Semitotal Block Double Domination in Graphs

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Abstract
For any graph $G = (V, E)$, the semitotal block graph $T_b(G) = H$, whose set of vertices is the union of the set of vertices and block of G and in which two vertices are adjacent if and if the corresponding vertices of G are adjacent or the corresponding members are incident in G. A subset D^d of $V[T_b(G)]$ is double dominating set of $T_b(G)$ if for every vertex $v \in V[T_b(G)], |N[v] \cap D^d| \geq 2$, that is v is in D^d and has at least one neighbor in D^d or v is in $V[T_b(G)] - D^d$ and has at least two neighbors in D^d. The semitotal block dominating number $\gamma_{ddtb}(G)$ is a minimum cardinality of the semitotal block double dominating set of G and is denoted by $\gamma_{ddtb}(G)$. In this paper, we establish some sharp bounds for $\gamma_{ddtb}(G)$. Also some upper and lower bounds on $\gamma_{ddtb}(G)$ in terms of elements of G and other dominating parameters of G are obtained.

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1. Introduction

In this paper we consider only finite, undirected, connected graphs with no loops and no multiple edges. Terms not defined here are used in the sense of Harary[2]. Let G be a graph with $V = V(G)$ is the vertex set of G and $E = E(G)$ is the edge set of G. The neighborhood of a vertex $v \in V$ is defined by $N(v) = \{u \in V / uv \in E\}$. The close neighborhood of a vertex$v$ is $N[v] = N(v) \cup \{v\}$. The order $|V(G)|$ of G is denoted by p. The degree of v is $d(v) = |N(v)|$. The maximum degree of a graph G is denoted by $\Delta(G)$ and the minimum degree is denoted by $\delta(G)$. A set D of vertices in a graph G is called a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. The domination number of G, denoted by $\gamma(G)$ is the minimum cardinality of a dominating set. The domination in graphs with many variations is now well studied in graph theory. The recent book of Chartrand and Lesniak[1] includes a chapter on domination. A thorough study of domination appears in [3]. For any graph $G = (V, E)$, the semitotal block graph $T_b(G) = H$, whose set of vertices is the union of the set of vertices and blocks of G and in which two vertices are adjacent if and if the corresponding vertices of G are adjacent or the corresponding members are incident in G. This concept was introduced in [4]. The vertex connectivity denoted as $\kappa(G)$ is the minimum number of vertices whose removal gives a disconnected graph. A dominating set D of G is called strong split dominating set of G if $< V(G) - D >$ is totally disconnected with at least two vertices. The strong split domination number $\gamma_{ss}(G)$ is the minimum cardinality of minimal strong split dominating set. Introduction and study of $\gamma_{ss}(G)$ appears in [5]. In this paper we, continue the study of a variation of the domination theme, namely that of semitotal block double domination in graph G. A subset D^d of $V[T_b(G)]$ is double dominating set of $T_b(G)$ if for every vertex $v \in V[T_b(G)], |N[v] \cap D^d| \geq 2$, that is v is in D^d and has at least one neighbour in D^d or v is in $V[T_b(G)] - D^d$ and has at least two neighbours in D^d. The semitotal block dominating number $\gamma_{ddtb}(G)$ is a minimum cardinality of the semitotal block double dominating set of G and is denoted by $\gamma_{ddtb}(G)$. In this paper, we establish some sharp bounds for $\gamma_{ddtb}(G)$. Also some upper and lower bounds on $\gamma_{ddtb}(G)$ in terms of elements of G and other dominating parameters of G are obtained.

2. Specific value of $\gamma_{ddtb}(G)$

In this section, we illustrate the semitotal block double domination number by giving the value of $\gamma_{ddtb}(G)$ for several classes of graphs. Also we found some constraints for which $\gamma_{ddtb}(G)$ follows the equality relations with other domination parameters of G. Some proofs are straightforward and are omitted.

We need the following theorem to prove one of our results.

Theorem A[5]: For any graph G, $\frac{p}{1+\Delta(G)} \leq \gamma(G)$.

