# A Family of Nonlinear Problems in Particle Suspension Mechanics with Some Special Characteristics

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**Abstract** - In this paper we present a novel class of nonlinear problems arising in low Reynolds number hydrodynamics, in which the analytical solution of the linearized equation yields an approximation to the numerical solution of the full nonlinear equation, when it is scaled and reduced suitably using the standard deviation and the mean of the numerical solution. The main interest in discussing these problems is that the analytical solution itself is sufficient to generate the numerical solution, when suitably scaled and reduced. Hence this system may be used as a test for software developed to solve complex integro – differential equations.

Keywords — Periodic force, mean, standard deviation, Reynolds number.

#### I. INTRODUCTION

The problem of determining the steady flow past fixed bodies in a slow uniform stream of a viscous incompressible fluid was originally considered by Stokes [1]. Stokes obtained his solution by neglecting the effect of inertia. His solution was at zero Reynolds numbers. Later, Whitehead [2] attempted to improve upon this solution by obtaining higher order approximations to the flow when the Reynolds number is not negligibly small. In his method he proposed a technique which is equivalent to expanding the solutions in terms of powers of Re, Reynolds numbers. However in this iterative solution, the boundary conditions had to be considered at every iteration and hence this led to a situation in which it is impossible to satisfy the boundary conditions of the problem in all terms except the leading one. This mathematical phenomenon appears to be common to all problems of uniform streaming past bodies of finite length - scale and is called 'Whitehead's paradox'. The paradox was resolved by Oseen [3], [4]. Oseen computed the first correction to the Stokes drag for small but finite values of the Reynolds numbers for a sphere held fixed in a steady uniform flow U. Goldstein [5] obtained a basic solution using Oseen's technique. Later, Lagerstrom and Cole [6] solved Oseen's equations to obtain higher approximations of the flow in two and three dimensional cases. Proudman and Pearson [7] described in detail an alternative procedure involving simultaneous consideration of locally valid expansions close to and far from the singularity of the perturbation. These expansions were called the 'Oseen' and the 'Stokes' expansions, respectively, since their leading orders are closely related to the original approximations of these authors. However the numerical 'convergence' of the expansion of Proudman and Pearson was so poor that its utility was limited to  $\text{Re} < \frac{1}{2}$ . Hence Chester and Breach [8] obtained an alterative expression for drag of order Re in the expansion of the drag coefficient for a sphere at small values of Re. Bentwich and Miloh [9] considered the problem of unsteady viscous incompressible flow past a solid sphere when a finite rectilinear velocity U is suddenly imparted to the sphere. They obtained an asymptotic solution for small Reynolds numbers by using the method of matched asymptotic expansions. Their work generalized the work of Proudman and Pearson [7].

There exist only a few solutions for the force acting on accelerating bodies. The best known is the Basset [10] solution for the sphere. Arminski and Weinbaum [11] considered the motion of a sphere, which started to move from rest under the action of an external force which ceases to act after some time. They showed that the Basset term does not contribute to the total displacement and the form of velocity may be very different from the quasi static velocity. Basset's result was extended to conditions where the flow far from the particle was other than uniform (Maxey and Riley, [12]) and to particles of non – spherical shape (Lawrence and Weinbaum [13], [14]; Gavze [15]). Sano [16]) extended the Proudman and Pearson [7] results to the unsteady flow case where U(t)=UH(t) and the fluid is stationary everywhere for t<0. (H(t) is the Heaviside function). Lovalenti and Brady [17] extended his result to conditions where the particle and the far field flow can have general time dependence and to particles of arbitrary shape. It is this expression for the hydrodynamic force given by Lovalenti and Brady [17] for spherical particle in which we are interested. In the next section we describe the problem and solution

methodology. In section three we present the comparison between the solutions of the linear part and the full numerical solutions for various problems considered. In the last section we present the conclusions.

