Description of Some Right Unit Elements of the Complete Semigroups of Binary Relations of the Class $\Sigma_6(X,8)$.

Didem Yeşil Sungur[#1],GiuliTavdgiridze[*2]

[1]Canakkale Onsekiz Mart University, Faculty of Science and Art, Department of Mathematics,TURKEY
[2]Shota Rustaveli BatumiState University, Faculty of Mathematics, Physics and Computer Sciences,GEORGİA

Abstract—*Let* D *be any* $X-$ *semilattice of unions and* Q *be a subsemilattice of* D *which satisfies the following conditions;* $Q=\{T_6,T_5,T_4,T_3,T_2,T_1,T_0\}$ *be a* $X-$ *semilattice where* $T_7\subset T_6\subset T_5\subset T_3\subset T_1\subset T_0$, $T_7\subset T_6\subset T_5\subset T_2\subset T_1\subset T_{0,}$ $T_7\subset T_6\subset T_4\subset T_2\subset T_1\subset T_0$, $T_2\cup T_3=T_1$, $T_4\cup T_3=T_1$, $T_4\cup T_5=T_2$, $T_2\setminus T_3\neq\varnothing$, $T_3\setminus T_2\neq\varnothing$, $T_5\setminus T_4\neq\varnothing$, $T_4\setminus T_5\neq\varnothing$, $T_4\setminus T_3\neq\varnothing$, $T_3\setminus T_4\neq\varnothing$. *In this paper, we investigate a binary relation* α *which is right unit element.*

Keywords—*Semigroups, Binary relation, Right unit elements.*

I. INTRODUCTION

It is well known that in the theory of semigroups, any semigroup is isomorphically embeddable in some semigroups of binary relations. Many studies related with this fact are available in literature. Among them, we refer the papers of Diasamidze [1 - 4]. In general he studied semigroups but, in particular, he investigates the semigroups of the binary relations. Moreover, in his papers he presents how the idempotents, one-sided units, regular, irreducible and externally adjoined elements of semigroups have exclusively important role in the investigation of the abstract properties of semigroups.

To construct all these work we first give some basic definitions.

A partially ordered set A is called a *semilattice* if there exists a greatest lower bound for every pair of elements of A.

Let X be an arbitrary nonempty set, D be some nonempty set of subsets of the set X and also let D closed under the union of its subsets, i.e., $\cup D'\in D$ for any nonempty subset D' of the set $D.$ In that case, the set D is called a *complete* $X-$ *semilattice of unions*. The union of all elements of the set D is denoted by the symbol $\breve{D}.$ Clearly, $\breve{D}\in D$ is the largest element. We say that an element Y *covers* Z in the semilattice D if $Y\supset Z$ and there is no other element $T\in D$ such that $Y\supset T\supset Z.$ Using the cover relation, we can obtain a graphic representation of the semilattice of unions D. Each element of D is represented in the form of a small circle. If $Y\supset Z$ then Y is located above Z and also Y and Z are connected by a rectilinear line.

Recall that a binary relation on the set X is a subset of the cartesian product $X\times X.$ If α and β are binary relations on the set X with the elements $x,\ y,z\in X$ the condition $(x,y)\in\alpha$ is denoted as $x\alpha y$ and $x\alpha y\beta z$ means the conditions $x\alpha y$ and $y\beta z$ are satisfied simultaneously. The binary relation $\alpha^{-1}=\{(x,y)\mid y\alpha x\}$ is usually called the *binary relation inverse* to $\alpha.$ The empty binary relation which is an empty subset of $X\times X$ is denoted by $\varnothing.$ The binary relation $\delta=\alpha\circ\beta$ is called the *product of the binary relations* α and $\beta.$ A pair (x,z) belongs to δ if and only if there exists $y\in X$ such that $x\alpha y\beta z.$ The binary operation $\circ$ is associative. So, B_X, the set of all binary relations on X, is therefore a semigroup with respect to the operation $\circ.$ This semigroup is called the *semigroup of all binary relations* on the set $X.$

Further, let $x,y\in X$, $Y\subseteq X$, $\alpha\in B_X(D)$, $T\in D$, $\varnothing\neq D'\subseteq D$ and $t\in\breve{D}.$ Then we have the following notations,

$$y\alpha = \{x \in X \mid y\alpha x\} \quad , Y\alpha = \bigcup_{y \in Y} y\alpha,$$
$$2^X = \{Y \mid Y \subseteq X\} \quad , V(D, \alpha) = \{Y\alpha \mid Y \in D\},$$
$$D_t = \{Z' \in D \mid t \in Z'\} \quad , D'_T = \{Z' \in D' \mid T \subseteq Z'\},$$
$$X^* = 2^X \setminus \{\varnothing\} \quad , \ddot{D}_T = \{Z' \in D' \mid Z' \subseteq T\}.$$

Let f be an arbitrary mapping from X into D. Then one can construct such a mapping f with a binary relation α_f on X provided by the condition below,

$$\alpha_f = \bigcup_{x \in X} (\{x\} \times f(x)).$$

The set of all such binary relations is denoted by $B_X(D)$. It is easy to prove that $B_X(D)$ is a semigroup with respect to the product operation of binary relations. This semigroup, $B_X(D)$, is called a *complete semigroup of binary relations* defined by an $X-$semilattice of unions D.

Now, let's take any $\alpha, \beta \in B_X(D)$. If $\beta \circ \alpha = \beta$ then α is called a *right unit element* of semigroup $B_X(D)$. If $\alpha \circ \alpha = \alpha$ then α is called an *idempotent element* of semigroup $B_X(D)$. And if $\alpha \circ \beta \circ \alpha = \alpha$ for some $\beta \in B_X(D)$ then a binary relation α is called a *regular element* of semigroup $B_X(D)$.

Note that the semilattice D is partially ordered with respect to the set-theoretic inclusion. Let $\varnothing \neq D' \subseteq D$ and $N(D, D') = \{Z \in D \mid Z \subseteq Z' \text{ for any } Z' \in D'\}$. It is clear that $N(D, D')$ is the set of all lower bounds of a nonempty subset D' included in D. If $N(D, D') \neq \varnothing$ then $\cup N(D, D')$ belongs to D and it is the greatest lower bound of D' and is denoted by $\Lambda(D, D') = \cup N(D, D')$.

