Fixed Points and Mappings

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Abstract

In this paper we give some important types of mappings related to the fixed point concept. We begin with the basic definition of mappings. Then we define self maps and commutative maps. We discuss about the existence of the fixed points of such mappings with examples. The main part of this research article deals mainly with the common fixed points of a class of polynomial functions. The polynomials considered here are self compositions of a given polynomial of degree n. We prove that if a polynomial and its first composition with itself have an identical set of fixed points, then the polynomial and its n^{th} composition with itself also have an identical set of fixed points. Examples are provided to demonstrate the results. While considering the fixed point results in this article, we have not considered the metric space settings. All the fixed point theorems cited and proved in this paper are free from the distance concept.

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1 Introduction

The fixed point problem is the basis of the fixed point theory. The fixed point problem is stated as follows.
Let X be a set and let A and B be any two non-empty subsets of X such that $A \cap B \neq \phi$ and $T : A \to B$ be a map. Under what conditions there exists a point $c \in A$ such that $T(c) = c$?
This basic problem forms the basis of the entire fixed point theory. Many conjectures, theorem and open problem are established in order to solve this problem. According to the well-known conjecture by E. Dyre in 1954, there must be a common fixed point for the two continuous commuting functions T_1 and T_2 on the interval $[0, 1]$ into itself. This conjecture is proved partially in many special cases. H. Cohen [7] in 1962, proved the conjecture for the commuting full functions. It is observed that if $c_1, c_2, c_3, \ldots, c_n$ are the finite number of fixed points of $T_1 \circ T_2$, then $T_1(c_1), T_1(c_2), T_1(c_3), \ldots, T_1(c_n)$

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is a permutation of $c_1, c_2, c_3, \ldots, c_n$ [2]. In 1963, Glen Baxter [2] threw more light in this direction. Ralph DeMarr [8] provided an affirmative answer to the conjecture for one more special case involving Lipschtiz constants for both T_1 and T_2. DeMarr [9] also explored common fixed points for commuting contractions. Further literature in this direction is available in [[4],[6],[10]-[13],[18],[20],[21]]. G. Jungck [[14],[15]] in 1976 studied common fixed points for commuting mappings in more general perspective. He studied this concept in metric space settings first time. J. Cano [5] considered this conjecture in a general interval $[a, b]$. The concept of commuting mappings has been generalized to weak commutativity, compatibility, weak compatibility and occasionally weak compatibility. Study of common fixed points in all these concepts is an active area of research. It also finds its applications to system of functional equations [19].
Commuting polynomials are of particular interest in this research area. An entire set of commuting polynomials is a set of polynomials which contains at least one polynomial of each degree and in which any two polynomials commute with each other. H. D. Block and H. P. Thielman [3] gave all the entire sets of commuting polynomials. In this research article we consider a special case of functions namely polynomials and its self compositions. As polynomials are continuous everywhere, we can extend the theorem of J. E. Maxfield and W. J. Mourant [17], which states that, if the fixed points of $T \circ T$ and T on $[0, 1]$ are necessarily same then so are of $T \circ T \circ T \cdots \circ T$ (n times) and T.

2 Mapping

Definition 2.1. *[1] Let A and B any two nonempty sets. Then the product of A and B is denoted by $A \times B$ and is defined by the set $A \times B = \{(a, b) : a \in A, b \in B\}$.*

Definition 2.2. *[1] Let A and B be any two nonempty sets. A relation T from A to B is a subset of $A \times B$. We denote it by $T : A \rightsquigarrow B$. For $(x, y) \in T$ we write xTy.*

Definition 2.3. *[1] A relation T is called left-total if for all $x \in A$ there exists a $y \in B$ such that xTy. T is called right-total if for all $y \in B$ there exists an $x \in A$ such that xTy. T is called a functional if xTy, xTz implies $y = z$, for $x \in A$ and $y, z \in B$.*

Definition 2.4. *[1] Let A and B be any two nonempty sets. A mapping T from the set A to the set B is a relation from A to B such that it is both functional and left-total. We denote the map T from the set A to the set B as $T : A \to B$.*

In the following two sections we define self mappings and commuting mappings. We also state some important fixed point theorems relating to these mappings. We note that these results are free of distance concept.

3 Fixed Point Theorems for Self Maps

Definition 3.1. *A self map T is a map from a set A to itself. That is $T : A \to A$.*

Theorem 3.2. *[16] If $[a,b]$ is a closed interval in $\mathbb{R}$ and if $T:[a,b]\to[a,b]$ is a continuous self map, then T has a fixed point.*

The following is the fundamental theorem in the fixed point theory of self maps.

Theorem 3.3. *(Brouwer's Fixed Point Theorem)[16] Let $\mathbf{B}$ be a closed ball in $\mathbb{R}^n$, then any continuous self mapping $T:\mathbf{B}\to\mathbf{B}$ has at least one fixed point.*

The following theorems can be easily proved.