#### **II. THE PROBLEM**

The Lovalenti and Brady [17] formalism for the hydrodynamic force on a rigid sphere undergoing arbitrary time dependent motion in an arbitrary time dependent uniform flow field at small Reynolds numbers is given by the following expression.

$$F^{H}(t) = \frac{4\pi}{3} \operatorname{ReSl}\dot{U}^{\infty}(t) - 6\pi U_{s}(t) - \frac{2\pi}{3} \operatorname{ReSl}\dot{U}_{s}(t) + \frac{3}{3} \left(\frac{\operatorname{ReSl}}{\pi}\right)^{\frac{1}{2}} \left\{ \int_{-\infty}^{t} \left[ \frac{2}{3} F_{s}^{H_{\parallel}}(t) - \left\{ \frac{1}{|A|^{2}} \left( \frac{\pi^{\frac{1}{2}}}{2|A|} \operatorname{erf}\left(|A|\right) - \exp(-|A|^{2}) \right) \right\} F_{s}^{H_{\parallel}}(s) + \frac{2}{3} F_{s}^{H_{\perp}}(t) - \left\{ \exp(-|A|^{2}) - \frac{1}{2|A|^{2}} \left( \frac{\pi^{\frac{1}{2}}}{2|A|} \operatorname{erf}\left(|A|\right) - \exp(-|A|^{2}) \right) \right\} F_{s}^{H_{\perp}} \right] \times \frac{2ds}{(t-s)^{\frac{3}{2}}} + o(\operatorname{Re}).$$
(1)

This expression is obtained by using the reciprocal theorem. The details of the derivation can be found in Lovalenti and Brady [17]. Here,  $U_s = U_p - U^\infty$  is the slip velocity of the fluid.  $U_p$  is the velocity of the particle.  $U_s$  has been non-dimensionalized by  $U_c$ . The acceleration terms  $\dot{U}_s$  and  $\dot{U}^\infty$  are non - dimensionalized by  $U_c/\tau_c$ , where  $\tau_c$  is the characteristic timescale defined as  $\tau_c = a/U_c$ .  $U^\infty$  is the velocity of the fluid as  $|r| \to \infty$ . Re is the Reynolds number, defined as  $Re = aU_c/v$  based on a characteristic particle slip velocity,  $U_c$ , a denotes the characteristic particle dimension and v is the kinematic viscosity of the fluid.  $F_s^{H_{\parallel}} = -6\pi U_s \cdot \underline{pp}$  and  $F_s^{H_{\perp}} = -6\pi U_s \cdot (\delta - \underline{pp})$ , where  $\delta$  is the idem tensor of order 2 and unit vector  $\underline{p} = Y_s(t) - Y_s(s)/|Y_s(t) - Y_s(s)|$ .

Here  $Y_s(t) - Y_s(s)$  is the integrated displacement of the particle relative to the fluid from time *s* to the current time *t*. *Sl* is the Strouhal number defined as  $Sl = (a/U_c)/\tau_o$  is the measure of the time scale of variation or unsteadiness, relative to the convective time  $a/U_c$ .

$$\mathbf{A} = (Re/2)(t - s/ReSl)^{1/2}(\mathbf{Y}_s(t) - \mathbf{Y}_s(s)/(t - s))$$
 Here,  $\mathbf{Y}_s = \mathbf{Y}_p - \mathbf{Y}^{\infty}$  and  $\mathbf{F}^H$  is scaled by  $\mu a U_c$ .

We use equation (1) to obtain the equation governing the motion of a sphere in a fluid, starting with zero velocity at time t = 0, with  $U_s = U_p - U^{\infty}$  where  $U_p$  is the velocity of the particle, scaled with respect to the size of the particle and the frequency of the external periodic force,  $\omega_l$ , i.e., we take  $U_c = a\omega_l$ . Under these conditions, equation (1) reduces to

$$F^{H}(t) = -6\pi U_{s}(t) - \frac{2\pi}{3} \operatorname{Re} Sl\dot{U}_{s}(t) + \frac{4\pi}{3} \operatorname{Re} Sl\dot{U}^{\infty}(t) + \frac{4\pi}{3} \operatorname{Re} Sl\dot{U}^{\infty}(t) + \frac{3}{8} \left(\frac{\operatorname{Re} Sl}{\pi}\right)^{\frac{1}{2}} \int_{0}^{t} \left[\frac{-8\pi U_{s}(t)ds}{(t-s)^{\frac{3}{2}}} - \left\{\frac{1}{|A|^{2}} \left(\frac{\pi^{\frac{1}{2}}}{2|A|}\operatorname{erf}(|A|) - \exp(-|A|^{2})\right)\right\} - \frac{-12\pi U_{s}(s)ds}{(t-s)^{\frac{3}{2}}}\right]$$

$$(2)$$

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We can generate a new class of problems to solve for the velocity of the particle by using Newton's laws as follows

$$\frac{m_p \boldsymbol{U}_p(t)}{\mu a^2 \omega_1} = \boldsymbol{F}^{ext}(t) + \boldsymbol{F}^{H}(t), \quad (3)$$