We say that a nonempty element T is a *nonlimiting element* of D' if $T \setminus l(D', T) \neq \varnothing$. A nonempty element T is *limiting element* of D' if $T \setminus l(D', T) = \varnothing$.

In this work, we use complete $X-$ semilattices of unions to define of all binary relations on a set X, which are called complete semigroups of binary relations. Let Q be a subsemilattice of D which satisfies the following conditions. $Q = \{T_6, T_5, T_4, T_3, T_2, T_1, T_0\}$ be a $X-$ semilattice where $T_7 \subset T_6 \subset T_5 \subset T_3 \subset T_1 \subset T_0$, $T_7 \subset T_6 \subset T_5 \subset T_2 \subset T_1 \subset T_0$, $T_7 \subset T_6 \subset T_4 \subset T_2 \subset T_1 \subset T_0$, $T_2 \cup T_3 = T_1$, $T_4 \cup T_3 = T_1$, $T_4 \cup T_5 = T_2$, $T_2 \setminus T_3 \neq \varnothing$, $T_3 \setminus T_2 \neq \varnothing$, $T_5 \setminus T_4 \neq \varnothing$, $T_4 \setminus T_5 \neq \varnothing$, $T_4 \setminus T_3 \neq \varnothing$, $T_3 \setminus T_4 \neq \varnothing$. Mainly, we investigate and determine right unit elements of $B_X(Q)$ for a finite set X and in particular, we give formulas for the calculation of the number of right unit elements.

Now, we continue with some essential definitions and theorems given by the cited references.

Definition 1.1 *[2, Definition 1] Let* $\alpha \in B_X(D)$, $T \in V(X^*, \alpha)$, $Y_T^\alpha = \{y \in X \mid y\alpha = T\}$. *Then a representation of a binary relation* α *of the form* $\alpha = \bigcup_{T \in V(X^*, \alpha)} (Y_T^\alpha \times T)$ *is called quasinormal.*

Note that, if $\alpha = \bigcup_{T \in V(X^*, \alpha)} (Y_T^\alpha \times T)$ is a quasinormal representation of the binary relation α, then the following conditions are true,

1. $X = \bigcup_{T \in V(X^*, \alpha)} Y_T^\alpha$,

2. $Y_T^\alpha \cap Y_{T'}^\alpha \neq \varnothing$ for $T, T' \in V(X^*, \alpha)$ and $T \neq T'$.

Definition 1.2 *[3, Definition 2] Let* $\breve{D}$ *and* D' *be some nonempty subsets of the complete* $X-$ *semilattices of unions. We say that a subset* $\breve{D}$ *generates a set* D' *if any element from* D' *is a set-theoretic union of the elements from* $\breve{D}$ *.*

Definition 1.3 *[4, Definition 1.14.2] We say that a complete* $X-$*semilattice of unions* D *is an* $XI-$ *semilattice of unions if it satisfies the following two conditions,*

- $\Lambda(D, D_t) \in D$ for any $t \in \breve{D}$,
- $Z = \bigcup_{t \in Z} \Lambda(D, D_t)$ for any nonempty element Z of D.

Theorem 1.4 *[4, Corollary 1.18.1] Let* $Y = \{y_1, y_2, \ldots, y_k\}$ *and* $D_j = \{T_1, \ldots, T_j\}$ *be some sets, where* $k \geq 1$ *and* $j \geq 1$*. Then the numbers* $s(k, j)$ *of all possible mappings of the sets* Y *on any subset* D_j' *of the set* D_j *and* $T_j \in D_j'$ *can be calculated by the formula*

$$s(k, j) = j^k - (j-1)^k.$$

Theorem 1.5 *[4, Theorem 6.1.3]Let* D *be a complete* $X-$ *semilattice of unions. The semigroup* $B_X(D)$ *possesses a right unit iff* D *is an* $XI-$ *semilattice of unions.*

Lemma 1.6 *[1, Lemma 3.1] Let* D *complete* $X-$ *semilattices of unions. If a binary relation* ε *having the form*

$$\varepsilon = \varepsilon(D, f) = \bigcup_{t \in \breve{D}} (\{x\} \times \Lambda(D, D_t)) \cup ((X \setminus \breve{D}) \times \breve{D})$$

is a right unit of the semigroup $B_X(D)$, then it is the largest right unit of this semigroup.

Definition 1.7 *[5, Definition 7] A one-to-one mapping* φ *between the complete* $X-$ *semilattices of unions* D' *and* D'' *is called a complete isomorphism if the condition*

$$\varphi(\cup D_1) = \bigcup_{T' \in D_1} \varphi(T')$$

is fulfilled for each nonempty subset D_1 of the semilattice D'.

Definition 1.8 *[5, Definition 8] Let* α *be some binary relation of the semigroup* $B_X(D)$*. We say that a complete isomorphism* φ *between* $XI-$*semilattice of unions* Q *and* D *is a complete* $\alpha-$ *isomorphism if*

1. $Q = V(D, \alpha)$,
2. $\varphi(\varnothing) = \varnothing$ for $\varnothing \in V(D, \alpha)$ and $\varphi(T)\alpha = T$ for any $T \in V(D, \alpha)$.

Theorem 1.9 *[4, Theorem 6.3.3] Let* X *be a finite set,* $\alpha \in B_X(D)$ *and* $D(\alpha)$ *be the set of those elements* T *of the semilattice* $D = V(D, \alpha) \setminus \{\varnothing\}$ *which are nonlimiting elements of the set* $\ddot{D}(\alpha)_T$*. A binary relation* α *having a quasinormal representation of the form* $\alpha = \bigcup_{T \in V(D, \alpha)} (Y_T^\alpha \times T)$ *is a right unit element of the semigroup* $B_X(D)$ *iff*

1. $V(D, \alpha)$ is a complete $XI-$ semilattice of unions and $V(D, \alpha) = D$,

2. $\bigcup_{T' \in \ddot{D}(\alpha)_T} Y_{T'}^{\alpha} \supseteq \varphi(T)$ for any $T \in D(\alpha)$,

3. $Y_T^{\alpha} \cap T \neq \varnothing$ for any nonlimiting element T of the set $\ddot{D}(\alpha)_T$.

