The Stress-Intensity Factors for Four Griffith-Cracks Opened by Asymmetrical Force at Crack Faces in Isotropic Infinite Medium

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ABSTRACT:

We have obtained closedform solution for stress-intensity factors and of crack shape for four Griffith cracks opened by asymmetrical forces at crack faces by using Fourier transform method which reduces the problem to integral equations.

KEYWORDS: *Stress Intensity Factors (SIF), Crack-opening displacement (C.O.D.), Fourier transforms, Integral equation (I.E.).*

1. INTRODUCTION

In the present research endeavour we tried to solvefor isotropic multiply connected infinite medium under plane-strain conditions. The presence of four Griffith-cracks makes the medium multiply connected and mathematically the problem becomes mixed-boundary value problem, because at the crack face stresses are prescribed and in remaining portion of boundary displacements components are prescribed.

Modern time is of smart devices. If cracks are developed in them then their efficiency will be reduced. Zhonget. al. [1] discussed that temperature change will affect the overall performance of smart devices. Liu [2] discussed the effects of temperature dependent material properties on stress and temperature fields in a cracked metal plate under electric current load. Hasanyan et.al. [3] discussed cracked plates carrying non-stationary electrical current. They reduced the problem to a system of singular integral equation with Cauchy-type singular Kernel.

Sneddon [4] used Fourier transform which reduced the problem to dual integral equation. Srivastava and Lowengrub[5] solved for triple integral equations. Kushwaha [6] obtained the solution for quintuple integral equation.

There are good accounts of crack problems [7, 8, 9, 10] with different types of geometries. Sneddon and Ejike [11] had solved for a crack opening due to asymmetrical forces at crack faces. Tweed [12] extended for two Griffith-cracks by extending the method of [11]. There are few more problems [13, 14, 15]. These problems can be solved by the method of Kushwaha and Awasthi [14]. But in present research paper we shall be using the method of [6].

There are few more related problems of crack in different geometries, medium and using different methods, see [17, 18, 19, 20, 21].

The cracks occupy the space in infinite isotropic medium as $y=0,\ b<|x|<c,\ d<|x|<e<\infty$ which gives that inner pair of cracks are equal in length and equally spaced with respect to *y*-axis at *x*-axis while outer pair of cracks i.e. $y=0,\ d<|x|<e<\infty$ is of equal lengths & equally spaced at different places than the inner pair. The physical problem of four Griffith-cracks opened by asymmetrical forces at crack faces is reduced to the following mixed-boundary value problem as:

$$\sigma_{xy}\left(x,0^{\pm}\right)=\begin{cases} q_1^{\pm}(x), & x\in I_1^{\pm} \\ q_2^{\pm}(x), & x\in I_2^{\pm} \\ q_3^{\pm}(x), & x\in I_3^{\pm} \\ q_4^{\pm}(x), & x\in I_4^{\pm} \end{cases} \quad (1.1)$$

$$\sigma_{yy}\left(x,0^{\pm}\right)=\begin{cases} p_1^{\pm}(x), & x\in I_1^{\pm} \\ p_2^{\pm}(x), & x\in I_2^{\pm} \\ p_3^{\pm}(x), & x\in I_3^{\pm} \\ p_4^{\pm}(x), & x\in I_4^{\pm} \end{cases} \quad (1.2)$$

See figure 1.

where, $I_1^{\pm}=(b,c),\ I_2^{\pm}=(-c,-b),\ I_3^{\pm}=(d,e),\ I_4^{\pm}=(-e,-d)$ (1.3)

and ($\pm$) over quantities refer to $y>0$ and $y<0$ respectively, and the continuity conditions are satisfied at y = 0 over boundary other than the crack faces.

$$u_x(x,0^+)=u_x(x,0^-),\qquad u_y(x,0^+)=u_y(x,0^-) \quad (1.4)$$

$$\sigma_{xy}(x,0^+)=\sigma_{xy}(x,0^-),\ \ \sigma_{yy}(x,0^+)=\sigma_{yy}(x,0^-) \quad (1.5)$$

for $0\le|x|\le b,\quad c\le|x|\le d,\quad e\le|x|<\infty$. (u_x,u_y) and $(\sigma_{xx},\sigma_{xy},\sigma_{yy})$ and displacement and stress components at general point (x,y) of the medium. It is also assumed that all physical equations vanish as $\sqrt{x^2+y^2}\to\infty$. We checked throughout that,seeBurniston[22].

$$[u_y(x,0^+),\,u_y(x,0^-)]>0,\,x\in I_1^{\pm}\cup I_2^{\pm}\cup I_3^{\pm}\cup I_4^{\pm} \quad (1.6)$$

which means that cracks are really opening. The Fourier transform is defined as,

$$f_{c,s}(\xi)=\int_0^{\infty} f(x)\{\cos\xi x,\ \sin\xi x\}\,dx \quad (1.7)$$

with usual inversion formula.

The plan of the paper is as follows; section 1 deals with the history of problem and of method to solve these. Section 2 formulates the problem. The problem is reduced to quintuple integral equations in section 3. The solution of integral equation is given in section 4. Physical quantifies of interest in fracture mechanics are reported in terms of solution of integral equation in section 5. Section 6 presents two special cases of loading and then physical quantities are reported for two special cases. Discussion and conclusion is in section 7. The references are in the last.

