ODD VERTEX MAGIC TOTAL LABELING OF TREES

CT. NAGARAJ[#1],C.Y. PONNAPPAN[⋆1], G. PRABAKARAN[#2]

[1] Research Scholar, Department of Mathematics
Research and Development centre
Bharathiar University, Coimbatore-641046, Tamilnadu, India.

[2] Department of Mathematics
Government Arts College, Melur - 625106, Tamilnadu, India.

[3] Department of Mathematics
Thiagarajar College, Madurai - 625009, Tamilnadu, India.

Abstract. Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$ and let $m = |E(G)|$ and $n = |V(G)|$. A one-to-one map f from $V \cup E$ onto the integers $\{1, 2, 3, ..., m+n\}$ is called vertex magic total labeling if there is a constant k so that for every vertex u, $f(u) + \sum f(uv) = k$ where the sum is over all vertices v adjacent to u. Let us call the sum of labels at vertex u the weight $w_f(u)$ of the vertex under labeling f; we require $w_f(u) = k$ for all u. The constant k is called the magic constant for f. Such a labeling is odd if $f(V(G)) = \{1, 3, 5, ..., 2n-1\}$. In this paper we present the odd vertex magic total labeling of trees.

Key words: Odd vertex magic labeling, Odd vertex magic graph, Tree.
2010 Mathematics Subject Classification: 05C78.

---

1. Introduction

All graphs considered in this paper are finite, simple and undirected. The graph G has vertex set $V = V(G)$ and edge set $E = E(G)$ and let $n = |V|$ and $m = |E|$.The degree of a vertex v is the number of edges that have v as an end point and the set of neighbours of v is denoted by $N(v)$.

A one-to-one map $f(V \cup E) = \{1, 2, 3, ..., m+n\}$ is a vertex magic total labeling of G if there is a constant k so that for every vertex u,$w_f(u) = f(u) + \sum_{v \in N(u)} f(uv) = k$.So the magic requirement is the associated weight $w_f(u) = k$ for all $u \in V(G)$. The fixed integer k is called the magic constant of f. A graph is called vertex magic if the graph has a vertex magic total labeling. Vertex magic total labeling first appeared in 2002 in [3] . For more details of vertex magic graphs see the book by Wallis [10] , and for other types of graphs labeling see the dynamic survey by Gallian [2].

Summing the weights of all vertices yields $nk = \sum_{u \in V} w_f(u) = S_n + 2S_m$, where S_n is the sum of all vertex labels and S_m is the sum of all edge labels. This observation always leads to upper and lower bounds for k.

In Particular, if f assigns the smallest integers $1, 2, ..., n$ to the vertices, then k will be the largest possible, and if f assigns the largest integers $m+1, m+2, ..., m+n$ to the vertices, k will be the smallest possible.

MacDougall et al. [4] introduced the notion of super vertex magic total labeling. They call a vertex magic total labeling is super if $f(V(G)) = \{1, 2, 3, ..., n\}$. We call it as $V-$ Super vertex magic total labeling.In this labeling, the smallest labels are assigned to the vertices.

Swaminathan and Jeyanthi [8] introduced a concept with same name of super vertex magic labeling. They call a vertex magic total labeling is super if $f(E(G)) = \{1, 2, 3, ..., m\}$. Note that the smallest labels are assigned to the edges. They proved the following: P_n is super vertex magic if and only if n is odd and $n \geq 3$; C_n is super vertex magic if and only if n is odd; the star graph is super vertex magic if and only if it is P_3; and rc_s is super vertex magic if and only if r and s are odd. In [9], they proved the following: no super vertex magic total graph has two or more isolated vertices or an isolated edge; a tree with s internal edges and st leaves is not super vertex magic total if $t > \frac{s+1}{s}$; the graph obtained from a comb by attaching a pendant edge to each vertex of degree 2 is super vertex magic total; the graph obtained by attaching a path with t edges to a vertex of an $n-$ cycle is super vertex magic total if and only if $n+t$ is odd. The use of the word "super" has introduce in [1].

MacDougall et al. [4] and Swaminathan and Jeyanthi [8] introduced different labelings with same name of super vertex magic total labeling. To avoid cofusion, Marimuthu and Balakrishan [5] called a vertex magic total labeling is E-super if $f(E(G)) = \{1, 2, 3, ..., m\}$. Note that the smallest labels are assigned to the edges. A graph G is called E-super vertex magic if it admits an $E-$ super vertex magic labeling.

Nagaraj et al. [6] introduced the concept of an Even vertex magic total labeling. They call a vertex magic total labeling is even if $f(V(G)) = \{2, 4, 6, ..., 2n\}$.

Nagaraj et al. [7] introduced the concept of an Odd vertex magic total labeling. They call a vertex magic total labeling is odd if $f(V(G)) = \{1, 3, 5, ..., 2n - 1\}$.

The following results that will subsequently be very useful to prove some theorems.