Definition 1.10 *[4, Definition 6.3.4] Let* $\Phi(Q, D')$ *be a set of all complete isomorphism of the* $XI-$ *semilattice of unions* Q *on the semilattice* D' *such that* $\varphi \in \Phi(Q, D')$ *only if* φ *is a* $\alpha-$ *isomorphism for some relation* $\alpha \in B_X(D)$ *and* $V(D, \alpha) = Q$.

$\Omega(Q)$ is the set of all $XI-$ subsemilattices of the complete $X-$ semilattice of unions D such that $Q' \in \Omega(Q)$ only if there exists some complete isomorphism between the semilattices Q' and Q.

Theorem 1.11 *[5, Theorem 22] Let* $D = \{\breve{D}, T_1, T_2, \ldots, T_m\}$ *be some finite* X *- semilattice of unions and* $C(D) = \{P_0, P_1, P_2, \ldots, P_{m-1}, P_m\}$ *be the family of sets of pairwise disjoint subsets of the set* X, φ *is a mapping of the semilattice* D *on the family sets* $C(D)$ *that satisfies the condition* $\varphi(\breve{D}) = P_0$ *and* $\varphi(T_i) = P_i$ *for any* $i = 1, 2, \ldots, m$ *and* $\hat{D}_Z = D \setminus \{T \in D \mid Z \subseteq T\}$, *then the following equalities are valid:*

$$\breve{D} = P_0 \cup P_1 \cup P_2 \cup \cdots \cup P_m \tag{1.1}$$

and

$$Z_i = P_0 \cup \bigcup_{T \in \hat{D}_{Z_i}} \varphi(T) \; for all i = 1, 2, \ldots, m.$$

In the sequel these equalities will be called formal.

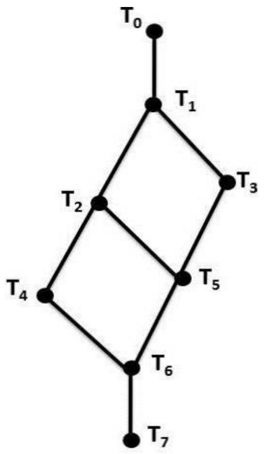
It is proved that if the elements of the semilattice D are represented in the form 1.1, then among the parameters P_i $(i = 0, 1, 2, \ldots, m-1)$ there exists such parameters thatcannot be empty sets for D. Such sets P_i $(0 \le i \le m-1)$ are called bases sources, whereas sets P_j $(0 \le j \le m-1)$ which can be empty sets too are called completeness sources.

It is proved that under the mapping φ the number of covering elements of the pre-image of a basis source is always equal to one, while under the mapping φ the number of covering elements of the pre-image of a completeness source either does not exists or is always greater than one.

Note that the set P_0 is always considered to be a source of completeness.

II. RESULTS

Xcvxcvxcv Let X be a finite set, $D = \{T_7, T_6, T_5, T_4, T_3, T_2, T_1, T_0\}$ be a $X-$subsemilattice of unions of D satisfies the following conditions.The diagram of the D is shown in the following figure.



$$T_7 \subset T_6 \subset T_5 \subset T_3 \subset T_1 \subset T_0$$
$$T_7 \subset T_6 \subset T_5 \subset T_2 \subset T_1 \subset T_0$$
$$T_7 \subset T_6 \subset T_4 \subset T_2 \subset T_1 \subset T_0$$
$$T_2 \cup T_3 = T_1, T_4 \cup T_3 = T_1,$$
$$T_4 \cup T_5 = T_2,$$
$$T_2 \setminus T_3 \neq \varnothing, T_3 \setminus T_2 \neq \varnothing,$$
$$T_5 \setminus T_4 \neq \varnothing, T_4 \setminus T_5 \neq \varnothing,$$
$$T_4 \setminus T_3 \neq \varnothing, T_3 \setminus T_4 \neq \varnothing.$$

Let

$$\varphi = \begin{pmatrix} T_0 & T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 \\ P_0 & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 \end{pmatrix}$$

is a mapping of the semilattice D onto the family sets $C(D)$. Then for the formal equalities of the semilattice D we have a form,

$$\begin{aligned} T_0 &= P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7 \\ T_1 &= P_0 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7 \\ T_2 &= P_0 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7 \\ T_3 &= P_0 \cup P_2 \cup P_4 \cup P_5 \cup P_6 \cup P_7 \\ T_4 &= P_0 \cup P_3 \cup P_5 \cup P_6 \cup P_7 \\ T_5 &= P_0 \cup P_4 \cup P_6 \cup P_7 \\ T_6 &= P_0 \cup P_7 \\ T_7 &= P_0 \end{aligned} \qquad (2.1)$$

Here the elements P_1, P_2, P_3, P_4, P_7 are basis sources, the elements P_0, P_5, P_6 is sources of completeness of the semilattice D.