2. FORMULATION

The solution of present linear elasticity problem is obtained by the method of Sneddon [4]. We split the physical quantities into symmetrical and anti-symmetrical quantities at every point of the medium as

$$\sigma_{ij}(x,y)=\sigma_{ij}^{s_k}(x,y)+\sigma_{ij}^{a_k}(x,y),\ \ i,j=x,y \quad (2.1)$$

$$u_i(x,y)=u_i^{s_k}(x,y)+u_i^{a_k}(x,y),\quad i=x,y \quad (2.2)$$

and$k$ = 3,4 which refer to y> 0 and y< 0, respectively.

s_k and a_k refer to symmetric and anti-symmetric part of the problem. The solution space $(-\infty,\infty)\cup(-\infty,\infty)$ i.e., whole two-dimensional space, is reduced to $[0,\infty)\cup[0,\infty)$ i.e. $x\geq 0, y=0.$ Now, we assume the displacement components as

$$u_x^{s_k}(x,y)=\alpha_0\int_0^\infty \frac{\sin(\xi x)}{\xi}\left[(1-\eta)\frac{\partial^2 G^{s_k}}{\partial y^2}+\eta\xi^2 G^{s_k}\right]d\xi \tag{2.3}$$

$$u_x^{a_k}(x,y)=\alpha_0\int_0^\infty \frac{\cos(\xi x)}{\xi}\left[(1-\eta)\frac{\partial^2 G^{a_k}}{\partial y^2}+\eta\xi^2 G^{a_k}\right]d\xi \tag{2.4}$$

$$u_y^{s_k}(x,y)=\alpha_0\int_0^\infty \frac{\cos(\xi x)}{\xi^2}\left[(1-\eta)\frac{\partial^3 G^{s_k}}{\partial y^3}+(\eta+2)\xi^2\frac{\partial G^{s_k}}{\partial y}\right]d\xi \tag{2.5}$$

$$u_y^{a_k}(x,y)=\alpha_0\int_0^\infty \frac{\sin(\xi x)}{\xi^2}\left[(1-\eta)\frac{\partial^3 G^{a_k}}{\partial y^3}+(\eta+2)\xi^2\frac{\partial G^{a_k}}{\partial y}\right]d\xi,\alpha_0=\frac{2(1+\eta)}{\pi E} \tag{2.6}$$

$$G^{s_k}(\xi,y)=(A_{k-2}+yB_{k-2})e^{\mp\xi y},G^{a_k}(\xi,y)=(C_{k-2}+yD_{k-2})e^{\mp\xi y} \tag{2.7}$$

Where (+) and (-) are to taken for k = 3, 4, respectively. i.e., y > 0 and y< 0. The above assumptions for displacement components satisfy continuity, compatibility relations along with satisfying equation of equilibrium. The function G is called Airy's stress function. η andE arePoisson ratio and Young's modulus of the medium. Mathematically the problem is solved if eight arbitrary constants in (2.7) – (2.8) are determined. Using relations (2.3) – (2.7) and the stress-strain relations we get the stress components as

$$\sigma_{xy}^{s_k}(x,y)=\mp\int_0^\infty \xi\sin(\xi x)[\xi A_{k-2}+B_{k-2}(\xi y-1)]e^{\mp\xi y}d\xi \tag{2.8}$$

$$\sigma_{xy}^{a_k}(x,y)=\pm\int_0^\infty \xi\cos(\xi x)[\xi C_{k-2}+D_{k-2}(\xi y-1)]e^{\mp\xi y}d\xi \tag{2.9}$$

$$\sigma_{yy}^{s_k}(x,y)=-\int_0^\infty \xi^2\cos(\xi x)[A_{k-2}+yB_{k-2}]e^{\mp\xi y}d\xi \tag{2.10}$$

$$\sigma_{yy}^{a_k}(x,y)=-\int_0^\infty \xi^2\sin\xi x[C_{k-2}+yD_{k-2}]e^{\mp\xi y}d\xi \tag{2.11}$$

$$\sigma_{xx}^{s_k}(x,y)=\int_0^\infty \xi\cos(\xi x)[\xi(A_{k-2}+yB_{k-2})-2B_{k-2}]e^{\mp\xi y}d\xi \tag{2.12}$$

$$\sigma_{xx}^{a_k}(x,y)=\int_0^\infty \xi\cos(\xi x)[\xi(C_{k-2}+yD_{k-2})-2D_{k-2}]e^{\mp\xi y}d\xi \tag{2.13}$$

3. REDUCTION TO QUINTUPLE INTEGRAL EQUATION

If continuity conditions are satisfied separately for symmetrical and anti-symmetrical, than the continuity of physical quantities are satisfied. Thecontinuity conditions (1.6) – (1.7) and (1.9) – (2.14) and using boundary conditions (1.1) – (1.2). Then using inversion of Fourier integral transform we get,

$$A_1-A_2=\xi^{-2}\left[\left\langle\int_b^c p_1^{e_1}(x)+\int_d^e p_1^{e_2}(x)\right\rangle\cos(\xi x)dx\right] \tag{3.1}$$

$$C_1-C_2=\xi^{-2}\left[\left\langle\int_b^c p_1^{0_1}(x)+\int_d^e p_1^{0_2}(x)\right\rangle\sin(\xi x)dx\right] \tag{3.2}$$

$$\xi(A_1+A_2)-(B_1-B_2)=\xi^{-1}\left[\left\langle\int_b^c Q_1^{e_1}(x)+\int_d^e Q_1^{e_2}(x)\right\rangle\sin(\xi x)dx\right] \tag{3.3}$$

$$\xi(C_1+C_2)-(D_1-D_2)=\xi^{-1}\left[\left\langle\int_b^c Q_1^{0_1}(x)+\int_d^e Q_1^{0_2}(x)\right\rangle\cos(\xi x)dx\right] \tag{3.4}$$

where

$$p_1^{e_1}(x)=p_1^+ + p_2^+ -(p_1^- + p_2^-),\quad p_1^{e_2}(x)=p_3^+ + p_4^+ -(p_3^- + p_4^-) \tag{3.5}$$

$$p_1^{0_1}(x)=p_1^+ - p_2^+ -(p_1^- - p_2^-),\quad p_1^{0_2}(x)=p_3^+ - p_4^+ -(p_3^- - p_4^-) \tag{3.6}$$

$$Q_1^{e_1}(x)=q_1^- - q_2^- -(q_1^+ - q_2^+),\quad Q_1^{e_2}(x)=q_3^- - q_4^- -(q_3^+ + q_4^+) \tag{3.7}$$

$$Q_1^{0_1}(x)=q_1^- + q_2^- -(q_1^+ + q_2^+),\quad Q_1^{0_2}(x)=q_3^- + q_4^- -(q_3^+ + q_4^+), \tag{3.8}$$

