Oscillation Theorem for Second Order Neutral Delay Differential Equations with Impulses

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Abstract: *Consider the second-order linear neutral delay impulsive ordinary differential equations of the form*

$$\begin{cases} \left[y(t)+p(t)y(t-\tau)\right]''+q(t)y(t-\sigma)=0,\ t\neq t_k \\ \Delta\left[y(t_k)+p(t_k)y(t_k-\tau)\right]'+q_k y(t_k-\sigma)=0,\ t=t_k, \end{cases}$$

(*)

where $0\le t_0<t_1<\cdots<t_k<\cdots$ *with* $\lim_{k\to+\infty} t_k=+\infty$ *and* τ *and* σ *are non-negative real numbers. We establish a theorem giving conditions for the oscillation of all solutions of equation (*) for the case where the coefficient* $q(t)$ *is* $\tau-$*periodic.*

Keywords: *Impulsive, Neutral, Oscillation, Second-order, differential equations*

1. INTRODUCTION

During the last thirty years, research in oscillation theory of impulsive differential equations has undergone a period of exciting growth. Werefertheredertothemonographsby Lakshmikantham*etal*andSamoilenkoandPerestyuk ([4], [2]),where properties of their solutions are studied and extensive bibliographies given. To a large extent, this is due to the realization that differential equations in general and indeed, impulsive differential equations, are important in various applications. In particular, new applications which involve the oscillations in delay and neutral impulsive differential equations continue to arise with increasing frequency in modelling of diverse phenomena in physics, biology, ecology, medicine, economics, control theory, industrial robotics, biotechnologies, to mention just a few. However, inspite of the large number of investigations of impulsive differential equations, their oscillation theory has not yet been fully elaborated, unlike the case of oscillation theory for delay differential equations. Them on o graphs by Erbe*etal*, G y o r i and Ladas, and Ladde*etal* ([3], [5], [6]) contain excellent survey so f known results for delay and neutral delay differential equations.

In this study we seek conditions for the oscillation of all solutions of a certain second order neutral delay impulsive differential equation of the form

$$\begin{cases} \left[y(t)+p(t)y(t-\tau)\right]''+q(t)y(t-\sigma)=0,\ t\neq t_k \\ \Delta\left[y(t_k)+p(t_k)y(t_k-\tau)\right]'+q_k y(t_k-\sigma)=0,\ t=t_k, \end{cases} \tag{1.1}$$

where $\tau,\sigma\ge 0$, $0\le t_0<t_1<\cdots<t_k<\cdots$ with $\lim_{k\to+\infty} t_k=+\infty$, $\Delta y^{(i)}(t_k)=y^{(i)}(t_k^+)-y^{(i)}(t_k^-)$, $i=0,1$ and $y(t_k^-)$, $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t=t_k$, respectively.

A second order neutral impulsive differential equation such as that in (1.1) is a system consisting of a differential equation to gether with animpulsive condition in which these cond order derivativeof the unknown function appears in the equation both with and without delay. The above definition becomes more meaningful if we define other related terms and concepts that will continue to be useful as we progress through the article.

In ordinary differential equations, the solutions are continuously differentiable sometimes at least once, whereas the impulsive differential equations generally possess non-continuous solutions. Since the continuity properties of the solutions play an important role in the analysis of the behaviour, the techniques used to handle the solutions of impulsive differential equations are fundamentally different, including the definitions of some of the basic terms. In this section, we examine some of these changes.

Notation 1.1: Let $J=(\alpha,\beta)\subset\mathbb{R}$, $-\infty<\alpha<\beta<+\infty$ is our domain of investigation

Definition 1.1: Let $S:=\{t_k\}_{k\in E}\subset J$ be a strictly ascending sequence of the time moments of impulse effects and let E be a subscript set which can be the set of natural numbers $\mathbb{N}$ or the set of integers $\mathbb{Z}$ such that

- $t_k\to\infty$ if $k\to\infty$ and if $E=\mathbb{Z}$, then $t_k\to-\infty$ if $k\to-\infty$;
- $t_k\geq 0$ if $k\geq 0.$

Our equation under consideration then has the form

$$\begin{cases} \left[y(t)+p(t)y(t-\tau)\right]''+q(t)y(t-\sigma)=0,\ t\geq t_0,\ t\in J\setminus S \\ \Delta\left[y(t_k)+p(t_k)y(t_k-\tau)\right]'+q_k y(t_k-\sigma)=0,\ t_k\geq t_0, \forall\, t_k\in S, \end{cases} \tag{1.2}$$

where $1\leq k\leq\infty$.

In order to simplify the statements of the assertions, we introduce the set of functions PC and PC^r which are defined as follows: Let $D:=[T,\bar{T})\subset J\subset R$ and let the set of impulse points S be fixed.

Definition 1.2: Let $PC(D,\mathbb{R}):=\{\varphi\,|\,\varphi:D\to\mathbb{R},\ \varphi\in C(D\setminus S),\ \exists\ \varphi(t-0),\ \varphi(t+0),\ \forall\, t\in D\}$.

From the studies in Bainov and Simeonov (1998), Lakshmikantham *et al.* (1989) and Isaac *et al.* (2011) ([1], [4], [8]), we define the function space $\forall\, r\in\mathbb{N}$:

Definition 1.3: Let $PC^r(D,\mathbb{R}):=\left\{\varphi\,|\,\varphi\in PC(D,\mathbb{R}),\ \dfrac{d^j\varphi}{dt^j}\in PC(D,\mathbb{R}),\ \forall\ 1\leq j\leq r\right\}$.