Theorem 3.4. *Let $T:\mathbb{R}\to\mathbb{R}$ be differentiable self map and let $T'(x)<1$ for all x. Then T can have at most one fixed point.*

Theorem 3.5. *Let $T:\mathbb{R}\to\mathbb{R}$ be a differentiable self map, then T must have a fixed point.*

Theorem 3.6. *Let $T:[0,1]\to[0,1]$ be a continuous self map such that*
(1) $(T\circ T)(x)=x$ for all $x\in[0,1]$ and
(2) $T(x)\neq x$ for at least one $x\in[0,1]$.
Then T has unique fixed point.

4 Fixed Point Theorems in Commuting Maps

Definition 4.1. *If $T_1:A\to B$ and $T_2:B\to C$ are two mappings, then the map $T_1\circ T_2:A\to C$, defined as $(T_1\circ T_2)(x)=T_1(T_2(x))$ is called as the composition of the maps T_1 and T_2.*

We shall denote the n^{th} self composition of a map T with itself by T^n. That is $T^n=T\circ T\circ\cdots\circ T$ (n times).

Definition 4.2. *[7] Two self maps T_1 and T_2 defined on a set A are said to commute or to be commutative if for each $x\in A$ we have $T_1(T_2(x))=T_2(T_1(x))$.*

Definition 4.3. *Two self maps T_1 and T_2 are said to have a common fixed point c if $T_1(c)=T_2(c)=c$.*

Theorem 4.4. *[7] Commuting full self maps (A self map on $[0,1]$ is said to be a full map if the interval can be divided into a finite number of subintervals on each of which the map is a homeomorphism onto $[0,1]$) must have a common fixed point.*

Let $T_1(x)$ and $T_2(x)$ be continuous commuting self mappings of $[0,1]$ into $[0,1]$ and let $T(x)=T_1(T_2(x))$. Then it can be observed that any fixed point c of $T(x)$ is one of the following type [2]
Type (i) Up-crossing: $T(x)$ passes from below to above the diagonal as x increases through c.

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Type (ii) Down-crossing: $T(x)$ passes from above to below the diagonal as x increases through c.
Type (iii) Touching: $T(x)$ does not cross the diagonal at c.

Theorem 4.5. *[2] Let $T_1(x)$ and $T_2(x)$ be continuous commuting self mappings of $[0,1]$ into $[0,1]$ and let $T(x)=(T_1\circ T_2)(x)$. If $T(x)$ have a finite number of fixed points, then $T_1(x)$ and $T_2(x)$ permute the fixed points of each type (i), (ii), (iii) listed above.*

Theorem 4.6. *[8] If T_1 and T_2 are commuting self mappings of $[0,1]$ into itself such that $|T_1(x)-T_1(y)|\le \alpha|x-y|$ and $|T_2(x)-T_2(y)|\le \beta|x-y|$ for all $x,y\in[0,1]$, where $\beta<\frac{\alpha+1}{\alpha-1}$, then there exists a common fixed point for both T_1 and T_2.*

Theorem 4.7. *[6] Let T be a continuous self mapping of the interval $[a,b]$ into itself. Then the following conditions are equivalent:*
(1) For each $x\in[a,b]$ such that $T(x)\neq x$ we have $T^n(x)\neq x$ for each $n>1$.
(2) For each $x\in[a,b]$ such that $T(x)>x$ we have $T^n(x)>x$ for each $n>1$ and for each $x\in[a,b]$ such that $T(x)<x$ we have $T^n(x)<x$ for each $n>1$.

Theorem 4.8. *[17] If T be continuous self map on $[0,1]$ into $[0,1]$, then if the fixed points of $T\circ T$ and T are necessarily same, then the fixed points of T^n and T are necessarily the same for all n.*

Theorem 4.9. *[10] Let T_1 and T_2 be self mappings on the unit interval $[0,1]$. If T_1 is a full function and if T_2 commutes with T_1, then T_1 and T_2 have a common fixed point.*

5 Main Results

In this section we discuss the common fixed points of a class of polynomials which are self compositions of a given polynomial. We begin with the following simple result.