Theorem 2.1 *The semigroup $B_X(D)$ always has a right unit element.*

Proof. Let $t \in D$, $D_t = \{Z \in D \mid t \in Z\}$ and $\wedge(D, D_t)$ is the exact lower bound of the set D_t in D. Then the formal equalities follows that,

$$D_t = \begin{cases} D, \text{if} t \in P_0 \\ T_0, \text{if} t \in P_1 \\ \{T_3, T_1, T_0\}, \text{if} t \in P_2 \\ \{T_4, T_2, T_1, T_0\}, \text{if} t \in P_3 \\ \{T_5, T_3, T_2, T_1, T_0\}, \text{if} t \in P_4 \\ \{T_4, T_3, T_2, T_1, T_0\}, \text{if} t \in P_5 \\ \{T_5, T_4, T_3, T_2, T_1, T_0\}, \text{if} t \in P_6 \\ \{T_6, T_5, T_4, T_3, T_2, T_1, T_0\}, \text{if} t \in P_7 \end{cases} \qquad \wedge(D; D_t) = \begin{cases} T_7, \text{if} t \in P_0 \\ T_0, \text{if} t \in P_1 \\ T_3, \text{if} t \in P_2 \\ T_4, \text{if} t \in P_3 \\ T_5, \text{if} t \in P_4 \\ T_6, \text{if} t \in P_5 \\ T_6, \text{if} t \in P_6 \\ T_6, \text{if} t \in P_7 \end{cases}$$

We have $D^{\wedge}=\{T_7,T_6,T_5,T_4,T_3,T_0\}$, $\wedge(D;D_t)\in D$ for all $t\in D$ and $T_2=T_4\cup T_5$, $T_1=T_2\cup T_3$. So from the Definition 1.3follows that the semilattice D is $XI-$semilattice. In view of the Theorem 1.5 $B_X(D)$ always has a right unit element.

Lemma 2.2 *For the semilattice* D, *the following equalities are true.*

$$P_0=T_7$$
$$P_5\cup P_6\cup P_7=(T_3\cap T_4)\setminus T_7$$
$$P_4=T_5\setminus T_4$$
$$P_3=T_4\setminus T_3$$
$$P_2=T_3\setminus T_2$$
$$P_1=T_0\setminus T_1$$

Proof. The given Lemma immediately follows from the formal equalities 2.1 of the semilattice D.

Theorem 2.3 *The binary relation*

$$\varepsilon=(T_7\times T_7)\cup(((T_3\cap T_4)\setminus T_7)\times T_6)\cup((T_5\setminus T_4)\times T_5)\cup((T_4\setminus T_3)\times T_4)\cup((T_3\setminus T_2)\times T_3)\cup((T_0\setminus T_1)\times T_0)\cup((X\setminus T_0)\times T_0)$$

is the largest right unit of the semigroup $B_X(D)$.

Proof. Using Lemma 1.6 and Theorem 2.1 we have that

$$\begin{aligned}\varepsilon &= \bigcup_{t\in \breve{D}}(\{t\}\times\wedge(D;D_t))\cup((X\setminus T_0)\times T_0)\\ &= (P_0\times T_7)\cup((P_5\cup P_6\cup P_7)\times T_6)\cup(P_4\times T_5)\cup(P_3\times T_4)\cup(P_2\times T_3)\cup(P_1\times T_0)\cup((X\setminus T_0)\times T_0)\\ &= (T_7\times T_7)\cup(((T_3\cap T_4)\setminus T_7)\times T_6)\cup((T_5\setminus T_4)\times T_5)\cup((T_4\setminus T_3)\times T_4)\\ &\quad\cup((T_3\setminus T_2)\times T_3)\cup((T_0\setminus T_1)\times T_0)\cup((X\setminus T_0)\times T_0)\end{aligned}$$

Corollary 2.4 *A binary relation* α *having quasinormal representation as*

$$\begin{aligned}\alpha &= (Y_7^\alpha\times T_7)\cup(Y_6^\alpha\times T_6)\cup(Y_5^\alpha\times T_5)\cup(Y_4^\alpha\times T_4)\\ &\quad\cup(Y_3^\alpha\times T_3)\cup(Y_2^\alpha\times T_2)\cup(Y_1^\alpha\times T_1)\cup(Y_0^\alpha\times T_0),\end{aligned}$$

where $Y_7^\alpha,Y_6^\alpha,Y_5^\alpha,Y_4^\alpha,Y_3^\alpha,Y_0^\alpha\notin\{\varnothing\}$, is a right unit of the semigroup $B_X(D)$ iff the binary relation α satisfies the following conditions,

$$Y_7^\alpha\supseteq T_7, Y_7^\alpha\cup Y_6^\alpha\supseteq T_6, Y_7^\alpha\cup Y_6^\alpha\cup Y_5^\alpha\supseteq T_5, Y_7^\alpha\cup Y_6^\alpha\cup Y_4^\alpha\supseteq T_4,$$
$$Y_7^\alpha\cup Y_6^\alpha\cup Y_5^\alpha\cup Y_3^\alpha\supseteq T_3, Y_6^\alpha\cap T_6\neq\varnothing, Y_5^\alpha\cap T_5\neq\varnothing, Y_4^\alpha\cap T_4\neq\varnothing$$
$$Y_3^\alpha\cap T_3\neq\varnothing, Y_0^\alpha\cap T_0\neq\varnothing.$$

Proof. It is easy to see, that the set $D(\alpha)=\{T_7,T_6,T_5,T_4,T_3,T_2,T_1\}$ is a generating set of the semilattice D. Then the following equalities,

$$\ddot{D}(\alpha)_{T_7}=\{T_7\}, \ddot{D}(\alpha)_{T_6}=\{T_7,T_6\}, \ddot{D}(\alpha)_{T_5}=\{T_7,T_6,T_5\}, \ddot{D}(\alpha)_{T_4}=\{T_7,T_6,T_4\},$$
$$\ddot{D}(\alpha)_{T_3}=\{T_7,T_6,T_5,T_3\}, \ddot{D}(\alpha)_{T_2}=\{T_7,T_6,T_5,T_4,T_2\},$$
$$\ddot{D}(\alpha)_{T_1}=\{T_7,T_6,T_5,T_4,T_3,T_2,T_1\},$$
$$\ddot{D}(\alpha)_{T_1}=\{T_7,T_6,T_5,T_4,T_3,T_2,T_1,T_0\},$$

are hold. By statement $b)$ of the Theorem 1.9 follows that

$$Y_7^\alpha \supseteq T_7, Y_7^\alpha \cup Y_6^\alpha \supseteq T_6, Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \supseteq T_5, Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \supseteq T_4,$$
$$Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_3^\alpha \supseteq T_3, Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_2^\alpha \supseteq T_2,$$
$$Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_3^\alpha \cup Y_2^\alpha \cup Y_1^\alpha \supseteq T_1.$$

For the last conditions we have

$$Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_2^\alpha = \left(Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha\right) \cup \left(Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha\right) \cup Y_2^\alpha \supseteq T_5 \cup T_4 \cup Y_2^\alpha$$
$$= T_2 \cup Y_2^\alpha \supseteq T_2$$
$$Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_3^\alpha \cup Y_2^\alpha \cup Y_1^\alpha = \left(Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha\right) \cup \left(Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha\right) \cup \left(Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_3^\alpha\right)$$
$$\cup Y_2^\alpha \cup Y_1^\alpha \supseteq T_5 \cup T_4 \cup T_3 \cup Y_2^\alpha \cup Y_1^\alpha$$
$$= T_1 \cup Y_2^\alpha \cup Y_1^\alpha \supseteq T_1$$

is always satisfied.