Theorem 1.1. *Let G be a non trivial graph. If G is an odd vertex magic, then the magic constant k is given by*

$$k = 1 + 2m + \frac{m}{n} + \frac{m^2}{n}.$$

Theorem 1.2. *A path P_n is an odd vertex magic total labeling if and only if n is odd.*

2. Main Results

Theorem 2.1. *A star graph $K_{1,r}$ is odd vertex magic if and only if $r = 2$.*

Proof. Let

$$V(K_{1,r}) = \{c, u_1, u_2, u_r\}$$

and

$$E(K_{1,r}) = \{cu_i : 1 \leq i \leq K\}$$

.

Let f be an odd vertex magic labeling of $K_{1,r}$. Then by 1.1, $k = 3r+1$ The minimum possible weight of c is $r^2 + r + 1$.

$$\text{ie. } w_f(c) \geq r^2 + r + 1.$$

When $r \neq 2$, which is a contradiction to $w_f(c) = k$

When $r = 2$, Then by Theorem 1.2, $K_{1,2} = P_3$ is an odd vertex magic graph with a magic constant $k = 7$. □

Theorem 2.2. *If a tree T is odd vertex magic then n is odd.*

Proof. For a tree T, $n = m + 1$.

They by Theorem 1.1,$k = 3n - 2$.

To prove that n is odd

Suppose n is even.

Then $k = 3n - 2$ is even.

For any odd vertex magic labeling f,

$f(u) + \sum_{v \in N(u)} f(uv) = k, \forall u \in V$. In particular if u is pendant vertex of T then,

$f(u) + f(uv) = k$,

Which is a contradiction.
Since $f(u)$ is odd and $f(uv)$ is even.
Therefore n is odd ☐

Theorem 2.3. *If T has s internal vertices and st leaves then T does not admit an odd vertex magic labeling if $t > \frac{(s+1)}{s}$.*

Proof. If T has s internal vertices and st leaves then
$n = (t+1)s$ and $m = ts + s - 1$
So the labels used for the veritices $\{1, 3, 5, ..., 2st + 2s - 1\}$
and for the edges are $\{2, 4, 6, ..., 2st + 2s - 2\}$.
The maximum possible sum of weights on the leaves is
$\{(2s+1) + (2s+3) + ... + (2s + 2st - 1)\} +$
$\{(2s) + (2s+2) + (2s+4) + ... + (2s + 2st - 2)\}$
$= st(2st + 4s - 1)$
Since there are st leaves, we set

$$stk \leq st(2st + 4s - 1)$$
$$k \leq (2st + 4s - 1)$$

On the other hand, the minimum possible sum of weights on the internal vertices occurs when the smallest labels $\{2, 4, 6, ..., 2s - 2\}$ are assigned to internal edges (because they will be added twice), the labels $\{2s, 2s + 2, ..., 2s + 2st - 2\}$ are assigned to the remaining edges and the labels $\{1, 3, 5, ..., 2s - 1\}$ are assigned to the vertices.
Hence the minimum possible sum of weights on the internals is

$$= 2(2 + 4 + 6 + ... + 2s - 2)$$
$$+ (2s + (2s+2) + ... + (2s + 2st - 2))$$
$$+ (1 + 3 + 5 + ... + 2s - 1).$$
$$= s(t^2 s + t(2s - 1) + 3s - 2)$$

Since there are s internal vertices,

$$sk \geq s(t^2 s + t(2s - 1) + 3s - 2)$$
$$k \geq t^2 s + t(2s - 1) + 3s - 2$$

Therefore no labeling will be possible when $t^2 s + t(2s - 1) + 3s - 2 > 2st + 4s - 1$ when $st^2 - t - (s+1) > 0$

$$t > \frac{1 + \sqrt{1 + 4s(s+1)}}{2s} = \frac{s+1}{s}$$
☐

Corollary 2.4. *The odd vertex magic labeling is impossible for a graph with s internal vertices and more than $s + 1$ leaves.*

Theorem 2.5. *If Δ is the largest degree of any vertex in a tree T with n vertices and m edges then T doesnot admit an odd vertex magic labeling whenever $\Delta > \frac{-3+\sqrt{1+16n}}{2}$.*

Proof. Let c be the vertex of maximum degree Δ.
The minimum possible weight of c is $2 + 4 + 6 + ... + 2\Delta + 1$.
Therefore, $k \geq 1 + \Delta(\Delta + 1)$.
Since there is an internal vertex of degree Δ, there are atleast Δ leaves in T.

so the maximum possible sum of weights on the leaves is atmost the sum of the Δ largest labels from $f(E)$ and the Δ largest labels from $f(v)$. Hence,

$$\Delta k \leq (2n-2\Delta)+(2n-2\Delta+2)+...+(2n-2\Delta+2\Delta-2)+(2n-2\Delta+1)+(2n-2\Delta+3)+...+(2n-2\Delta+2\Delta-1).$$

$$\Delta k \leq \Delta(4n-2\Delta-1)$$

$$k \leq 4n-2\Delta-1.$$

So a labeling will be impossible whenever

$$4n-2\Delta-1 < 1+\Delta(\Delta+1)$$

.