Hence

$$\dot{\boldsymbol{U}}_{p}(t) = \frac{1}{\mathbf{Re}^{*}} \left( \frac{\boldsymbol{F}^{ext}}{\mu a^{2} \omega_{1}} - 6\pi \boldsymbol{U}_{s}(t) + 2\pi \,\mathbf{Re} \,Sl\dot{\boldsymbol{U}}^{\infty}(t) + \frac{3}{8} \left( \frac{\mathbf{Re} \,Sl}{\pi} \right)^{\frac{1}{2}} \right)$$

$$\int_{0}^{t} \left[ \frac{-8\pi \boldsymbol{U}_{s}(t)ds}{(t-s)^{\frac{3}{2}}} - \left\{ \frac{1}{|\boldsymbol{A}|^{2}} \left( \frac{\pi^{\frac{1}{2}}}{2|\boldsymbol{A}|} \operatorname{erf}\left(|\boldsymbol{A}|\right) - \exp\left(-|\boldsymbol{A}|^{2}\right) \right\} \right\} \frac{-12\pi \boldsymbol{U}_{s}(s)ds}{(t-s)^{\frac{3}{2}}} \right] \right) \qquad (4)$$

$$\dot{\boldsymbol{Y}}_{p}(t) = \boldsymbol{U}_{p}(t) \qquad (5)$$

Here **Re**\* =  $4\pi/3$ **Re**+ $2\pi/3$ **ReSl.** 

We note that there is a singularity at s = t and this can be removed by splitting the nonlinear integral into two intervals  $[0, t-\varepsilon]$  and  $[t-\varepsilon, t]$ . The integral in the interval  $[t-\varepsilon, t]$  can be then transformed with respect to "A" and this integral goes to zero. Thus we are left with the integral in the interval  $[0, t-\varepsilon]$ . For details of derivation one can refer Ramamohan et al [18], [19].

We solved the above system of equations numerically using an adaptive step size Runge Kutta Method (Press et al, 1992, [20]) and the nonlinear integral was calculated using the Romberg integration (Press et al, 1992, [20]) for the following test cases:

- 1. Periodically forced spherical particle in a quiescent Newtonian fluid at low Reynolds numbers.
- 2. Periodically forced spherical particle with a constant bias force applied externally in an undisturbed uniform flow
- 3. Periodically forced spherical particle in an oscillating Newtonian fluid.

The details of the methodology can be found in Ramamohan et al [18], [19].

The interesting feature of this class of problems is that in all the three problems considered at low values of external periodic force the solution of the linear part is a good approximation of the full numerical solution and in other regions a suitable scaling and reduction given by the expression below resulted in an excellent match between the numerical and the scaled and reduced analytical solution.

$$U_{p_{x}}(num soln) = \left(\frac{SD(U_{p_{x}}(num))}{SD(U_{p_{x}}(analytic))}\right) \left(U_{p_{x}}(analytic) + mean(U_{p_{x}}(num))\right)$$
$$Y_{p_{x}}(num soln) = \left(\frac{SD(Y_{p_{x}}(num))}{(1 - 1)^{2}}\right) \left(Y_{p_{x}}(analytic) + mean(U_{p_{x}}(num))^{*}t\right)$$
(6)

$$\left(SD(Y_{p_x}(analytic))\right)^{(x_{p_x}(analytic)) + mean(e_{p_x}(analytic)) + mean(e_{p_x}(analytic)))^{(x_{p_x}(analytic)) + mean(e_{p_x}(analytic)))^{(x_{p_x}(analytic)))^{(x_{p_x}(analytic)))^{(x_{p_x}(analytic)))^{(x_{p_x}(analytic)))^{(x_{p_x}(analytic)))^{(x_{p_x}(analytic)))^{(x_{p_x}(analytic)))^{(x_{p_x}(analytic))}}$$

