an edge in E_1^f satisfy its Greek labeling condition only at its end vertices, while an edge in E_2^f satisfy its Greek labeling condition at all the vertices adjacent to each of its end vertices. So,$n - 2a - 2b \le \sum_x d(x)$, where x is adjacent to an edge in E_2^f

Proposition 3.8. For a connected graph G, T(G) =1 if and only if G = K_2.

Proof. We know that T(G) = 1 if and only if G has a Greeklabeling, f such that f(e) = 1 for exactly one edge e and f(x) = 0 for any other edges x. This is possible if and only G contains no edge other than e. Hence G = K_2.

Proposition 3.9. For a connected graph G with n vertices, T(G) = 2 if and only if there exist an edge e={u, v} such that d(u)+d(v) = n−1.

Proof. Suppose that, T(G) = 2. Then we have to consider two cases.

Case.1: There exists a Greek labeling, f such that f(e) = 2 for exactly one edge e={u, v} and f(u) = 0 for any other edge u. Then, all vertices other than u and v are adjacent to either u or v, so that d(u) + d(v) =n − 1.

Case.2: There exists aGreek labeling, f such that f(a)=f(b)=1. Let a={u, v} and b={x, y}.Since G is connected, there exists an edge, say c, connecting a, b. We can define a new Greek labeling, g such that g(c)=2 and g(x)=0 for all other edges. So, it reduces to case 1.

To prove the converse, suppose there exist an edge e={u, v} such that d(u)+d(v) = n−1. Then the function g defined by g(e) = 2 and g(x) = 0 for all other edges, is a Greek labeling. So T(G) = 2.

Proposition 3.10. For a connected graph G, T(G) = 3 if and only if G contains an edge e={u, v} and a K_2 or K_1 such that there is no edge connecting the vertices of K_2 or K_1 to either u or v in G. All edges from K_2 or K_1 are connected to a vertex adjacent to u or v, other than these vertices. The graph G is the union of the induced subgraph$< N[u] \cup N[v] >$, K_2 or K_1 and allthe edges connecting the vertices of K_2 or K_1 to the induced subgraph.

Proof. If T(G) = 3, then we have to consider 2 cases.

Case.1: There exists a Greek labeling f such that f(e) = 2 for exactlyone edge e={u, v}, f(a) = 1 for an edge a={x, y} and other edges have function value zero. Since f(e) = 2, The second soldier can guard all the places connected to the places u or v. Since a separate soldier is needed to guard places represented by K_2 or K_1, definitely the vertices of this portion of the graph are neither adjacent to u nor adjacent to v. Hence we get the result.

Case.2: There exists a Greek labeling f such that f(a) = f(b) = f(c) =1, a, b and c are non-adjacent edges of G. Let a={u, v}, b={x, y} and c={o, p}. Then G does not contain another vertex. So n = 6. Since G is connected, a pair of edges among a, b and c are joined by an edge. Assume that the edges a, b are joined by the edge ux.

Then it is possible to define a new Greek labeling g such that g(a) = 2,g(c) = 1 and g(d) = 0, for all other edges d in G. Thus this case reduces tocase 1.

To prove the converse, suppose the graph G is the union of the induced subgraph < N[u] [N[v] >, K_2 or K_1 and all the edges connecting thevertices of K_2 or $K_1$to the induced subgraph. First suppose K_2 is in G and V (K_2) = {x, y}. The function f(uv) = 2, f(xy) = 1 and f(d) = 0 forall other edges, d is Greek labeling of G. On the other hand, if K_1 is present in G, then the function f(uv) = 2, f(xy) = 1 where xy is an edge connecting $K_1$to < N[u] [N[v] > and f(d) = 0 for all other edges, d is Greek labeling of G.

Proposition 3.11. $T(K_n) = 2$ where n > 2 and $T(K_1) = 1$.

Proof. Clearly, $T(K_1) = 1$. When n>2, let e={u, v} be any edge of K_n. Every other vertex is adjacentto both u and v. So $T(K_n) = 2$.

Proposition 3.12.$T(K_{n_1,n_2 \dots n_r}) = 2$

Proof. Let e be an edge connecting a vertex in one partition to a vertex inanother partition. If we label 2to this edge and 0 to all other edges, then this function is aGreek labeling, so that $T(K_{n_1,n_2 \dots n_r}) = 2$

Proposition 3.13. If G = P_n, then

$$EVS(P_n) = \begin{cases} 2r & if n = 4r \\ 2r = 1 & if n = 4r + 1 \; or \; 4r + 2 \\ 2r = 2 & if n = 4r + 3 \end{cases}$$

Proof.

Case 1: When n = 4r.

Along the path we take a block of three edges, containing four vertices. We take the middle edge from each blockand assign label 2. All remaining edges are assigned zero. The function obtained is a Greek labeling of G.

Case 2: When n = 4r + 1.

After making the blocks of four vertices one vertex is left, which is connected to one of the blocks by an edge e. In addition to the labels assigned above we give one to the edge e toobtain a Greek labeling of G.

Case 3: When n = 4r + 2

A K_2 is connected to one of the blocks by an edge. We can give label 1 to the edge in K_2, which make that edge satisfy the Greek labeling condition for the vertices of K_2.

Case 4: When n = 4r + 3

A P_3 is connected to one of the blocks by an edge. We have to give label 2 to one of the edges in P_3 in addition to the labels assigned to the edges in the blocks. The function defined is a Greek labeling.

Similarly, we can prove the following:

Proposition 3.14.

$$T(C_n) = \begin{cases} 2r & if n = 4r \\ 2r + 1 & if n = 4r + 1 \; or \; 4r + 2 \\ 2r + 2 & if n = 4r + 3 \end{cases}$$

The product of two graphs G_1 and G_2 is a the graph $G_1 \times G_2$ whose vertex set is $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and two vertices u = (u_1, u_2) and v = (v_1, v_2) are adjacent in $G_1 \times G_2$ whenever $u_1 = v_1$ and u_2 is adjacent to u_2 or $u_2 = v_2$ and u_1 is adjacent to u_1.

Proposition 3.15. If $T(G_1)$ and $T(G_2)$ be Greek numbers of G_1 and G_2 respectively and n_1 and n_2 be the orders of the graphs, then $T(G_1 \times G_2) \leq \lim\{n_1T(G_2), n_2T(G_1)\}$.

Proof. It is clear that $G_1 \times G_2$ contains n−2 copies of G_1 and n_1 copies of G_2. So the copies of the minimal Greek labeling of G_1 and G_2 are enough to satisfy its Greek labeling condition for all the vertices in $G_1 \times G_2$. So the result follows.

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