Main Results

Proposition 2.1: For any nontrivial tree T, $\gamma_{ddtb}(T) = n + 1$ where n is the number of blocks.

Proposition 2.2: For any nontrivial tree T, $\gamma_{ddtb}(T) = p$

Proposition 2.3: For any star graph $K_{1,n}$ $n \geq 2$, $\gamma_{ddtb}(K_{1,n}) = n + 1$.

Proposition 2.4: For any bipartite graph $K_{m,n}, m \leq n$, $\gamma_{ddtb}(K_{m,n}) = m + 1$.

Proposition 2.5: For any nontrivial tree T, $\gamma_{ddtb}(T) = \beta_0(G) + 2$.

Proposition 2.6: For any (p, q) connected graph G, $\gamma_{ddtb}(G) \leq \gamma_{dd}(G)$.

Theorem 2.7: For any nontrivial tree T with $p \geq 3$ vertices, $\gamma_{ddtb}(T) = C_0 + V_e$ where C_0 is the number of cut vertices and V_e is the number of end vertices.

Proof: Let $B = \{B_1, B_2, B_3, \dots, B_n\}$ be the set of blocks of T and $H = \{b_1, b_2, b_3, \dots, b_n\}$ be the corresponding block vertices of the set B in $T_b(T)$. Let $H_1 = \{v_1, v_2, v_3, \dots, v_k\}$ be the set of end vertices of T and $C = \{c_1, c_2, c_3, \dots, c_l\}$ be the set of cut vertices of T such that $|H_1| = V_e$ and $|C| = C_0$. Then in $T_b(T)$, $V[T_b(T)] = C \cup H \cup H_1$. Let D^d be a double dominating set of $T_b(G)$ such that $D^d = C \cup H_1$ then any vertex $v \in V[T_b(T)] - D^d = H$ has two neighborhood in D^d. Hence D^d is a γ_{ddtb} - set of T, which gives $|D^d| = \gamma_{ddtb}(T)$. Thus $\gamma_{ddtb}(T) = |C \cup H_1|$, which implies that $\gamma_{ddtb}(G) = C_0 + V_e$.

Theorem 2.8: For any nontrivial tree T with $p \geq 3$, $\gamma_{ddtb}(T) = n + \kappa(T)$, where n is the number of blocks and $\kappa(T)$ is the vertex connectivity.

Proof: Let T be a graph, in which each block is complete. Then in $T_b(T)$, each block is also complete. Let $B = \{B_1, B_2, B_3, \dots, B_n\}$ be the set of blocks of T and $H = \{b_1, b_2, b_3, \dots, b_n\}$ be the corresponding block vertices of the set B in $T_b(T)$. Let $H_1 = \{v_1, v_2, v_3, \dots, v_k\}$ be the set of end vertices of T and $C = \{c_1, c_2, c_3, \dots, c_k\}$ be the set of cut vertices of T such that $|H| = n$. Hence $|H_1| = V_e$ and $|C| = C_0$. Then in $T_b(T)$, $V[T_b(T)] = H \cup H_1 \cup C$. We consider $D^d = C \cup H_1$ be the set such that for any vertex $v \in V[T_b(T)] - \{C \cup H_1\}$ is dominated by at least two vertices of $T_b(T)$. Thus $\{C \cup H_1\}$ is a double dominating set of $T_b(T)$. Thus $|C \cup H_1| = |H \cup \kappa|$, since $\kappa = 1$ for every nontrivial tree, then we have $\gamma_{ddtb}(T) = n + \kappa(T)$

3. LOWER BOUNDS FOR $\gamma_{ddtb}(G)$.

Here we establish lower bounds for $\gamma_{ddtb}(G)$ in terms of elements of G.

For two vertices x and y of a graph G, the distance between x and y is denoted by $d(x, y)$.

Theorem 3.1: Let D^d be a double dominating set of $T_b(G)$ and $u, v \in D^d$, then $|D^d| \geq 1 + d(u, v)$.