Now for using limiting element definition we find,

$$l\left(\ddot{D}(\alpha)_{T_7}, T_7\right) = \cup\left(\ddot{D}(\alpha)_{T_7} \setminus \{T_7\}\right) = \varnothing, T_7 \setminus l\left(\ddot{D}(\alpha)_{T_7}, T_7\right) = T_7 \setminus \varnothing = T_7 \neq \varnothing,$$
$$l\left(\ddot{D}(\alpha)_{T_6}, T_6\right) = \cup\left(\ddot{D}(\alpha)_{T_6} \setminus \{T_6\}\right) = T_7, T_6 \setminus l\left(\ddot{D}(\alpha)_{T_6}, T_6\right) = T_6 \setminus T_7 \neq \varnothing,$$
$$l\left(\ddot{D}(\alpha)_{T_5}, T_5\right) = \cup\left(\ddot{D}(\alpha)_{T_5} \setminus \{T_5\}\right) = T_6, T_5 \setminus l\left(\ddot{D}(\alpha)_{T_5}, T_5\right) = T_5 \setminus T_6 \neq \varnothing,$$
$$l\left(\ddot{D}(\alpha)_{T_4}, T_4\right) = \cup\left(\ddot{D}(\alpha)_{T_4} \setminus \{T_4\}\right) = T_6, T_4 \setminus l\left(\ddot{D}(\alpha)_{T_4}, T_4\right) = T_4 \setminus T_6 \neq \varnothing,$$
$$l\left(\ddot{D}(\alpha)_{T_3}, T_3\right) = \cup\left(\ddot{D}(\alpha)_{T_3} \setminus \{T_3\}\right) = T_5, T_3 \setminus l\left(\ddot{D}(\alpha)_{T_3}, T_3\right) = T_3 \setminus T_5 \neq \varnothing,$$
$$l\left(\ddot{D}(\alpha)_{T_2}, T_2\right) = \cup\left(\ddot{D}(\alpha)_{T_2} \setminus \{T_2\}\right) = T_2, T_2 \setminus l\left(\ddot{D}(\alpha)_{T_2}, T_2\right) = T_2 \setminus T_2 = \varnothing,$$
$$l\left(\ddot{D}(\alpha)_{T_1}, T_1\right) = \cup\left(\ddot{D}(\alpha)_{T1} \setminus \{T_1\}\right) = T_1, T_1 \setminus l\left(\ddot{D}(\alpha)_{T_1}, T_1\right) = T_1 \setminus T_1 = \varnothing,$$
$$l\left(\ddot{D}(\alpha)_{T_0}, T_0\right) = \cup\left(\ddot{D}(\alpha)_{T_0} \setminus \{T_0\}\right) = T_1, T_0 \setminus l\left(\ddot{D}(\alpha)_{T_0}, T_0\right) = T_0 \setminus T_1 \neq \varnothing.$$

Therefore, T_7, T_6, T_5, T_4, T_3 and T_0 are nonlimiting elements of the sets $\ddot{D}(\alpha)_{T_7}, \ddot{D}(\alpha)_{T_6}, \ddot{D}(\alpha)_{T_5}, \ddot{D}(\alpha)_{T_4}, \ddot{D}(\alpha)_{T_3}$ and $\ddot{D}(\alpha)_{T_0}$ respectively. By the statement $c)$ Theorem 1.9it follows, that the conditions

$$Y_6^\alpha \cap T_6 \neq \varnothing, Y_5^\alpha \cap T_5 \neq \varnothing, Y_4^\alpha \cap T_4 \neq \varnothing, Y_3^\alpha \cap T_3 \neq \varnothing, Y_0^\alpha \cap T_0 \neq \varnothing$$

are hold. As a result of these we have,

$$Y_7^\alpha \supseteq T_7, Y_7^\alpha \cup Y_6^\alpha \supseteq T_6, Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \supseteq T_5, Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \supseteq T_4,$$
$$Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_3^\alpha \supseteq T_3, Y_6^\alpha \cap T_6 \neq \varnothing, Y_5^\alpha \cap T_5 \neq \varnothing, Y_4^\alpha \cap T_4 \neq \varnothing, \quad (2.2)$$
$$Y_3^\alpha \cap T_3 \neq \varnothing, Y_0^\alpha \cap T_0 \neq \varnothing.$$

Theorem 2.5 $E_X^{(r)}(D)$ *are the set of all right unit elements of the semigroup* $B_X(D)$. *If* X *be a finite set, then we calculate the number of right unit elements through this formula*

$$\left|E_X^{(r)}(D)\right| = 2^{\left|(T_3 \cap T_4) \setminus T_7\right|} \cdot \left(2^{|T_6 \setminus T_7|} - 1\right) \cdot \left(3^{|T_5 \setminus T_4|} - 2^{|T_5 \setminus T_4|}\right) \cdot \left(3^{|T_4 \setminus T_3|} - 2^{|T_4 \setminus T_3|}\right)$$
$$\cdot \left(4^{|T_3 \setminus T_2|} - 3^{|T_3 \setminus T_2|}\right) \cdot \left(8^{|T_0 \setminus T_1|} - 3^{|T_0 \setminus T_1|}\right) \cdot 8^{|X \setminus T_0|}.$$