To specify the points of discontinuity of functions belonging to PC and PC^r, we shall sometimes use the symbols $PC(D,R;S)$ and $PC^r(D,R;S),\ r\in\mathbb{N}$ [7].

Definition 1.4 The solution $y(t)$ of an impulsive differential equation is said to be

i) finally positive (finally negative) if there exist $T\geq 0$ such that $y(t)$ is defined and is strictly positive (negative) for $t\geq T$ [8];

ii) non-oscillatory, if it is either finally positive or finally negative; and

iii) oscillatory, if it is neither finally positive nor finally negative ([1], [9]).

Remark 1.1: All functional inequalities considered in this paper are assumed to hold finally, that is, they are satisfied for all t large enough.

2. STATEMENT OF THE PROBLEM

We are concerned with the oscillatory properties of the second order linear neutral delay impulsive differential equation with variable coefficients and constant deviating arguments of the form

$$\begin{cases} \left[y(t)+p(t)y(t-\tau)\right]''+q(t)y(t-\sigma)=0,\ t\geq t_0,\ t\in J\setminus S \\ \Delta\left[y(t_k)+p(t_k)y(t_k-\tau)\right]'+q_k y(t_k-\sigma)=0,\ t_k\geq t_0,\forall\, t_k\in S, \end{cases} \tag{2.1}$$

where $p(t), q(t)\in C\left([t_0,\infty),\square\right)$ and τ and σ are non-negative real numbers.

Our aim is to establish some sufficient conditions for every bounded (unbounded) solution of equation (2.1) to be oscillatory. Throughout the discussion of this work, except specified otherwise, we shall assume the following conditions:

C2.1: $q_k\geq 0\ \forall\, k\in\square$,

C2.2: $p(t)\in PC([t_0,\infty),\square),\ p_1\leq p(t)\leq p_2$ for $t\in[t_0,\infty)$, where $p_1, p_2\in\square$,

C2.3: $q(t)\in PC([t_0,\infty),\square),\ q(t)\geq q_1>0$ for $t\in[t_0,\infty)$.

Here, we demonstrate how well-known mathematical techniques and methods, after suitable modifications, is extended in proving an oscillation theorem for impulsive delay differential equations. We shall restrict ourselves to the study of impulsive differential equations for which the impulse effects take place at fixed moments of time $\{t_k\}$.

3. MAIN RESULTS

The following theorem extends Theorem 3.1.6of the monograph by Bainov and Mishev[10] by imposing impulsive constant jumps as appropriate.

Theorem 3.1: In addition to conditions C2.1—C2.3, further assume thatthe conditions

i) $\tau,\sigma\geq 0$,

ii) $p(t)\equiv p>0$, where p is a constant,

iii) $q(t)\geq 0,\ q(t)\not\equiv 0$ and $\tau-$periodic

are finally fulfilled.Then everysolution of equation (2.1) is oscillatory.

Proof: By contradiction, we assume that $y(t)$ is a finally bounded positive solution of equation (2.1). Set

$$z(t)=y(t)+p(t)y(t-\tau).$$

Then

$$z(t)>0 \tag{3.1}$$

and

$$\begin{cases} z''(t)=-q(t)y(t-\sigma)\leq 0,\ t\notin S \\ \Delta z'(t_k)=-q_k\, y(t_k-\sigma)\leq 0,\forall\, t_k\in S. \end{cases}$$

Thus, $z'(t)$ is a decreasing function of t . We claim that

$$z'(t)\geq 0.$$

Otherwise,

$$z'(t)<0 \text{ and } z''(t)\leq 0.$$

This implies that

$$\lim_{t\to\infty} z(t)=-\infty$$

which contradicts condition (3.1).We now set $w(t)=z(t)+pz(t-\tau)$ which is known to be positive. Then a direct substitution shows that $w(t)$ is a piece-wise continuously differentiable solution of the neutral delay impulsive differential equation

$$\begin{cases} w''(t)+pw''(t-\tau)+q(t)w(t-\sigma)=0, \ t\notin S \\ \Delta w'(t_k)+p\Delta w'(t_k-\tau)+q_k\, w(t_k-\sigma)=0, \forall\, t_k\in S. \end{cases}$$

(3.2)

Also, we have that $w'(t)>0$ and

$$\begin{cases} w''(t)=-q(t)z(t-\sigma)\le 0, \ t\notin S \\ \Delta w'(t_k)=-q_k\, z(t_k-\sigma)\le 0, \forall\, t_k\in S. \end{cases}$$

Since $z(t)$ is an increasing function, it follows that

$$w''(t-\tau)=-q(t)z(t-\sigma-\tau)\ge -q(t)z(t-\sigma)=w''(t)$$

and

$$\Delta w'(t_k-\tau)=-q_k z(t_k-\sigma-\tau)\ge -q_k z(t_k-\sigma)=\Delta w'(t_k).$$

Hence, from equation (3.2), we obtain

$$\begin{cases} w''(t)+\dfrac{q(t)}{1+p}w(t-\sigma)\le 0, \ t\notin S \\ \Delta w'(t_k)+\dfrac{q_k}{1+p}w(t_k-\sigma)\le 0, \forall\, t_k\in S. \end{cases}$$

Integrating, inequality (3.3) from T to t with T sufficiently large, we obtain

$$w'(t)-w'(T)+\frac{1}{1+p}w(T-\sigma)\int_T^t q(s)ds+\frac{1}{1+p}w(T-\sigma)\sum_{T\le k<t} q_k ,$$

which leads to a contradiction as $t\to\infty$. This completes the proof of Theorem 3.1.

Remark 3.1: Careful observation shows that the conclusion of Theorem 3.1 still remains valid even when the requirement of condition C2.3 is not met.

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