Thus we get four equations (3.1) – (3.4) which will determine four constants. The remaining four constants will be determined by the following mixed-boundary conditions. Now we make use of continuity conditions of displacement components given by (1.4) – (1.5) and splitting into symmetric and anti-symmetric part. Thus using (1.4) – (1.5) and (2.3) – (2.6) we get, after using (3.1) – (3.6), as

$$\int_0^\infty (B_1-B_2)\cos(\xi x)d\xi = t_1\begin{cases} F_{11}(x), & x\in I_1\\ F_{12}(x), & x\in I_3\\ F_{13}(x), & x\in I_5\end{cases} \tag{3.9}$$

$$\int_0^\infty (D_1-D_2)\sin(\xi x)d\xi = t_1\begin{cases} F_{21}(x), & x\in I_1\\ F_{22}(x), & x\in I_3\\ F_{23}(x), & x\in I_5\end{cases} \tag{3.10}$$

$$\int_0^\infty (B_1+B_2)\sin(\xi x)d\xi = t_1\begin{cases} F_{31}(x), & x\in I_1\\ F_{32}(x), & x\in I_3\\ F_{33}(x), & x\in I_5\end{cases} \tag{3.11}$$

$$\int_0^\infty (D_1+D_2)\cos(\xi x)d\xi = t_1\begin{cases} F_{41}(x), & x\in I_1\\ F_{42}(x), & x\in I_3\\ F_{43}(x), & x\in I_5\end{cases} \tag{3.12}$$

with

$$t_1=[2(1+\eta)/E]^{-1}, F_{13}=F_{21}=F_{31}=F_{43}=0, I_1=[0,b], I_3=[c,d], I_5=[e,\infty) \tag{3.13}$$

$$\left.\begin{aligned} &F_{12}=\frac{\pi}{2}\left\langle\int_b^c Q_1^{e_1}(x)dx+\int_d^e Q_1^{e_2}(x)dx\right\rangle, F_{12}=\frac{\pi}{2}\int_d^e Q_1^{e_2}(x)dx, F_{22}=\frac{\pi}{2}\int_b^c Q_1^{0_1}(x)dx\\ &F_{23}=\frac{\pi}{2}\int_d^e Q_1^{0_2}(x)dx,\ F_{32}=\frac{\pi}{2}\int_b^c p_1^{e_1}(x)dx,\ F_{33}=\frac{\pi}{2}\int_d^e p_1^{e_2}(x)dx+F_{32}\\ &F_{41}=\frac{\pi}{2}\int_b^c p_1^{0_1}(x)dx+F_{42}, F_{42}=\frac{\pi}{2}\int_d^e p_1^{0_2}(x)dx \end{aligned}\right\} \tag{3.14}$$

Now we evaluate $(\sigma_{yy}^{(3)}+\sigma_{yy}^{(4)})$ and $(\sigma_{xy}^{(3)}+\sigma_{xy}^{(4)})$ for symmetric and anti-symmetric by using (2.9) – (2.12) with (3.1) – (3.4) we get,

$$\int_0^\infty \xi(B_1 - B_2)\cos(\xi x)d\xi = \begin{cases} P_{11} + \int_b^c \frac{yQ_1^{e_1}(y)dy}{y^2 - x^2}, x \in I_2 \\ P_{12} + \int_b^e \frac{yQ_1^{e_2}(y)dy}{y^2 - x^2}, x \in I_4 \end{cases} \tag{3.15}$$

$$\int_0^\infty \xi(D_1 - D_2)\sin(\xi x)d\xi = \begin{cases} P_{21} + x\int_b^c \frac{Q_1^{0_1}(y)dy}{y^2 - x^2}, x \in I_2 \\ P_{22} + x\int_b^e \frac{Q_1^{0_2}(y)dy}{y^2 - x^2}, x \in I_4 \end{cases} \tag{3.16}$$

$$\int_0^\infty \xi(B_1 + B_2)\sin(\xi x)d\xi = \begin{cases} Q_{11} + x\int_b^c \frac{P_1^{e_1}(y)dy}{y^2 - x^2}, x \in I_2 \\ Q_{12} + x\int_d^e \frac{P_1^{e_2}(y)dy}{y^2 - x^2}, x \in I_4 \end{cases} \tag{3.17}$$

$$\int_0^\infty \xi(D_1 + D_2)\cos(\xi x)d\xi = \begin{cases} Q_{21} + \int_b^c \frac{P_1^{0_1}(y)dy}{y^2 - x^2}, x \in I_2 \\ Q_{22} + \int_d^e \frac{P_1^{0_2}(y)dy}{y^2 - x^2}, x \in I_4 \end{cases} \tag{3.18}$$

where

$$\left.\begin{array}{l} I_2 = (b,c), I_4 = (d,e) \quad \text{and} \\ P_{11} = p_1^+ + p_1^- + p_2^+ + b_2^-, \quad P_{12} = p_3^+ + p_3^- + p_4^+ + p_4^- \\ P_{21} = p_1^+ - p_1^- + (p_2^+ - p_2^-), P_{22} = p_3^+ - p_3^- + (p_4^+ - p_4^-) \\ Q_{11} = q_1^+ + q_1^- + q_2^+ + q_2^-, Q_{12} = q_3^+ + q_3^- + q_4^+ + q_4^- \\ Q_{21} = q_1^+ - q_1^- + q_2^+ - q_2^-, Q_{22} = q_3^+ - q_3^- + q_4^+ - q_4^- \end{array}\right\} \tag{3.19}$$

Thus remaining four constants $B_1, B_2;\ D_1, D_2$ will be determined through quintuple integral equations given by (3.9) – (3.12) & (3.15) – (3.18).