### *here* SD = *standard deviation and* t = *time*

We observe that in the solutions generated for this class of problems the effect of the nonlinearity is to reduce the magnitude of the solution. However, the trend of the analytical solution and the numerical solutions remains the same. At low values of 'A' the nonlinear term is cancelled by one of the linear terms. At high values of 'A' the nonlinearity goes to zero. Hence it is only at the intermediate values of 'A' that the nonlinearity plays a role and we observe that this yields in a reduction of the amplitude of the oscillation but not in its general form. Hence multiplying the analytical solution of the linearized equation by the standard deviation of the numerical solution of the full nonlinear equation and dividing the same by the standard deviation of the analytical solution removes the scaling deviations of the analytical solution. In the nonlinear problem we also observe a drift of the solution in the direction of the first motion. This drift can be removed by adding a drift term as shown in equation (6). Further, the reason why the numerical solutions and the analytical solutions have same trend is due to the fact that the nonlinear integral term at large times fades away and tends to zero.

#### III THE LINEAR PART

The linear part of equation (4) is given by the following expression:

$$\dot{\boldsymbol{U}}_{\boldsymbol{p}}(t) = \frac{1}{Re^*} \left( \frac{\boldsymbol{F}^{\boldsymbol{ext}}}{\mu a^2 \omega_1} - 6\pi \boldsymbol{U}_{\boldsymbol{s}}(t) + 2\pi Re \, Sl\dot{\boldsymbol{U}}^{\infty}(t) \right)$$
(7)

Its solution is given by

$$\boldsymbol{U}_{\boldsymbol{p}}(t) = exp\left(\frac{-6\pi t}{Re^{*}}\right)\left(\frac{1}{Re^{*}}\right)\int\left(\frac{\boldsymbol{F}^{ext}}{\mu a^{2}\omega_{1}} + 2\pi Re Sl\dot{\boldsymbol{U}}^{\infty}(t) + 6\pi \boldsymbol{U}^{\infty}(t)\right)exp\left(\frac{6\pi t}{Re^{*}}\right)ds$$
$$+ c exp\left(\frac{-6\pi t}{Re^{*}}\right) \tag{8}$$

Where, c is a constant of integration.

We note that  $F^{ext}$  and  $U^{\infty}$  are different for the different problems considered, which we shall now illustrate.

#### IIIa. Periodically forced spherical particle in a quiescent Newtonian fluid at low Reynolds numbers.

In this problem, we assume  $U^{\infty} = 0$ ,  $F^{ext} = F_0 \sin(t)$  along the x – direction. Hence the full nonlinear equation is of the form

$$\dot{U}_{p}(t) = \frac{1}{Re^{*}} \left( Re_{F} \sin(t) - 6\pi U_{p}(t) + \frac{3}{8} \left( \frac{Re Sl}{\pi} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \int_{0}^{t-\varepsilon} \left[ \frac{-8\pi U_{p}(t)ds}{(t-s)^{\frac{3}{2}}} - \left\{ \frac{1}{|A|^{2}} \left( \frac{\pi^{\frac{1}{2}}}{2|A|} \operatorname{erf}(|A|) - \exp(-|A|^{2}) \right) \right\} \frac{-12\pi U_{p}(s)ds}{(t-s)^{\frac{3}{2}}} \right] \right)$$
(9)  
$$\dot{Y}_{p}(t) = U_{p}(t)$$
(10)

Where  $Re_F = F_0 / \mu a^2 \omega_1$ 

Its linear part neglecting the first linear integral term and the nonlinear integral term is given by:

$$\dot{U}_{p}(t) = \frac{1}{Re^{*}} \left( Re_{F} \sin(t) - 6\pi U_{p}(t) \right)$$
(11)
The solution to equation (11) is given by

The solution to equation (11) is given by

$$U_{p} = \frac{Re_{F}}{36\pi^{2} + Re^{*2}} \left( 6\pi \sin(t) + Re^{*} \left( exp\left(\frac{-6\pi t}{Re^{*}}\right) - \cos(t) \right) \right)$$
$$Y_{p} = \frac{Re_{F}}{6\pi} \left( 1 - \frac{36\pi^{2} \cos(t) + 6\pi Re^{*} \sin(t) + Re^{*2} exp\left(\frac{-6\pi t}{Re^{*}}\right)}{36\pi^{2} + Re^{*2}} \right)$$
(12)

This analytical solution when plotted in the phase plane yields a limit cycle whereas the numerical solution of the full nonlinear problem yields an overlapping and weakly drifting circular trajectory as can be seen from figure(1).