Proof: Since $< D^d >$ is connected, then there exist a distance between every pair of vertices. Hence one can easily verify that $|D^d| \geq 1 + d(u, v)$.

For a vertex v of a graph G, the eccentricity $e(v)$ is the distance between v and a vertex farthest from v. The maximum eccentricity is its diameter, $diam(G)$. Now we have the following.

Theorem 3.2: For any connected (p, q) graph G, $\gamma_{ddtb}(G) \geq 1 + diam(G)$.

Proof: Let D^d be a γ_{ddtb}-set of G. We first notice that any two vertices $u, v \in D^d$ there is a path in which end vertices are u and v. Let x, y be two vertices of G such that $d(x, y) = diam(G)$. Since $|V(G) - D^d| \geq 0$, then $D^d = \{x, u, v, \dots, y\} \subseteq T_b(G)$. If $\{x, y\} \subseteq D^d$, by the Theorem 9, $\gamma_{ddtb}(G) \geq 1 + diam(G)$.

Theorem 3.3: Let G be a graph with $\delta(G) \geq 2$. If D^d is a minimal double dominating set of $T_b(G)$, then $V[T_b(G)] - D^d$ contains a minimal dominating set.

Proof: Let D^d be a minimal double dominating set of $T_b(G)$. Suppose there exists a vertex $v \in D^d$ which is adjacent to no vertex in $V[T_b(G)] - D^d$. Then v is adjacent to at least two vertices of D^d itself. Therefore $D^d - \{v\}$ is a double dominating set, which is a contradiction. Thus every vertex in D^d must be adjacent to at least one vertex in $V[T_b(G)] - D^d$. Thus $V[T_b(G)] - D^d$, is a dominating set of $T_b(G)$ and hence it contains a minimal dominating set.

Theorem 3.4: For any connected (p, q) graph G, $\frac{1}{2}(2q - p(p-3)) \leq \gamma_{ddtb}(G)$.

Proof: Let D^d be a γ_{ddtb}-set of G. Then there exists a vertex $v \in D^d$ which is not adjacent to any vertex in $V[T_b(G)] - D^d$. Since from proposition 6, $\gamma_{dd}(G) \geq \gamma_{ddtb}(G)$. This implies that $q \leq \frac{p(p-1)}{2} - (p - \gamma_{ddtb}(G))$. It follows that $\frac{1}{2}(2q - p(p-3)) \leq \gamma_{ddtb}(G)$.

Theorem 3.5: For any connected (p, q) graph G, $p - q + \delta(G) \leq \gamma_{ddtb}(G)$.

Proof: Let D^d be a γ_{ddtb}-set of G. Then there exists a vertex $v \in D^d$. But in G $\deg(v) = \Delta(G)$. Since $\gamma_{dd}(G) \geq \gamma_{ddtb}(G)$. Thus $q \geq |V(G) - D^d| + \deg(u) \geq |V(G) - D^d| + \delta(G)$. Which implies that $p - q + \delta(G) \leq \gamma_{ddtb}(G)$.

Theorem 3.6: For any non-trivial tree T with $p \geq 2$, $\delta(T) + 1 \leq \gamma_{ddtb}(T)$.

Proof: Since each edge of T is K_2 and each block in $T_b(T)$ is K_3. Let γ_{ddtb}-set of T. Then there is a vertex $v \in D^d$ such that v is not a adjacent to any vertex of $V[T_b(T)] - D^d$. It follows that $degv \leq \gamma_{ddtb}(G)$. Since in $T_b(T)$, $\delta(G) + 1 \leq degv$. This implies that $\delta(T) + 1 \leq \gamma_{ddtb}(T)$.

Theorem 3.7: For any connected (p, q) graph G, $\gamma_{ss}(G) \leq \gamma_{ddtb}(G)$.