4. SOLUTION OF QUINTUPLE INTEGRAL EQUATIONS

The trial solution of above quintuple integral equations are assumed as, see [6]

$$B_1 - B_2 = \frac{2}{\pi\xi}\left[\left\langle \int_b^c g_{11}(t) + \int_d^e g_{12}(t) \right\rangle \sin(\xi t)dt - t_1 \left\langle \int_0^b F_{11}'(t) + \int_c^d F_{12}'(t) \right\rangle \sin(\xi t)dt\right] \tag{4.1}$$

$$D_1 - D_2 = \frac{2}{\pi\xi}\left[\left\{\left\langle \int_b^c g_{21}(t) + \int_d^e g_{22}(t) \right\rangle - t_1 \left\langle \int_c^d F_{22}'(t) + \int_e^\infty F_{23}'(t) \right\rangle\right\}(1 - \cos\xi t)dt\right] \tag{4.2}$$

$$B_1 + B_2 = \frac{2}{\pi\xi}\left[\left\{\left\langle \int_b^c g_{31}(t) + \int_d^e g_{32}(t) \right\rangle - t_1 \left\langle \int_c^d F_{32}'(t) + \int_e^\infty F_{33}'(t) \right\rangle\right\}(1 - \cos\xi t)dt\right] \tag{4.3}$$

$$D_1+D_2=\frac{2}{\pi\xi}\left[\left\{\left\langle\int_b^c g_{41}(t)+\int_d^e g_{42}(t)\right\rangle-t_1\left\langle\int_0^b F_{11}(t)+\int_c^d F'_{41}(t)+\int_e^\infty F'_{42}(t)\right\rangle\right\}\sin(\xi t)\,dt\right] \quad (4.4)$$

The above assumption (4.1) – (4.4) satisfies the equations (3.9) – (3.12) identically if

$$\int_b^c g_{11}(t)dt=t_1\langle F_{12}(d)-F_{11}(b)\rangle,\ \int_d^e g_{12}(t)dt=t_1F_{12}(d) \quad (4.5)$$

$$\int_b^c g_{21}(t)dt=-t_1F_{12}(d),\ \int_d^e g_{22}(t)=t_1\langle F_{22}(d)-F_{23}(e)\rangle \quad (4.6)$$

$$\int_b^c g_{31}(t)dt=-t_1F_{23}(c),\ \int_d^e g_{32}(t)dt=t_1\langle F_{32}(d)-F_{33}(e)\rangle \quad (4.7)$$

$$\int_b^c g_{41}(t)dt=t_1\langle F_{42}(c)-F_{41}(b)\rangle,\ \int_d^e g_{42}(t)dt=-t_1F_{42}(d) \quad (4.8)$$

where F_{ij} are given in (3.14). Now, the substation of (4.1) – (4.4) into (3.15) – (3.18) and using the method of Kushwaha [6] for inverting the integrals we get,

$$g_{11}(t)=\frac{\Delta_1(t)}{\pi^2\theta(t)},\ t\in I_2,\ g_{12}(t)=-\frac{\Delta_1(t)}{\pi^2\theta(t)},\ t\in I_4 \quad (4.9)$$

$$g_{21}(t)=\frac{t\Delta_2(t)}{\pi^2\theta(t)},\ t\in I_2,\ g_{22}(t)=-\frac{t\Delta_2(t)}{\pi^2\theta(t)},\ t\in I_4 \quad (4.10)$$

$$g_{31}(t)=\frac{t\Delta_3(t)}{\pi^2\theta(t)},\ t\in I_2,\ g_{32}(t)=-\frac{t\Delta_3(t)}{\pi^2\theta(t)},\ t\in I_4 \quad (4.11)$$

$$g_{41}(t)=\frac{\Delta_4(t)}{\pi^2\theta(t)},\ t\in I_2,\ g_{42}(t)=-\frac{\Delta_4(t)}{\pi^2\theta(t)},\ t\in I_4 \quad (4.12)$$

with $\theta(t)=\left\{\left|(t^2-b^2)(c^2-t^2)(d^2-t^2)(e^2-t^2)\right|\right\}^{1/2}$ (4.12)a

$$\Delta_1(t)=\left\langle\int_b^c T_{11}-\int_d^e T_{12}\right\rangle\frac{y\theta(y)dy}{y^2-t^2}+t_1\left\langle\int_c^d F'_{12}(y)+\int_e^\infty F'_{13}(y)+\right\rangle\frac{y\theta(y)}{y^2-t^2}dy+t^2L_1+L_2 \quad (4.13)$$

$$\Delta_2(t)=\left\langle\int_b^c T_{21}-\int_d^e T_{22}\right\rangle\frac{\theta(y)dy}{y^2-t^2}+t_1\left\langle\int_c^d F'_{22}-\int_e^\infty F'_{23}\right\rangle\frac{\theta(y)\,dy}{y^2-t^2}+t^2M_1+M_2 \quad (4.14)$$

$$\Delta_3(t)=\left\langle\int_b^c T_{31}-\int_d^e T_{32}\right\rangle\frac{\theta(y)dy}{y^2-t^2}+t_1\left\langle\int_c^d F'_{32}-\int_e^\infty F'_{33}\right\rangle\frac{\theta(y)\,dy}{y^2-t^2}+t^2R_1+R_2 \quad (4.15)$$