Proof. Assume that $\alpha \in E_X^{(r)}(D)$ and from Corollary 2.4, we have that α has a quasinormal representation as $\alpha = \bigcup_{T \in D}\left(Y_T^\alpha \times T\right)$, where $Y_7^\alpha, Y_6^\alpha, Y_5^\alpha, Y_4^\alpha, Y_3^\alpha, Y_0^\alpha \notin \{\varnothing\}$ (i.e., $D = V(D, \alpha)$)and satisfies the following conditions

$$Y_7^\alpha \supseteq T_7, Y_7^\alpha \cup Y_6^\alpha \supseteq T_6, Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \supseteq T_5, Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \supseteq T_4,$$
$$Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_3^\alpha \supseteq T_3, Y_6^\alpha \cap T_6 \neq \varnothing, Y_5^\alpha \cap T_5 \neq \varnothing, Y_4^\alpha \cap T_4 \neq \varnothing,$$
$$Y_3^\alpha \cap T_3 \neq \varnothing, Y_0^\alpha \cap T_0 \neq \varnothing.$$

Further, let f_α be a mapping from the set X in the semilattice D satisfying the condition $f_\alpha(t) = t\alpha$ for all $t \in X$. We can define $f_{0\alpha}$, $f_{1\alpha}$, $f_{2\alpha}$, $f_{3\alpha}$, $f_{4\alpha}$, $f_{5\alpha}$ and $f_{6\alpha}$ which are the restrictions of the mapping f_α on the sets $T_{7,}$ $(T_3 \cap T_4) \setminus T_7$, $T_5 \setminus T_4$, $T_4 \setminus T_3$, $T_3 \setminus T_2$, $T_0 \setminus T_1$ and $X \setminus T_0$, respectively. It is clear, that the intersection disjoint sets of the set

$$\{T_{7,}(T_3 \cap T_4) \setminus T_7, T_5 \setminus T_4, T_4 \setminus T_3, T_3 \setminus T_2, T_0 \setminus T_1, X \setminus T_0\}$$

are empty set and

$$T \cup ((T_3 \cap T_4) \setminus T_7) \cup (T_5 \setminus T_4) \cup (T_4 \setminus T_3) \cup (T_3 \setminus T_2) \cup (T_0 \setminus T_1) \cup (X \setminus T_0) = X$$

Now, we are going to find properties of maps $f_{0\alpha}$, $f_{1\alpha}$, $f_{2\alpha}$, $f_{3\alpha}$, $f_{4\alpha}$, $f_{5\alpha}$ and $f_{6\alpha}$.

1. Let $t \in T_7$. Then, by using Equation 2.2 and the definition of the set Y_7^α , we have $t \in T_7 \subseteq Y_7^\alpha \Rightarrow f_{0\alpha}(t) = T_7$ for all $t \in T_7$.

2. Let $t \in (T_3 \cap T_4) \setminus T_7$. Then, by using Equation 2.2 and definition of the sets Y_7^α and Y_6^α , we have $t \in (T_3 \cap T_4) \setminus T_7 \subseteq T_3 \cap T_4 \subseteq (Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha) \cap (Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_3^\alpha) = Y_7^\alpha \cup Y_6^\alpha \Rightarrow$ $f_{1\alpha}(t) \in \{T_7, T_6\}$ for all $t \in (T_3 \cap T_4) \setminus T_7$.

Suppose that $Y_6^\alpha \cap T_6 \neq \varnothing$ and $t_1 \in Y_6^\alpha \cap T_6 \Rightarrow t_1 \in Y_6^\alpha$ and $t_1 \in T_6 \Rightarrow t_1\alpha = T_6$ and $t_1 \in T_6$. If $t_1 \in T_7$ then $t_1 \in T_7 \subseteq Y_7^\alpha \Rightarrow f_{1\alpha}(t_1) = T_7$ for some $t_1 \in T_7$. Which contradicts with $t_1\alpha = T_6$ (T_6 is not equal to T_7 in D). So $Y_6^\alpha \cap T_6 \subseteq (T_3 \cap T_4) \setminus T_7$ and $f_{1\alpha}(t_1) = T_6$ for some $t_1 \in T_6 \setminus T_7$.

3. Let $t \in T_5 \setminus T_4$. Then, by using Equation 2.2 and definition of the sets Y_7^α, Y_6^α and Y_5^α , we have $t \in T_5 \setminus T_4 \subseteq T_5 \subseteq Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \Rightarrow f_{2\alpha}(t) = t\alpha \in \{T_7, T_6, T_5\}$ for all $t \in T_5 \setminus T_4$.

Suppose that $Y_5^\alpha \cap T_5 \neq \varnothing$ and $t_2 \in Y_5^\alpha \cap T_5 \Rightarrow t_2 \in Y_5^\alpha$ and $t_2 \in T_5 \Rightarrow t_2\alpha = T_5$. If $t_2 \in T_4$ then $t_2 \in T_4 \subseteq Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha$. Therefore, $t_2\alpha \in \{T_7, T_6, T_4\}$. Which contradicts of the equality $t_2\alpha = T_5$, since $T_5 \notin \{T_7, T_6, T_4\}$. So $f_{2\alpha}(t_2) = T_5$ for some $t_2 \in T_5 \setminus T_4$.

4. Let $t \in T_4 \setminus T_3$. Then, by using Equation 2.2 and definition of the sets Y_7^α, Y_6^α and Y_4^α , we have $t \in T_4 \setminus T_3 \subseteq T_4 \subseteq Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \Rightarrow t\alpha \in \{T_7, T_6, T_4\}$. Therefore $f_{3\alpha}(t) \in \{T_7, T_6, T_4\}$ for all $t \in T_4 \setminus T_3$.