Fig 1: The numerical solution of the above problem in III(a).

However the velocity time series of the analytical solution and the numerical solution can be matched at low  $Re_F$  and at high  $Re_F$  using the scaling and reduction formula given in equation (6). This is shown in figure (2). At low  $Re_F$  and at large t, the integral terms cancel and hence the nonlinearity is removed and at large 'A' in equation (9) the nonlinear integral becomes negligible.



Fig 2: The match between the numerical solution of the velocity time series and the solution due to scaling and reduction of analytical solution via equation 6.

# IIIb. Periodically forced spherical particle with a constant bias force applied externally in an undisturbed uniform flow

In this problem we apply an additional external force on the spherical particle, namely the constant bias force  $F_b$ , along with the external periodic force. Also, here  $U^{\infty} = (u_x, u_y, u_z)$  constant velocity at infinity.

Hence, 
$$\mathbf{F}^{ext} = \mathbf{F}_{b} + \mathbf{F}_{\theta} \sin(t)$$
,  $\mathbf{F}_{b} = \left( F_{x}', F_{y}' \right)$ . Upon making these substitution we get:  

$$\frac{dY_{p_{x}}}{dt} = U_{p_{x}} \qquad (13a)$$

$$\frac{dY_{p_{y}}}{dt} = U_{p_{y}} \qquad (13b)$$

$$\frac{dY_{p_{z}}}{dt} = U_{p_{z}} \qquad (13c)$$

1

$$\frac{dU_{p_x}}{dt} = \frac{1}{Re^*} \left[ Re_F \sin(t) + F_x - 6\pi \left( U_{p_x} - u_x \right) + \frac{3}{8} \left( \frac{Re\,Sl}{\pi} \right)^{1/2} \left( J_1 + I_1 \right) \right]$$
(14a)

$$\frac{dU_{p_y}}{dt} = \frac{1}{Re^*} \left[ F_y - 6\pi \left( U_{p_y} - u_y \right) + \frac{3}{8} \left( \frac{Re\,Sl}{\pi} \right)^{\frac{1}{2}} \left( J_2 + I_2 \right) \right]$$
(14b)

$$\frac{dU_{p_z}}{dt} = \frac{1}{Re^*} \left[ -6\pi \left( U_{p_z} - u_z \right) + \frac{3}{8} \left( \frac{Re\,Sl}{\pi} \right)^{1/2} \left( J_3 + I_3 \right) \right]$$
(14c)

Where,

$$J_{1} = 16\pi \left( U_{p_{x}}(t) - u_{x} \right) \left[ \frac{1}{\sqrt{t}} - \frac{1}{\sqrt{\varepsilon}} \right]$$
(14d)  
$$I_{1} = \int_{0}^{t-\varepsilon} \left\{ \frac{1}{A^{2}} \left( \frac{\sqrt{\pi}}{2|A|} \operatorname{erf}(A) - \exp(-A^{2}) \right) \right\} \frac{12\pi \left( U_{p_{x}}(s) - u_{x} \right)}{(t-s)^{3/2}} ds$$
(14e)  
Similarly

Similarly,

$$J_{2} = 16\pi \left( U_{p_{y}}(t) - u_{y} \right) \left[ \frac{1}{\sqrt{t}} - \frac{1}{\sqrt{\varepsilon}} \right]$$
(14f)  
$$I_{2} = \int_{0}^{t-\varepsilon} \left\{ \frac{1}{A^{2}} \left( \frac{\sqrt{\pi}}{2|A|} \operatorname{erf}(A) - \exp(-A^{2}) \right) \right\} \frac{12\pi \left( U_{p_{y}}(s) - u_{y} \right)}{(t-s)^{3/2}} ds$$
(14g)

$$J_{3} = 16\pi \left( U_{p_{z}}(t) - u_{z} \right) \left[ \frac{1}{\sqrt{t}} - \frac{1}{\sqrt{\varepsilon}} \right]$$

$$(14h)$$

$$I_{z} = \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \int_{$$

$$I_{3} = \int_{0}^{t-\varepsilon} \left\{ \frac{1}{A^{2}} \left( \frac{\sqrt{\pi}}{2|A|} \operatorname{erf}(A) - \exp(-A^{2}) \right) \right\} \frac{12\pi \left( U_{p_{z}}(s) - u_{z} \right)}{(t-s)^{2}} ds$$
(14*i*)