$$\Delta_4(t)=\left\langle\int_b^c T_{41}-\int_d^e T_{42}\right\rangle\frac{\theta(y)dy}{y^2-t^2}+t_1\left\langle\int_c^d F'_{41}\right\rangle\frac{\theta(y)\,dy}{y^2-t^2}+t^2N_1+N_2 \quad (4.16)$$

where $L_i,\ M_i,\ N_i,\ R_i,\ i=1,2$ are constants to be determined through (4.9) – (4.12) and (4.5) – (4.8). And,

$$T_{11}(x)=P_{11}(x)-\int_b^c\frac{yQ_1^e(y)dy}{y^2-x^2},T_{12}=P_{12}(x)-\int_d^e\frac{yQ_1^{e_2}(y)dy}{y^2-x^2} \quad (4.17)$$

$$T_{21}(x)=P_{21}(x)-\int_b^c \frac{xQ_1^{0_1}dy}{y^2-x^2},\ T_{22}(x)=P_{22}(x)-\int_d^e \frac{xQ_1^{0_2}dy}{y^2-x^2} \tag{4.18}$$

$$T_{31}(x)=Q_{11}(x)+x\int_b^c \frac{P_1^{e_1}dy}{y^2-x^2},\ T_{32}(x)=Q_{12}(x)+x\int_d^e \frac{P_1^{e_2}dy}{y^2-x^2} \tag{4.19}$$

$$T_{41}(x)=Q_{21}(x)+\int_b^c \frac{yP_1^{0_1}dy}{y^2-x^2},\ T_{42}(x)=Q_{22}(x)+\int_d^e \frac{yP_1^{0_2}dy}{y^2-x^2} \tag{4.20}$$

In next section we shall evaluate the physical quantities.

5. PHYSICAL QUANTITIES

The physical quantities, which are of practical importance in fracture mechanics, are stress-intensity factors at crack tips and the crack opening displacement.

Stress-Components

The stress components are evaluated for $y=0$ $x\in[0,b]\cup[c,d]\cup[e,\infty)$. The normal component of stress is $\sigma_{yy}(x,0)$ and shear stress is $\sigma_{xy}(x,0)$.

Normal Stress

The normal stress is evaluated through the value of integrals (3.15) – (3.16) for symmetric and anti-symmetric parts, respectively for $x\in I_2\cup I_3\cup I_5$.

$$\sigma_{yy}^{s_3}(x,0)=\sigma_{yy}^{s_4}(x,0),\quad x\in I_1\cup I_3\cup I_5$$

$$\sigma_{yy}^{s_3}(x,0)=\frac{2}{\pi}\left[\left\{\int_b^c g_{11}(t)+\int_d^e g_{12}(t)-t_1\left\langle\int_c^d F_{12}'(t)\right\rangle\right\}\frac{t\,dt}{t^2-x^2}+\frac{\pi}{2}F_1(x)\right] \tag{5.1}$$

with

$$F_1(x)=\begin{cases}\left\langle\int_b^c Q_1^{e_1}+\int_d^e Q_1^{e_2}\right\rangle\dfrac{ydy}{(y^2-x^2)}, x\in I_1\\ \int_d^e Q_1^{e_2}(y)\dfrac{dy}{(y^2-x^2)}, & x\in I_3\\ 0, & x\in I_5\end{cases} \tag{5.2}$$

$$\sigma_{yy}^{a_3}(x,0)=\sigma_{yy}^{a_4}(x,0)=\frac{1}{2\pi}\left[\int_b^c g_{21}(t)+\int_d^e g_{22}(t)-t_1\left\langle\int_c^d F_{22}'(t)+\int_e^\infty F_{23}(t)\right\rangle\right]\frac{xdt}{t^2-x^2}+\frac{\pi}{2}F_2(x) \tag{5.3}$$

$$F_2(x)=\begin{cases}\left\langle\int_b^c Q_1^{o_1}+\int_d^e Q_1^{o_2}\right\rangle\dfrac{tdt}{(t^2-x^2)}, x\in I_1\\ \int_d^e Q_1^{0_2}(t)\dfrac{tdt}{t^2-x^2}, & x\in I_3\\ 0, & x\in I_3\end{cases} \tag{5.4}$$

Shear Stress

The component of shear $\sigma_{xy}(x,0^+)=\sigma_{xy}(x,0^-)$ for $x\in I_1\cup I_3\cup I_5$ is evaluated through the value of integrals of left hand side of (3.17) – (3.18) which is given as,

$$\sigma_{xy}^{s_3}(x,0)=\sigma_{xy}^{s_4}(x,0)=\frac{x}{2\pi}\left[\left\{\int_b^c g_{31}(t)+\int_d^e g_{32}(t)+t_1\left\langle\int_c^d F_{32}(t)+\int_c^\infty F_{33}'(t)\right\rangle\right\}\frac{dt}{t^2-x^2}\right]$$

$$+\frac{\pi}{2}F_3(x),\quad x\in I_1\cup I_3\cup I_5 \qquad (5.5)$$

$$F_3(x)=\begin{cases}0\ ,x\in I_1\\ \int_b^c P_1^{e_1}(y)\dfrac{dy}{y^2-x^2},x\in I_3\\ \left\langle\int_b^c P_1^{e_1}(y)+\int_d^e P_1^{e_2}(y)\right\rangle\dfrac{dy}{(y^2-x^2)},x\in I_5\end{cases} \qquad (5.6)$$

And

$$\sigma_{xy}^{a_3}(x,0)=\sigma_{xy}^{a_4}(x,0)=-\frac{1}{2\pi}\left[\left\{\int_b^c g_{41}(t)+\int_{g_{42}}^e(t)dt+t_1\left\langle\int_c^d F_{42}'(t)+\int_e^\infty F_{43}'(t)\right\rangle\right\}\frac{tdt}{t^2-x^2}\right]+\frac{\pi}{2}F_4(x) \qquad (5.7)$$

$$F_4(x)=\begin{cases}0,x\in I_1\\ \int_b^c P_1^{0_1}(y)\dfrac{dy}{(y^2-x^2)},x\in I_3\\ \left\langle\int_b^c P_1^{0_1}(y)+\int_d^e P_1^{0_2}(y)\right\rangle\dfrac{dy}{(y^2-x^2)},x\in I_5\end{cases} \qquad (5.8)$$