Proof: Let G be a graph with $p \geq 3$ vertices and let $V = \{v_1, v_2, v_3, \ldots, v_p\}$ be the set of vertices of G. Let $B = \{B_1, B_2, B_3, \ldots, B_n\}$ be the set of blocks of G and let $H = \{b_1, b_2, b_3, \ldots, b_n\}$ be the corresponding block vertices of the set B in $T_b(G)$. Further let $D = \{v_1, v_2, v_3, \ldots, v_k\}$ be the set of vertices of G such that $\forall v_i \in V - D$ is an isolate then D is the γ_{ss}-set of G. Since $V[T_b(G)] = V(G) \cup H$. Let $D_1 \subseteq D$, now we consider $v_i, v_j \in V[T_b(G)]$ such that $\deg(v_i) \geq \deg(v_j)$ and again we consider $H_1 \subseteq H, \forall v_k \in H_1$, $\deg(v_k) \geq \deg(v_i)$ or $\deg(v_j)$. Then for any vertex $v \in V[T_b(G)] - \{H_1 \cup D_1\}$ is adjacent to at least two vertices of $T_b(G)$. Thus $D^d = H_1 \cup D_1$ is double dominating set of $T_b(G)$. Thus $|D| \leq |H_1 \cup D_1|$ and it follows that $\gamma_{ss}(G) \leq \gamma_{ddtb}(G)$.

Theorem 3.8: For any connected (p, q) graph G with $G \neq k_3$. Then $\gamma_{ddtb}(G) \geq \frac{5p-2q}{4}$.

Proof: Let D^d be a γ_{ddtb}-set of G. Every vertex in $V[T_b(G)] - D^d$ is adjacent to at least two vertices in D^d and any vertex in D^d must have at least one neighbor in D^d. Hence $q \geq 2|V(G) - D^d| + \frac{|V(G) - D^d|}{2} = \frac{5}{2}|V(G) - D^d| + \frac{\gamma_{ddtb}(G)}{2} = \frac{5}{2}(p - \gamma_{ddtb}(G)) + \frac{\gamma_{ddtb}(G)}{2}$. Thus $2q \geq 5p - 4\gamma_{ddtb}(G)$, so that $\gamma_{ddtb}(G) \geq \frac{5p-2q}{4}$.

Theorem 3.9: For any connected (p, q) graph G with $G \neq P_{p,}\ p = 5,6,7$, $\gamma_{ddtb}(G) \geq \frac{2p}{\Delta(G)+1}$.

Proof: Let D^d be a γ_{ddtb}-set of G and D be a γ_{dd}-set of G. Since $\gamma_{ddtb}(G) \leq \gamma_{dd}(G)$ if $G \neq P_{p,}\ p = 5,6,7$ so on. Let t be the number of edges with one end in D and other end in $V(G) - D$. Since every vertex in D has at least one neighbor in D, since for any connected graph G, there exists at least one vertex $u \in V(G)$ such that $\deg(u) = \Delta(G)$, then $t \leq (\Delta(G) - 1)|D^d| = (\Delta(G) - 1)\gamma_{ddtb}(G)$. Also every vertex in $V(G) - D$ is adjacent to at least two vertices in D and so $t \geq 2|V - D^d| = 2(p - \gamma_{ddtb}(G))$. Thus $2p - 2\gamma_{ddtb}(G) \leq (\Delta(G) - 1)\gamma_{ddtb}(G)$ and it follows that $\gamma_{ddtb}(G) \geq \frac{2p}{\Delta(G)+1}$.

Theorem 3.10: For any connected (p, q) graph G, $\gamma_{\mathrm{ddtb}}(\mathrm{G}) \geq \frac{2\mathrm{p}}{\delta(\mathrm{G})+1}$.