And  $F_x = F_x' / \mu a^2 \omega_1$  and  $F_y = F_y' / \mu a^2 \omega_1$ , where, *a* is the particle size,  $\omega_1$  is the frequency of the applied external periodic force,  $\mu$  is the viscosity of the fluid.

The linear part of the full differential equation is given by the following equations

$$\frac{dU_{p_x}}{dt} + \frac{6\pi}{Re^*}U_{p_x} = \frac{6\pi}{Re^*}u_x + \frac{Re_F}{Re^*}\sin t + \frac{F_x}{Re^*}$$
(15a)

$$\frac{dU_{p_{y}}}{dt} + \frac{6\pi}{Re^{*}}U_{p_{y}} = \frac{6\pi}{Re^{*}}u_{y} + \frac{F_{y}}{Re^{*}}$$
(15b)

$$\frac{dU_{p_z}}{dt} + \frac{6\pi}{Re^*}U_{p_z} = \frac{6\pi}{Re^*}u_z \tag{15c}$$

$$\frac{dY_{p_x}}{dt} = U_{p_x} \tag{15d}$$

$$\frac{dY_{p_y}}{dt} = U_{p_y} \tag{15e}$$

$$\frac{dY_{p_z}}{dt} = U_{p_z} \tag{15f}$$

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The analytical solution of these equations is given by:

$$U_{p_x} = \left(u_x + \frac{F_x}{6\pi}\right) \left(1 - exp\left(\frac{-6\pi t}{Re^*}\right)\right) + \frac{Re_F}{Re^{*2} + 36\pi^2} \left(6\pi \sin t + Re^*\left(exp\left(\frac{-6\pi t}{Re^*}\right) - \cos t\right)\right)$$
(15g)

$$Y_{p_{x}} = \frac{Re_{F}}{6\pi} - \frac{Re_{F}}{Re^{*2} + 36\pi^{2}} \left( \frac{36\pi^{2} \cos t + 6\pi Re^{*} \sin t + Re^{*2} \exp(-6\pi t / Re^{*})}{6\pi} \right) + \left( \frac{F_{x} + 6\pi u_{x}}{36\pi^{2}} \right) \left( \left( \exp\left(\frac{-6\pi t}{Re^{*}}\right) - 1 \right) Re^{*} + 6\pi t \right)$$
(15h)

$$U_{p_{y}} = \left(u_{y} + \frac{F_{y}}{6\pi}\right) \left(1 - exp\left(\frac{-6\pi t}{Re^{*}}\right)\right)$$
(15i)

$$Y_{p_{y}} = \left(\frac{F_{y} + 6\pi u_{y}}{36\pi^{2}}\right) \left(\left(exp\left(\frac{-6\pi t}{Re^{*}}\right) - 1\right)Re^{*} + 6\pi t\right)$$
(15j)

$$U_{p_z} = \left(u_z \left(1 - exp\left(\frac{-6\pi t}{Re^*}\right)\right)\right) \tag{15k}$$

$$Y_{p_z} = \left(\frac{u_z}{6\pi}\right) \left( \left( exp\left(\frac{-6\pi t}{Re^*}\right) - 1 \right) Re^* + 6\pi t \right)$$
(151)

We observe that for moderate to high values of the constant velocity at infinity and the constant bias force, the above analytical solution is a good approximation of the numerical solution as nonlinear effects are not significant in these regimes. However, at low values of the constant velocity at infinity and the constant bias force and high values of the periodic force, the effect of the periodic forcing is significant and nonlinear effects come into play. In these regimes we have observed that we can match the analytical solution and the numerical solution by scaling and reducing the analytical solution appropriately. This can be done using the standard deviations and the mean of the numerical and analytical solutions as in equation (6). The reason for this has been discussed earlier. Figures (3) and (4) show the match between analytical solution and the numerical solution with and without scaling respectively.