Now, using the value of $g_{ij}(t)$, i = 1, 2, 3, 4; j = 1, 2 from (4.9) – (4.12) and evaluating the integrals we get,

$$\sigma_{yy}^{s_3}(x,0)=\begin{cases}\dfrac{\Delta_1(x)}{2\pi\theta(x)}+F_1(x),x\in I_1\\ -\dfrac{\Delta_1(x)}{2\pi\theta(x)}+F_1(x),x\in I_3\\ -\dfrac{\Delta_1(x)}{2\pi\theta(x)}+F_1(x),x\in I_5\end{cases} \qquad (5.9)$$

Where $\theta(x)$ and $\Delta_1(x)$ are given by (4.12)a and (4.13) and

$$\sigma_{yy}^{a_3}(x,0)=\begin{cases}\frac{x}{2\pi}\frac{\Delta_2(x)}{\theta(x)}+\frac{\pi}{4}F_2(x), x\in I_1\\ -\frac{x}{2\pi}\frac{\Delta_2(x)}{\theta(x)}+\frac{\pi}{4}F_2(x), x\in I_3\\ -\frac{x}{2\pi}\frac{\Delta_2(x)}{\theta(x)}+\frac{\pi}{4}F_2(x), x\in I_5\end{cases} \tag{5.10}$$

Where $\Delta_2(x)F_2$ are given by (4.14) & (5.4), respectively

$$\sigma_{xy}^{s_3}(x,0)=\begin{cases}\frac{x}{2}\frac{\Delta_3(x)}{\theta(x)}+\frac{\pi}{2}F_3(x), x\in I_1\\ -\frac{x}{2}\frac{\Delta_3(x)}{\theta(x)}+\frac{\pi}{4}F_3(x), x\in I_3\\ -\frac{x}{2}\frac{\Delta_3(x)}{\theta(x)}+\frac{\pi}{4}F_3(x), x\in I_5\end{cases} \tag{5.11}$$

where $\Delta_3(x)$ and $F_3(x)$ are given by (4.15) and (5.3), respectively.

$$\sigma_{xy}^{a_4}(x,0)=\begin{cases}\frac{\Delta_4(x)}{2\pi\,\theta(x)}+F_4(x), x\in I_1\\ -\frac{\Delta_4(x)}{2\pi\theta(x)}+F_4(x), x\in I_3\\ -\frac{\Delta_4(x)}{2\pi\theta(x)}+F_4(x), x\in I_5\end{cases} \tag{5.12}$$

Where $\Delta_4(x)$ and $F_4(x)$ are given by (4.16) and (5.8) respectively.

CRACK OPENING DISPLACEMENT

The crack shape will be obtained through the evaluation of crack opening displacement. The crack opening displacement will be evaluated through the integrals of left hand side of equations (3.9) – (3.10) and using the relation (4.1) – (4.4) and evaluating certain integrals

$$u_y^{s_3}(x,0)-u_y^{s_4}(x,0)=(1-\eta)\alpha_0\begin{cases}\int_x^c g_{11}(t)dt+\int_d^e g_{12}(t)dt-U_1(x), x\in I_2\\ \int_x^e g_{12}(t)dt-U_1(x), x\in I_4\end{cases} \tag{5.13}$$

$$U_1(x)=\frac{\pi}{4}\left[\int_b^c p_1^{e_1}(t)+\int_a^e p_1^{e_2}(t)\right]\log\left|t^2-x^2\right|dt \tag{5.13)a}$$

$$u_y^{s_3}(x,0)+u_y^{s_4}(x,0)=\alpha_0\begin{cases}\int_x^c g_{11}(t)dt-t_1(1-\eta)g_{31}(x)-t_1(F_{12}(d)-F_{12}(c)), x\in I_2\\ \int_x^e g_{12}(t)dt-t_1(1-\eta)g_{32}(x), x\in I_4\end{cases} \tag{5.14}$$

Thus using (5.13) – (5.14) we can easily evaluate $u_y^{s_3}$ and $u_y^{s_4}$.

STRESS-INTENSITY FACTORS

The stress-intensity factors at crack tips are defined as

$$\left.\begin{aligned}[K_b,N_b]&=\lim_{x\to b^-}\sqrt{b-x}\left[\sigma_{yy},\sigma_{xy}\right]; \quad [K_c,N_c]=\lim_{x\to c^+}\sqrt{x-c}\left[\sigma_{yy},\sigma_{xy}\right]\\ [K_d,N_d]&=\lim_{x\to d^-}\sqrt{d-x}\left[\sigma_{yy},\sigma_{xy}\right]; \quad [K_e,N_e]=\lim_{x\to e^+}\sqrt{x-e}\left[\sigma_{yy},\sigma_{xy}\right]\end{aligned}\right\} \tag{5.15}$$