Proof: Let D^d be a γ_{ddtb}-set of G. Since $\gamma_{ddtb}(G) \leq \gamma_{dd}(G)$ (if $G \neq P_{p,}\ p = 5,6,7$ so on) and t denote the number of edges joining the vertices of D^d to the vertices of $V(G) - D^d$. Then by definition of double dominating set any vertex $v \in D^d$ has exactly $\deg(v) - 1$ neighbours in $V(G) - D^d$. Thus $t = \sum(\deg(v) - 1)$. Since for any graph G there exists at least one vertex $v \in V(G)$ such that $\deg(v) = \delta(G)$. Further $|D^d|(\delta - 1) \geq t = 2|V - D^d|$ it implies that $\gamma_{ddtb}(G)(\delta - 1) \geq 2p - 2\gamma_{ddtb}(G)$. Hence $\gamma_{ddtb}(G)\,\delta(G) - \gamma_{ddtb}(G) + 2\gamma_{ddtb}(G) \geq 2p$. Which gives, $\gamma_{ddtb}(G)(\,\delta(G) + 1) \geq 2p$. Hence $\gamma_{ddtb}(G) \geq \frac{2p}{\delta(G)+1}$.

Theorem 3.11: For any connected (p, q) graph G, $\gamma_{ddtb}(G) \geq \gamma(G) + 1$.

Proof: We consider the following two cases.

CaseI: Suppose G is acyclic graph. Let D^d be a γ_{ddtb}-set of G. Since $\gamma(G) \leq p - 1$ and if $D^d = V(G)$, then G is acyclic and $\gamma_{ddtb}(G) \geq \gamma(G) + 1$.

CaseII: Suppose G is cyclic graph. Then there exists a vertex $v \in V(G)$ such that v is adjacent to u and $u, w \in V(G)$ and u, w has a unique path joining the vertices. Hence $uv\{u, w\}$ forms a cycle. Further $v \in D$, where D is a dominating set of G. But either $\{v, u\}$ or $\{v, w\} \in D^d$ set. Clearly $\gamma_{ddtb}(G) \geq \gamma(G) + 1$.

4. UPPER BOUNDS FOR $\gamma_{ddtb}(G)$.

Here we establish upper bounds for γ_{ddtb} (G) in terms of elements of G.

Theorem 4.1: For any connected (p, q) graph G with $n \geq 1$ blocks, $\gamma_{ddtb}(G) \leq n + \frac{p.\Delta(G)}{1+\Delta(G)}$.

Proof: Let $B = \{B_1, B_2, B_3, \ldots, B_n\}$ be the set of blocks of G and $H = \{b_1, b_2, b_3, \ldots, b_n\}$ be the corresponding block vertices in $T_b(G)$. Since $V[T_b(G)] = V(G) \cup B$. Let D be a dominating set of G. Then each $v \in V[T_b(G)]$ is also dominating by some vertices of D. Then $V[T_b(G)] - D$ is also a dominating set of G. Let D^d be a double dominating set of $T_b(G)$. Then $|D^d| \leq |V[T_b(G) - D]|$. From Theorem A[5], $\gamma_{ddtb}(G) \leq \left|p + n - \frac{p}{1+\Delta(G)}\right|$ it follows that, $\gamma_{ddtb}(G) \leq n + \frac{p.\Delta(G)}{1+\Delta(G)}$.

The clique number $\omega(G)$ of G is the minimum order of a clique in the graph. Clearly $\omega(G) = \beta_0(\bar{G})$. We give an upper bound for $\gamma_{ddtb}(G)$.

Theorem 4.2: For any connected (p, q) graph G with n blocks and $p \geq 3$, $\gamma_{ddtb}(G) \leq p + n - 2$.

Proof: If $T_b(G) = H$ it implies that $\gamma_{ddtb}(G) = \alpha_0(H)$ which gives $\alpha_0(H) = p + n - \beta_0(H)$. Further it implies that $p + n - \omega(\bar{H})$. Since $\beta_0(\bar{H}) \geq 2$, then $\gamma_{ddtb}(G) \leq p + n - 2$.

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Theorem 4.3: For any connected (p, q) graph G with $p \geq 3$ vertices,

(I)$\gamma_{ddtb}(G) + \gamma_{ddtb}(\bar{G}) \leq 2p + 1$.
(II)$\gamma_{ddtb}(G).\gamma_{ddtb}(\bar{G}) \leq p(p + 1)$.

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