Fig 3: The match between the numerical method and the scaling and reduction through equation 6 for very low values of constant bias force and low values of the uniform undisturbed force and high values of  $Re_F$ , the amplitude of the external periodic force.



Fig 4: The match between the numerical method and the scaling and reduction through equation 6 for high values of constant bias force and uniform undisturbed force and low values of  $Re_F$ , the amplitude of the external periodic force.

## IIIc. Periodically forced spherical particle in an oscillating Newtonian fluid.

For this problem we consider  $U^{\infty} = (u_x \sin \omega t, u_y \sin \omega t)$ . Upon making these substitutions in equation (6), we obtain the component wise equations for  $U_p$  and  $Y_p$  as follows:

$$\frac{dY_{p_x}}{dt} = U_{p_x}$$
(16a)  
$$\frac{dY_{p_y}}{dt} = U_{p_y}$$
(16b)

$$\frac{dU_{p_x}}{dt} = \frac{1}{Re^*} \left[ Re_F \sin(t) + 2\pi Re Slu_x \omega \cos(\omega t) - 6\pi \left( U_{p_x} - u_x \sin(\omega t) \right) + \frac{3}{8} \left( \frac{Re Sl}{\pi} \right)^{\frac{1}{2}} (J_1 + I_1) \right] (17a)$$

$$\frac{dU_{p_y}}{dt} = \frac{1}{Re^*} \left[ 2\pi Re Slu_x \omega \cos(\omega t) - 6\pi \left( U_{p_y} - u_y \sin(\omega t) \right) + \frac{3}{8} \left( \frac{Re Sl}{\pi} \right)^{\frac{1}{2}} (J_2 + I_2) \right]$$

$$(17b)$$

Where,

$$J_{1} = 16\pi \left( U_{p_{x}}(t) - u_{x} \sin(\omega t) \right) \left[ \frac{1}{\sqrt{t}} - \frac{1}{\sqrt{\varepsilon}} \right]$$

$$I_{1} = \int_{0}^{t-\varepsilon} \left\{ \frac{1}{A^{2}} \left( \frac{\sqrt{\pi}}{2|A|} \operatorname{erf}(A) - \exp(-A^{2}) \right) \right\} \frac{12\pi \left( U_{p_{x}}(s) - u_{x} \sin(\omega t) \right)}{(t-s)^{3/2}} ds \quad (17d)$$

$$(17c)$$

Similarly,

$$J_{2} = 16\pi \left( U_{p_{y}}(t) - u_{y} \sin(\omega t) \right) \left[ \frac{1}{\sqrt{t}} - \frac{1}{\sqrt{\varepsilon}} \right]$$
(17e)  
$$I_{2} = \int_{0}^{t-\varepsilon} \left\{ \frac{1}{A^{2}} \left( \frac{\sqrt{\pi}}{2|A|} \operatorname{erf}(A) - \exp(-A^{2}) \right) \right\} \frac{12\pi \left( U_{p_{y}}(s) - u_{y} \sin(\omega t) \right)}{(t-s)^{3/2}} ds$$
(17f)

And  $Re^* = 4\pi/3Re + 2\pi/3ReSl$ ,  $Re_F = F_0/\mu a^2 \omega_1$  where, *a* is the particle size  $\omega_l$  is the frequency of the applied external periodic force,  $\mu$  is the viscosity of the fluid.

We linearized the equations (16) and (17) by neglecting the nonlinear and irreducible terms, in order to compute an analytical solution to compare with the numerical solutions. Thus we considered the following equation:

$$\frac{dU_{p_x}}{dt} = \frac{1}{Re^*} \Big[ Re_F \sin(t) - 6\pi \Big( U_{p_x} - u_x \sin(\omega t) \Big) + 2\pi Re Sl\omega u_x \cos(\omega t) \Big] \quad (18a)$$

$$\frac{dY_{p_x}}{dt} = U_{p_x} \qquad (18b)$$

$$\frac{dU_{p_y}}{dt} = \frac{1}{Re^*} \Big[ -6\pi \Big( U_{p_y} - u_y \sin(\omega t) \Big) + 2\pi Re Sl\omega u_y \cos(\omega t) \Big] \qquad (18c)$$