6. SPECIAL TYPE OF LOADING

This makes the analysis and the method of integral equation as workable we consider special loading over crack faces

$$p_1^{\pm}=p_2^{\pm}=p_3^{\pm}=p_4^{\pm}=p_0 \tag{6.1}$$

$$q_1^{\pm}=q_2^{\pm}=q_3^{\pm}=q_4^{\pm}=0 \tag{6.2}$$

It means that cracks are opening by constant and uniform force over crack faces. Using (6.1) – (6.2) in (3.5) – (3.8) we get

$$p_1^{e_1}(x)=0,\, p_1^{e_2}(x)=0,\, p_1^{0_1}(x)=0,\, p_1^{0_2}(x)=0 \tag{6.3}$$

$$Q_1^{e_1}(x)=0=Q_1^{e_2}(x)=Q_1^{0_1}(x)=Q_1^{0_2}(x) \tag{6.4}$$

Now we use (6.3) – (6.4) in (3.14) we get,

$$F_{11}(x)=F_{12}(x)=F_{13}(x)=0, F_{21}(x)=F_{22}(x)=F_{23}(x)=0 \tag{6.5}$$

$$F_{31}(x)=F_{32}(x)=F_{33}(x)=0, F_{41}(x)=F_{42}(x)=F_{43}(x)=0 \tag{6.6}$$

Now using (6.3) – (6.4) in (3.19) we get

$$P_{11}=4p_0=P_{12}, P_{21}=P_{22}=0 \tag{6.7}$$

$$Q_{11}=Q_{12}=Q_{21}=Q_{22}=0 \tag{6.8}$$

Using (6.7) – (6.8) and (6.3) – (6.4) in (4.17) – (4.18)

$$T_{11}(x)=4p_0, T_{12}(x)=T_{21}(x)=T_{22}(x)=0, \tag{6.9}$$

$$T_{31}(x)=T_{32}(x)=T_{41}(x)=T_{42}(x)=0 \tag{6.10}$$

Then using (6.9) and (6.5) – (6.6) in (4.13) – (4.16) we get,

$$\Delta_1(t)=4p_0\left(\int_b^c-\int_d^e\right)\frac{y\theta(y)dy}{y^2-t^2}+t^2L_1+L_2, \tag{6.11}$$

$$\Delta_2(t)=t^2M_1+M_2 \tag{6.12}$$

$$\Delta_3(t)=t^2R_1+R_2 \tag{6.13}$$

$$\Delta_4(t)=t^2N_1+N_2 \tag{6.14}$$

Now evaluating integrals in (6.11) we get,

$$\Delta_1(t)=4p_0\left\{s_1+s_2(|t^2-b^2|)+s_3(|(t^2-b^2)(c^2-t^2)|)+s_4(|(t^2-b^2)(c^2-t^2)(d^2-t^2)|)+\frac{\pi}{2}\theta(t)\right\} \tag{6.15}$$

$$\left.\begin{array}{l} s_1=\left(\int_b^c-\int_d^e\right)y\theta_3(y)dy, s_2=\left(\int_b^c-\int_d^e\right)y\theta_2(y)dy \\ s_3=\left(\int_b^c-\int_d^e\right)y\theta_1(y)dy, s_4=\left(\int_b^c-\int_d^e\right)\dfrac{ydy}{\theta(y)} \\ \theta_1(y)=(e^2-y^2)^{1/2}, \theta_2(y)=\left[\left|(d^2-y^2)\right|\left|\theta_1(y)\right|\right]^{1/2} \\ \theta_3=\left[\left|(c^2-y^2)\right|\left|\theta_2(y)\right|\right]^{1/2} \end{array}\right\} \tag{6.16}$$

Above integrals can easily be evaluated numerically.

Now, we use(6.3) – (6.6) in (5.2), (5.4), (5.6), (5.8), (5.13)a, we get

$$U_1(x)=0, F_1(x)=0, F_2(x)=0, F_3(x)=0, F_4(x)=0$$

$$s_3=\frac{1}{3}[(e^2-c^2)^{3/2}-(e^2-b^2)^{3/2}-(e^2-d^2)^{3/2}]$$

Now using (6.12)-(6.16) and the relations (4.5) – (4.8) in (4.10) – (4.16) we get

$$g_{21}(t)=g_{22}(t)=g_{31}(t)=g_{32}(t)=g_{41}(t)=g_{42}(t)=0$$

$$\left.\begin{array}{l} g_{11}(t)=\dfrac{2p_0}{\pi^2\theta(t)}\left[\Delta_1(t)+t^2L_1+L_2\right], t\in I_2 \\ g_{12}(t)=-\dfrac{p_0}{\pi^2\theta(t)}\left[\Delta_1(t)+t^2L_1+L_2\right], t\in I_4 \end{array}\right\} \tag{6.17}$$

$\Delta_1(t)$ is defined in (6.15). Now evaluating the integrals in (5.1) after using (6.17)

$$\sigma_{yy}^{s_3}(x,0)=\frac{p_0}{\pi}\begin{cases} \dfrac{\Delta_5(x)}{\theta(x)}, x\in I_1 \\ -\dfrac{\Delta_5(x)}{\theta(x)}, x\in I_3 \\ -\dfrac{\Delta_5(x)}{\theta(x)}, x\in I_5 \end{cases}$$

$$\sigma_{yy}^{a_3}(x,0)=\sigma_{yy}^{a_4}(x,0)=0$$

$$\sigma_{xy}^{s_3}(x,0)=\sigma_{xy}^{s_4}(x,0)=\sigma_{xy}^{a_3}(x,0)=\sigma_{xy}^{a_4}(x,0)=0$$

Using the definition of stress-intensity factor from (5.15)

$$\left[K_b, K_c, K_d, K_e\right]=\frac{p_0}{\pi}\left[\frac{\Delta_5(b)}{n_1(b)}, -\frac{\Delta_5(c)}{n_1(c)}, -\frac{\Delta_5(d)}{n_2(d)}, \frac{\Delta_5(e)}{n_2(e)}\right] \tag{6.18}$$

$$\left.\begin{array}{l} n_1(x)=\sqrt{2x(c^2-b^2)(d^2-x^2)(e^2-x^2)} \\ n_2(x)=\sqrt{2x(e^2-d^2)(x^2-b^2)(x^2-c^2)} \end{array}\right\} \tag{6.19}$$

The crack opening displacement will be evaluated numerically from the following integrals

$$u_y^{s_3}(x,0)=u_y^{s_4}(x,0)=\frac{2(1+\eta)}{\pi E}\begin{cases}\int_x^c g_{11}(t)dt\\ \int_x^d g_{12}(t)dt\end{cases} \tag{6.20}$$

CASE - II

$$[p_1^{\pm},p_2^{\pm}]=p_0[\delta(x-d_1),\delta(x+d_1)]\delta(y),b<d_1<c \tag{6.21}$$

$$[p_3^{\pm},p_4^{\pm}]=p_0[\delta(x-d_2),\delta(x+d_2)]\delta(y),d<d_2<e \tag{6.22}$$

$$q_i^{\pm}=0,i=1,2,3,4$$

The problem is reduced to symmetric case only.