Theorem 5.1. *Let $T(x)=Ax+B$. Then the fixed point of $T(x)$ is the fixed point of $T^n(x)$ and it is given by $\frac{B}{1-A}$, $A\neq 1$.*
Proof We observe that
$T^2(x)=T[T(x)]=T(Ax+B)=A(Ax+B)+B=A^2x+AB+B$
$T^3(x)=T[T^2(x)]=T(A^2x+AB+B)=A(A^2x+AB+B)+B=A^3x+A^2B+AB+B$
In general $T^n(x)=A^nx+A^{n-1}B+A^{n-2}B+\cdots+AB+B$.
First consider the case $A=1$. Then we have $T(x)=x+B$. Then $T(x)$ has a fixed point only if $B=0$ and then all the points are fixed. Also then $T^n(x)=x$. Thus the fixed points of $T(x)$ and $T^2(x)$ are same.
Let $A\neq 1$. Then the fixed point of $T(x)=Ax+B$ is $x=\frac{B}{1-A}$.
Consider the equation

$$\begin{aligned} T^n(x) &= x \\ A^n x + A^{n-1}B + A^{n-2}B + \cdots + AB + B &= x \\ (A^n - 1)x &= -[A^{n-1}B + A^{n-2}B + \cdots + AB + B] \\ x &= \frac{B[A^{n-1} + A^{n-2} + \cdots + A + 1]}{1 - A^n} \\ x &= \frac{B[A^{n-1} + A^{n-2} + \cdots + A + 1]}{(1-A)(1 + A + A^2 + \cdots + A^{n-1})} \\ x &= \frac{B}{1-A} \end{aligned}$$

Hence the fixed points of $T(x)$ and $T^n(x)$ are same.

Theorem 5.2. *Let $T(x)$ be a polynomial. If the fixed points of $T^2(x)$ and $T(x)$ are same, then the fixed points of $T^n(x)$ and $T(x)$ are also the same.*
Proof. Given that the fixed points of $T^2(x)$ and $T(x)$ are same. Let c be a fixed point of $T(x)$. That is $T(c) = c$. Then

$$\begin{aligned} T^n(c) &= (T \circ T \circ T \cdots \circ T(n \ times))(c) \\ &= (T \circ T \circ T \cdots \circ T((n-1) \ times))(c) \\ &= \dots \\ &= c \end{aligned}$$

Thus the fixed point of $T(x)$ is the fixed point of $T^n(x)$.
Now let c be a fixed point of $T^n(x)$. On the contrary suppose that c is not a fixed point of $T(x)$. We can select n to be the smallest integer such that $T^n(c) = c$. Consider the elements

$$T(c), T^2(c), T^3(c), \dots, T^n(c)$$

All these elements are distinct, since $T^i(c) = T^j(c)$ ($i < j$) imply $T^{j-i}(c) = c$ and as $j - i < n$, this contradicts the selection of the integer n. Let

$$T^k(c) = max\{T(c), T^2(c), T^3(c), \dots, T^n(c)\}$$

Then $T^k(c) > T^{k-1}(c)$ and $T^{k+1}(c) < T^k(c)$. That is $T[T^{k-1}(c)] > T^{k-1}(c)$ and $T[T^k(c)] < T^k(c)$. Thus the point $(T^{k-1}(c), T[T^{k-1}(c)])$ lie above the line $x = y$ and the point $(T^k(c), T[T^k(c)])$ lie below the line $x = y$. As $T(x)$ is continuous, $T(x)$ cuts the line $x = y$ at least ones between $T^{k-1}(C)$ and $T^k(c)$. That is there is at least one fixed point of $T(x)$ between $T^{k-1}(c)$ and $T^k(c)$. As $T^{k-1}(c)$ is not a fixed point of $T(x)$ (since $T[T^{k-1}(c)] = T^k(c) \neq T^{k-1}(c)$), there exists a smallest element b greater than

$T^{k-1}(c)$. That is $b \in (T^{k-1}(c), T^k(c)]$. Since $T(x)$ is a continuous function, for every $x \in (b, T^k(c)]$ there exists an element $y \in [T^{k-1}(c), b)$ such that $T(y) = x$.