That is when $\Delta^2+3\Delta+(2-4n)>0$.

$$\Delta > \frac{-3+\sqrt{1+16n}}{2}$$

.

□

Remark 2.6. *The following table shows the maximum degree permitted by the restriction given in 2.5 for some small values of n.*

n	3	5	7	9	11
Δ	2	3	3	4	5

Theorem 2.3 and 2.5 donot provide sufficient condition, for a graph to be an odd vertex magic, since we can prove that there is a tree with 7 vertices and $\Delta = 3$, shown in Figure 1, which doesnot admit any odd vertex magic labeling

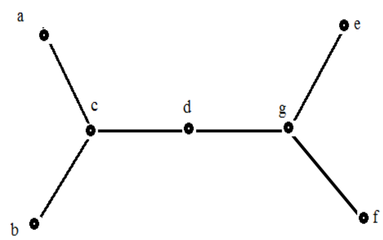

Figure 1:

The reason is as follows: The magic constant $k = 3n-2 = 19$. The labels 2 and 4 can be assigned only to the internal edges. The only possible vertex label of d is 13. The labels 1, 3 and 5 can be assigned only two internal vertices namely c and g. Which is a contradiction.

Theorem 2.7. *Let G be a graph obtained by joining a pendant vertex with a vertex of degree 2 of a comb graph. Then G admits an odd vertex magic labeling.*

Proof. Let the vertex $V = \{a_1, a_2, ..., a_t\} \cup \{a_{11}, a_{12}, a_{21}, a_{31}, ..., a_{t1}\}$ and the edge set. $E = \{a_1a_{11}, a_1a_{12}, a_2a_{21}, ..., a_ta_{t1}\} \cup \{a_ia_{i+1} \ / \ 1 \leq i \leq t-1\}$.
Here $n = 2t+1$ and $m = 2t$.
Define $f : E \to \{1, 2, 3, ..., m\}$ as follows.

$$f(a_ia_{i+1}) = 2t-2i \quad if \quad 1 \leq i \leq t-1$$

$$f(a_1a_{12}) = 2t$$

$$f(a_ia_{i1}) = 2t+2i \quad if \quad 1 \leq i \leq t$$

The vertex labelings are as follows.

$$f(a_{i1}) = 4t + 1 - 2i \quad if \quad 1 \leq i \leq t$$
$$f(a_{12}) = 4t + 1$$
$$f(a_i) = 2i - 1 \quad if \quad 1 \leq i \leq t$$

It can be easily verified that f is an odd vertex magic labeling with a vertex magic constant $k = 6t + 1$. □

Example 2.8. *Example if an odd vertex magic labeling of a graph G with $k = 31$ given in Figure 2.*

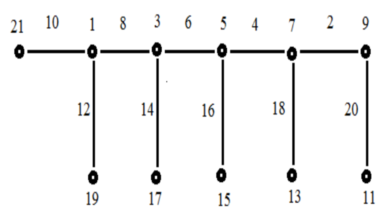

Figure 2:

References

[1] H.Enomoto, A.S.Llado, T.Nakamigawa G.Ringel, Super edge magic graphs, SUTJ.Math.2(1998) 105-109.

[2] J.A. Gallian, *A dynamic survey of graph labeling*, Electron J.Combinatorics 16 (2009) DS6.

[3] J.A. MacDougall, M.Miller, Slamin, W.D.Wallis, *Vertex magic total labeling of graphs*, Utilitas Math 61 (2002) 3-21.

[4] J.A.MacDougall, M.Miller,K.A.Sugeng,*Super vertex magic total labelling of graphs*, in: Proceedings of the 15^{th} Australian Workshop on Combinatorial Algorithms,2004,pp.222-229.

[5] G.Marimuthu, M.Balakrishnan, *E-Super vertex magic labelings of graphs, Discrete Applied Mathematics*, 160, 1766-1744, 2012.

[6] CT.Nagaraj, C.Y.Ponnappan, G.Prabakaran, *Even vertex magic total labeling, International Journal of Pure and Applied Mathematics*, Volume 115 , No. 9, (2017),363-375

[7] CT.Nagaraj, C.Y.Ponnappan, G.Prabakaran, *Odd vertex magic total labeling of some graphs*, Communicated.

[8] V.Swaminathan, P.Jeyanthi, *Super vertex magic labeling*, Indian J.Pure Appl.Math 34(6)(2003) 935-939.

[9] V.Swaminathan, P. Jeyanthi, *On Super vertex magic labeling*, J.Discrere Math.Sci.Cryptogr.8(2005) 217-224.

[10] W.D.Wallis, Magic Graphs, Birkhauser, Basel,2001.