Suppose that $Y_4^\alpha \cap T_4 \neq \varnothing$ and $t_3 \in Y_4^\alpha \cap T_4 \Rightarrow t_3 \in Y_4^\alpha$ and $t_3 \in T_4 \Rightarrow t_3\alpha = T_4$. If $t_3 \in T_3$ then $t_3 \in T_3 \subseteq Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_3^\alpha$. Therefore, $t_3\alpha \in \{T_7, T_6, T_5, T_3\}$. Which contradicts of the equality $t_3\alpha = T_4$, since $T_4 \notin \{T_7, T_6, T_5, T_3\}$. So $f_{3\alpha}(t_3) = T_4$ for some $t_3 \in T_4 \setminus T_3$.

5. Let $t \in T_3 \setminus T_2$. Then, by using Equation 2.2 and definition of the sets $Y_7^\alpha, Y_6^\alpha, Y_5^\alpha$ and Y_3^α , we have $t \in T_3 \setminus T_2 \subseteq T_3 \subseteq Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_3^\alpha \Rightarrow t\alpha \in \{T_7, T_6, T_5, T_3\}$. Therefore $f_{4\alpha}(t) \in \{T_7, T_6, T_5, T_3\}$ for all $t \in T_3 \setminus T_2$.

Suppose that $Y_3^\alpha \cap T_3 \neq \varnothing$ and $t_4 \in Y_3^\alpha \cap T_3 \Rightarrow t_4 \in Y_3^\alpha$ and $t_4 \in T_3 \Rightarrow t_4\alpha = T_3$. If $t_4 \in T_2$ then $t_4 \in T_2 \subseteq Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha$. Therefore, $t_4\alpha \in \{T_7, T_6, T_5, T_4\}$. Which contradicts of the equality $t_4\alpha = T_3$, since $T_3 \notin \{T_7, T_6, T_5, T_4\}$. So $f_{4\alpha}(t_4) = T_3$ for some $t_4 \in T_3 \setminus T_2$.

6. Let $t \in T_0 \setminus T_1$. Then, by using Equation 2.2 and definition of the sets $Y_i^\alpha, i = 0,1,\ldots,7$, we have $t \in T_0 \setminus T_1 \subseteq T_0 \subseteq \bigcup_{i=0}^{7} Y_i^\alpha \Rightarrow t\alpha \in \{T_7, T_6, T_5, T_4, T_3, T_2, T_1, T_0\}$. Therefore $f_{5\alpha}(t) \in \{T_7, T_6, T_5, T_4, T_3, T_2, T_1, T_0\}$ for all $t \in T_0 \setminus T_1$.

Suppose that $Y_0^\alpha \cap T_0 \neq \varnothing$ and $t_5 \in Y_0^\alpha \cap T_0 \Rightarrow t_5 \in Y_0^\alpha$ and $t_5 \in T_0 \Rightarrow t_5\alpha = T_0$. If $t_5 \in T_1$ then $t_5 \in T_1 \subseteq Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_3^\alpha \cup Y_2^\alpha \cup Y_1^\alpha$. Therefore, $t_5\alpha \in \{T_7, T_6, T_5, T_4, T_3, T_2, T_1\}$. Which contradicts of the equality $t_5\alpha = T_0$, since $T_0 \notin \{T_7, T_6, T_5, T_4, T_3, T_2, T_1\}$. So $f_{5\alpha}(t_5) = T_0$ for some $t_5 \in T_0 \setminus T_1$.

7. Let $t \in X \setminus T_0$ Then, by using Equation 2.2 and the definition of the sets $Y_6^\alpha, Y_5^\alpha, Y_4^\alpha, Y_3^\alpha, Y_2^\alpha, Y_1^\alpha$ and Y_0^α, we have $t \in X \setminus T_0 \subseteq X \subseteq Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_3^\alpha \cup Y_2^\alpha \cup Y_1^\alpha \cup Y_0^\alpha \Rightarrow t\alpha \in \{T_6, T_5, T_4, T_3, T_2, T_1, T_0\}$. Therefore $f_{6\alpha}(t) \in \{T_6, T_5, T_4, T_3, T_2, T_1, T_0\}$ for all $t \in X \setminus T_{0}$.

Therefore, there is an ordered system $(f_{0\alpha}, f_{1\alpha}, f_{2\alpha}, f_{3\alpha}, f_{4\alpha}, f_{5\alpha}, f_{6\alpha})$ of every binary relation α in which is the element of $E_X^{(r)}(D)$.

Now, let

$$
\begin{aligned}
&f_0 : T_7 \to \{T_7\},\\
&f_1 : (T_3 \cap T_4) \setminus T_7 \to \{T_7, T_6\},\\
&f_2 : T_5 \setminus T_4 \to \{T_7, T_6, T_5\},\\
&f_3 : T_4 \setminus T_3 \to \{T_7, T_6, T_4\},\\
&f_4 : T_3 \setminus T_2 \to \{T_7, T_6, T_5, T_3\},\\
&f_5 : T_0 \setminus T_1 \to \{T_7, T_6, T_5, T_4, T_3, T_2, T_1, T_0\},\\
&f_6 : X \setminus T_0 \to \{T_7, T_6, T_5, T_4, T_3, T_2, T_1, T_0\}.
\end{aligned}
$$

are such mappings which satisfies the following conditions:

1. $f_0(t) = T_7$ for all $t \in T_7$,
2. $f_1(t) \in \{T_7, T_6\}$ for all $t \in (T_3 \cap T_4) \setminus T_7$ and $f_1(t_1) = T_6$ for some $t_1 \in T_6 \setminus T_7$,
3. $f_2(t) \in \{T_7, T_6, T_5\}$ for all $t \in T_5 \setminus T_4$ and $f_2(t_2) = T_5$ for some $t_2 \in T_5 \setminus T_4$,
4. $f_3(t) \in \{T_7, T_6, T_4\}$ for all $t \in T_4 \setminus T_3$ and $f_3(t_3) = T_4$ for some $t_3 \in T_4 \setminus T_3$,
5. $f_4(t) \in \{T_7, T_6, T_5, T_3\}$ for all $t \in T_3 \setminus T_2$ and $f_4(t_4) = T_3$ for some $t_4 \in T_3 \setminus T_2$,
6. $f_5(t) \in \{T_7, T_6, T_5, T_4, T_3, T_2, T_1, T_0\}$ for all $t \in T_0 \setminus T_1$ and $f_5(t_5) = T_0$ for some $t_5 \in T_0 \setminus T_1$,
7. $f_6(t) \in \{T_7, T_6, T_5, T_4, T_3, T_2, T_1, T_0\}$ for all $t \in X \setminus T_0$.