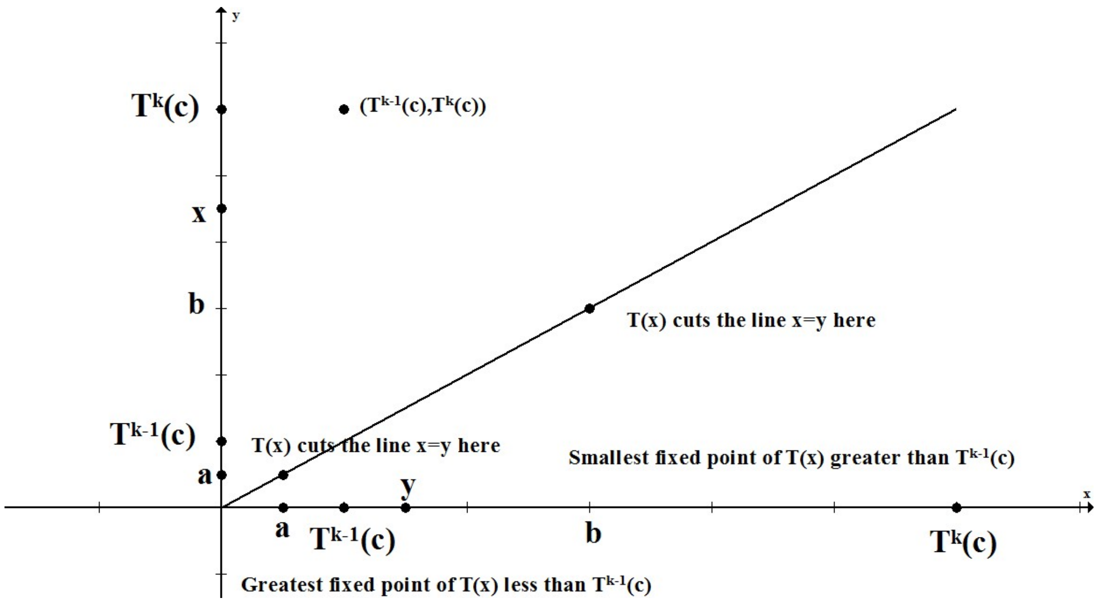

Figure 1: Characteristics of graph of $T(x)$

Now we consider the list of elements

$$\begin{aligned} T^{k-1}(c), \\ T[T^{k-1}(c)] = T^k(c) > T^{k-1}(c), \\ T^2[T^{k-1}(c)] = T^{k+1}(c), \\ \ldots, \\ T^n[T^{k-1}(c)] = T^{k-1}(c). \end{aligned}$$

This list begins with $T^{k-1}(c)$ and after successive iterations of T, n times, again ends with $T^{k-1}(c)$. Thus there exist a smallest integer j such that $T^j[T^{k-1}(c)] \leq T^{k-1}(c)$. Therefore $T^{k-1}(c) < T^{j-1}[T^{k-1}(c)] \leq T^k(c)$. But if $x \in [T^{k-1}(c), b)$ then $T(x) > T^{k-1}(c)$. Now as $T[T^{j-1}(T^{k-1}(c))] = T^j[T^{k-1}(c)] \leq T^{k-1}(c)$, we must have $T^{j-1}[T^{k-1}(c)] \notin [T^{k-1}(c), b)$. We must have $T^{j-1}[T^{k-1}(c)] \in (b, T^k(c)]$. For this $x = T^{j-1}[T^{k-1}(c)]$ there exists $y \in [T^{k-1}(c), b)$ such that $T(y) = T^{j-1}[T^{k-1}(c)]$. Applying T on both sides we get $T^2(y) = T^j[T^{k-1}(c)] \leq T^{k-1}(c) \leq y$ (since $y \in [T^{k-1}(c), b)$). Thus $T^2(y) \leq y$.
Now if there are any fixed points of $T(x)$ less than $T^{k-1}(c)$, suppose a is the largest fixed point of $T(x)$ less than $T^{k-1}(c)$. By the continuity of $T(x)$ there are values of $y \in (a, b)$ with $T(y) \in (a, b)$. Therefore, $T^2(y) > y$. Hence $T^2(y) = y$. Thus $y \in (a, b)$ is a fixed point of $T^2(x)$ which is not a fixed point of $T(x)$.

If there are no fixed points of $T(x)$ less than $T^{k-1}(c)$, consider

$$T^m(c) = \min\{T(c), T^2(c), \dots, T^n(c)\}$$

Then $0 \leq T^m(c) \leq T^{k-1}(c)$ and $T^2[T^m(c)] = T^{m+2}(c) \geq T^m(c)$. Thus $y = T^m(c) \in [0, b)$ is such that $T^2(y) \geq y$. Hence $T^2(y) = y$. Thus there is a fixed point $y = T^m(c)$ of $T^2(x)$ which is not the fixed point of $T(x)$. This contradiction proves the result.

Example 5.3. Consider the polynomial $T(x) = 2x + 3$. It has fixed point $x = -3$. Also $T[T(x)] = T(2x + 3) = 2(2x + 3) + 3 = 4x + 9$. We see that the fixed point of $T^2(x)$ is $x = -3$. Thus the fixed points of $T(x)$ and $T^2(x)$ are same. Thus the fixed points of the following polynomials are all same.

$$T^3(x) = 8x + 21, T^4(x) = 16x + 42, T^4(x) = 32x + 87, T^5(x) = 64x + 177, \dots$$

Example 5.4. Consider $T(x) = x^2$. Then $T(x)$ has two fixed points $x = 0, 1$. Consider now $T^2(x) = T[T(x)] = T(x^2) = (x^2)^2 = x^4$, whose fixed points are also $x = 0, 1$. Thus the fixed points of $T^3(x) = x^8, T^4(x) = x^{16} \dots$ are all same.

Conclusion. Theorem 5.2 extends the theorem of Maxfield and Mourant[17] to a class of polynomials on the whole real line.

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