$$\frac{dY_{p_y}}{dt} = U_{p_y} \qquad (18d)$$

The solutions of equation (18) are given by:

$$U_{p_{x}} = \frac{Re_{F}}{\left(36\pi^{2} + Re^{*2}\right)} \left[ 6\pi \sin(t) + \left(exp\left(\frac{-6\pi t}{Re^{*}}\right) - \cos(t)\right) Re^{*} \right] + \frac{6\pi u_{x}}{\left(36\pi^{2} + \omega^{2} Re^{*2}\right)} \left[ 6\pi \sin(\omega t) + \omega\left(2\pi Re Sl - Re^{*}\right) \left(\cos(\omega t) - exp\left(\frac{-6\pi t}{Re^{*}}\right)\right) \right] + \frac{2\pi\omega^{2}u_{x} Re Sl Re^{*} \sin(\omega t)}{\left(36\pi^{2} + \omega^{2} Re^{*2}\right)}$$
(19a)

$$Y_{p_{x}} = \frac{-Re_{F}}{\left(36\pi^{2} + Re^{*2}\right)} \left[ 6\pi \cos(t) + Re^{*} \left(\sin(t) + \frac{Re^{*}}{6\pi} \exp\left(\frac{-6\pi t}{Re^{*}}\right)\right) \right] \\ + \frac{Re_{F}}{6\pi} + \frac{6\pi u_{x}}{\left(36\pi^{2} + \omega^{2} Re^{*2}\right)} \left[ \frac{-6\pi}{\omega} \cos(\omega t) + \omega \left(2\pi Re Sl - Re^{*}\right) \left(\frac{\sin(\omega t)}{\omega} + \frac{Re^{*}}{6\pi} \exp\left(\frac{-6\pi t}{Re^{*}}\right)\right) \right] - \frac{2\pi \omega u_{x} Re Sl Re^{*} \cos(\omega t)}{\left(36\pi^{2} + \omega^{2} Re^{*2}\right)} + \frac{u_{x}}{\omega}$$
(19b)

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$$U_{p_{y}} = \frac{6\pi u_{y}}{(36\pi^{2} + \omega^{2} Re^{*2})} \left[ 6\pi \sin(\omega t) + \omega (2\pi Re Sl - Re^{*}) \left( \cos(\omega t) - exp\left(\frac{-6\pi t}{Re^{*}}\right) \right) \right] + \frac{2\pi \omega^{2} u_{y} Re Sl Re^{*} \sin(\omega t)}{(36\pi^{2} + \omega^{2} Re^{*2})}$$

$$(19c)$$

$$Y_{p_{y}} = \frac{6\pi u_{y}}{(36\pi^{2} + \omega^{2} Re^{*2})} \left[ \frac{-6\pi \cos(\omega t)}{\omega} + \omega (2\pi Re Sl - Re^{*}) \left( \frac{\sin(\omega t)}{\omega} + \frac{Re^{*}}{6\pi} exp\left(\frac{-6\pi t}{Re^{*}}\right) \right) \right] - \frac{2\pi \omega u_{y} Re Sl Re^{*} \cos(\omega t)}{(36\pi^{2} + \omega^{2} Re^{*2})} + \frac{u_{y}}{\omega}$$

$$(19d)$$

Here also a good match at low  $Re_F$  was obtained and at high  $Re_F$  equation (6) yields a good approximation for the full numerical solution. Figure (5) show the match between the numerical solution and the analytical solution without scaling and reduction. Figure (6) shows the match with scaling and reduction.



Fig 5: The match between the numerical method and analytical solution without scaling and reduction through equation 6.



Fig 6: The match between the numerical method and analytical solution with scaling and reduction through equation 6.

# **IV. CONCLUSION**

The specialty of this nonlinear problem is that the nonlinear integral term is weak at large times and it is only at large  $Re_F$  that the nonlinearity is significant. The scaling and reduction formula in these regimes yield the numerical solution of the full nonlinear equation. Hence this system is an ideal system where the analytical solution is sufficient for most regions in the parametric regimes considered in this work. Hence this system can be an ideal system to check software for solving complex integro – differential equation.

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