Then, we can define a map f from X in the semilattice D by following way:

$$f(t)=\begin{cases} f_0(t), \text{if } t\in T_7, \\ f_1(t), \text{if } t\in (T_3\cap T_4)\setminus T_7, \\ f_2(t), \text{if } t\in T_5\setminus T_4, \\ f_3(t), \text{if } t\in T_4\setminus T_3, \\ f_4(t), \text{if } t\in T_3\setminus T_2, \\ f_5(t), \text{if } t\in T_0\setminus T_1, \\ f_6(t), \text{if } t\in X\setminus T_0. \end{cases}$$

Further, we identify the binary relation $\beta=\bigcup_{x\in X}(\{x\}\times f(x))$ which is originated with the mapping f. Since, $Y_i^\beta=\{t\in X\mid t\beta=T_i\}$ $(i=0,1,2,\ldots,6)$, then binary relation β is represented by following form

$$\beta=(Y_6^\beta\times T_6)\cup(Y_5^\beta\times T_5)\cup(Y_4^\beta\times T_4)\cup(Y_3^\beta\times T_3)\cup(Y_2^\beta\times T_2)\cup(Y_1^\beta\times T_1)\cup(Y_0^\beta\times T_0).$$

If the definitions of Y_i^β taken into consideration, then we have

$$Y_7^\beta\supseteq T_7, Y_7^\beta\cup Y_6^\beta\supseteq T_6, Y_7^\beta\cup Y_6^\beta\cup Y_5^\beta\supseteq T_5, Y_7^\beta\cup Y_6^\beta\cup Y_4^\beta\supseteq T_4,$$
$$Y_7^\beta\cup Y_6^\beta\cup Y_5^\beta\cup Y_3^\beta\supseteq T_3, Y_6^\beta\cap T_6\neq\varnothing, Y_5^\beta\cap T_5\neq\varnothing, Y_4^\beta\cap T_4\neq\varnothing,$$
$$Y_3^\beta\cap T_3\neq\varnothing, Y_0^\beta\cap T_0\neq\varnothing.$$

(By suppose $f_1(t_1)=T_6$ for some $t_1\in T_6\setminus T_7$, $f_2(t_2)=T_5$ for some $t_2\in T_5\setminus T_4$, $f_3(t_3)=T_4$ for some $t_3\in T_4\setminus T_3$, $f_4(t_4)=T_3$ for some $t_4\in T_3\setminus T_2$, $f_5(t_5)=T_0$ for some $t_5\in T_3\setminus T_2$.)

From these properties and Corollary 2.4, we have that the binary relation β is an element of $E_X^{(r)}(D)$.

Therefore for every binary relation $\alpha\in E_X^{(r)}(D)$ and ordered system $(f_{0\alpha}, f_{1\alpha}, f_{2\alpha}, f_{3\alpha}, f_{4\alpha}, f_{5\alpha}, f_{6\alpha})$ exist one to one mapping.

And also, by using the Theorem 1.4, the number of the mappings $f_{0\alpha}, f_{1\alpha}, f_{2\alpha}, f_{3\alpha}, f_{4\alpha}, f_{5\alpha}$ and $f_{6\alpha}$ are respectively as

$$1, 2^{|(T_3\cap T_4)\setminus T_7|}\cdot\left(2^{|T_6\setminus T_7|}-1\right), 3^{|T_5\setminus T_4|}-2^{|T_5\setminus T_4|},$$
$$3^{|T_4\setminus T_3|}-2^{|T_4\setminus T_3|}, 4^{|T_3\setminus T_2|}-3^{|T_3\setminus T_2|}, 8^{|T_0\setminus T_1|}-7^{|T_0\setminus T_1|}, 8^{|X\setminus T_0|}. \tag{2.3}$$

By using Equation 2.3, we calculate the number of ordered system $(f_{0\alpha}, f_{1\alpha}, f_{2\alpha}, f_{3\alpha}, f_{4\alpha}, f_{5\alpha}, f_{6\alpha})$ or the number of right unit elements through this formula

$$\left|E_X^{(r)}(D)\right|=2^{|(T_3\cap T_4)\setminus T_7|}\cdot\left(2^{|T_6\setminus T_7|}-1\right)\cdot\left(3^{|T_5\setminus T_4|}-2^{|T_5\setminus T_4|}\right)\cdot\left(3^{|T_4\setminus T_3|}-2^{|T_4\setminus T_3|}\right)$$
$$\cdot\left(4^{|T_3\setminus T_2|}-3^{|T_3\setminus T_2|}\right)\cdot\left(8^{|T_0\setminus T_1|}-7^{|T_0\setminus T_1|}\right)\cdot 8^{|X\setminus T_0|}.$$

REFERENCES

[1] Diasamidze Ya., Complete Semigroups of Binary Relations. Journal of Mathematical Sciences, Plenum Publ. Cor., New York, Vol. 117, No. 4, 4271-4319, 2003.

[2] Diasamidze Ya., Makharadze Sh., Fartenadze G., Maximal Subgroups of Complete Semigroups of Binary Relations. Proceedigs of A. Razmadze Mathematical Institute, Vol. 131, 21-38, 2003.

[3] Diasamidze Ya., To the The Theory of the Binary Relation Semigroups. Proceedigs of A. Razmadze Mathematical Institute, Vol. 128, 1-15, 2002.

[4] Diasamidze Ya., Makharadze Sh., Complete semigroups of binary relations. Kriter Yay nevi, İstanbul, 1- 520 , 2010

[5] Diasamidze Ya., Makharadze Sh., Complete Semigroups of Binary Relations Defined by X-Semilattices of Unions. Journal of Mathematical Sciences, Plenum Publ. Cor., New York, Vol. 166, No. 5, 615-633., 2010