The cracks (inner & outer) are opened by point forces of equal magnitude acting at points $(\pm d_1,0)$ and $(\pm d_2,0)$ of crack faces.

$$p_1^{e_1}=p_1^{e_2}=p_1^{0_1}=p_1^{0_2}=0,Q_1^{e_1}=Q_1^{e_1}=Q_1^{0_1}=Q_1^{0_2}=0 \tag{6.23}$$

$$F_{ij}=0,\quad i=1,2,3,4;\quad i=1,2$$

$$P_{11}=2p_0\left[\delta(x-d_1)+\delta(x+d_1)\right],P_{12}=2p_0\left[\delta(x-d_2)+\delta(x+d_2)\right],$$

$$P_{21}=P_{22}=Q_{11}=Q_{12}=Q_{21}=Q_{22}=0 \tag{6.24}$$

$$\int_b^c g_{i1}(t)dt=0,\int_d^e g_{i2}(t)dt=0,\ i=1,2,3,4 \tag{6.25}$$

Now we evaluate $g_{ij}(t)$, i = 1, 2, 3, 4; j = 1, 2 through (4.13) – (4.16) and (6.21) – (6.25) we get,

$$g_{11}(t)=\frac{2p_0}{\theta(t)\pi^2}\left[G(t)+\frac{d_1\theta(d_1)}{d_1^2-t^2}-\frac{d_2\theta(d_2)}{d_2^2-t^2}\right]+t^2M_1+M_2 \tag{6.26}$$

$$M_1=\frac{\alpha_7}{\alpha_8},M_2=\frac{\alpha_9}{\alpha_8} \tag{6.26)a}$$

$$\alpha_7=\alpha_3\alpha_4-\alpha_1\alpha_6,\alpha_8=\alpha_2\alpha_6-\alpha_3\alpha_5,\alpha_9=\alpha_1\alpha_5-\alpha_2\alpha_4 \tag{6.26)b}$$

$$\alpha_1=\int_b^c G(t)dt,\alpha_2=\frac{c^3-b^3}{3},\alpha_3=(c-b) \tag{6.26)c}$$

$$\alpha_4=\int_d^e\frac{G(t)dt}{\theta(t)},\alpha_5=\frac{e^3-d^3}{3},\alpha_6=(e-d) \tag{6.26)d}$$

$$\left.\begin{array}{l}g_{11}(t)=\dfrac{2p_0}{\pi^2}\dfrac{\Delta_5(t)}{\theta(t)},\Delta_5(t)=G(t)+t^2M_1+M_2,t\in I_2\\ g_{12}(t)=-\dfrac{2p_0\Delta_5(t)}{\pi^2\theta(t)},t\in I_4\end{array}\right\} \tag{6.26)e}$$

$$G(t)=\frac{d_1\theta(d_1)}{d_1^2-t^2}-\frac{d_2\theta(d_2)}{d_2^2-t^2},m_1=\frac{\alpha_7}{\alpha_8},m_2=\frac{\alpha_9}{\alpha_8} \tag{6.27}$$

Now using $g_{11}(t),\ g_{12}(t)$ into (5.1) and evaluating the integrals then using the definitions in (5.15) we get,

$$\left.\begin{aligned} k_b &= \frac{p_0}{\pi}\frac{\Delta_5(b)}{n_1(b)},\ k_c = -\frac{p_0}{\pi}\frac{\Delta_5(c)}{n_1(c)} \\ k_d &= -\frac{p_0}{\pi}\frac{\Delta_5(d)}{n_2(d)}, k_e = \frac{p_0}{\pi}\frac{\Delta_5(e)}{n_2(e)} \end{aligned}\right\} \tag{6.28}$$

$$\left.\begin{aligned} n_1(x) &= [2x(c^2-b^2)(e^2-x^2)(d^2-x^2)]^{1/2} \\ n_2(x) &= [2x(e^2-d^2)(x^2-b^2)(x^2-e^2)]^{1/2} \end{aligned}\right\} \tag{6.29}$$

The values for crack opening displacement, which will give crack shape, will be evaluated numerically from

$$u_y^{s_3}(x,0) = u_y^{s_4}(x,0) = \frac{2(1-\eta^2)}{\pi E}\begin{cases} \int_x^c g_{11}(t)dt \\ \int_x^e g_{12}(t)dt \end{cases} \tag{6.30}$$

7. DISCUSSION AND CONCLUSION

Thus we determined the stress and the displacement over multiply connected isotropic body by using the integral equation method.The method used for asymmetrical loading of crack faces can be extended to the analysis for crack opening due to heat or thermo elastic problem. Stress components are evaluated for region $x \in I_1 \cup I_3 \cup I_5$ and it is found that it has square root singularity at crack tips. The singularity at crack tips, it seems, may generate plastic region around crack tips. This type of problems will be discussed in future. The displacement distribution along *x*-axis for $x \in I_2 \cup I_4$ is smooth, i.e.;thereis no singularity anywhere in this region. The method used here will be extended to orthotropic multiply-connected region.

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