# Graphs Approach Hypermetric Inequalities via Topological Spaces

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**Abstract:** *Hypermetric Topology inequalities have many applications in the planar graphs and most recently in the approximate solution to the graphs by linear and semi definite programming. However, not much is known about the separation problem for these inequalities. In this paper we show that similar results holds for inequalities of negative type, even though the separation problem for negative type inequalities is well known to be solvable in polynomial time. We also show similar results hold for the more general k-konal and gap inequalities.*

**Keywords**: *Hypermetric, k-konal, gapinequality, negative type inequality, pure, cayley.*

## Introduction

Graph hypermetric topology is an active area of research in the fast growing field of topology. The motivation for this study comes from a variety of practical problems such as k-konal graph inequalities and hypermetric topology.

Let b = ($b_1$ , ..., $b_n$ ) be an integer vector, let k = $\Sigma_{1\le i\le n} |b_i|$ and s = $\Sigma_{1\le i\le n} b_i$ .

Let x = ($x_{ij}$ ), $1 \le i < j \le n$ be a vector in $R^{\binom{n}{2}}$ .

We say that b defines a k-konal

$$Q(b,x) = \sum_{1\le i<j\le n} b_i b_j x_{ij} \le \left\lfloor \frac{s^2}{4} \right\rfloor \quad \text{-----------------------(1)}$$

Since the vector −b generates the same inequality as the vector b, we will assume throughout the paper that $s \ge 0$. The inequality (1) is called ***hypermetric*** if s = 1, in which case k is necessarily odd. It is called ***negative type*** if s = 0, in which case k is even. It is called ***pure*** if bi ∈ {±1, 0}. Inequalities of type (1) have been well studied and rediscovered many times. The hypermetric inequalities appear in Deza [3], and the ***negative type inequalities*** in the work of ***Cayley***. The book of Deza and Laurent [5] collects a wealth of information about them and their applications.

For fixed n, it is easy to show that the cone formed by the negative type inequalities is not polyhedral. However, Deza, Grishukhin and Laurent [4] showed that the hypermetric inequalities do form a polyhedral cone. Furthermore, each 2k + 2-gonal inequality can be obtained by a non-negative combination of 2k + 1-konal inequalities. This was proved by Deza [3] for negative type inequalities, and for the general k-gonal inequalities by Avis and Umemoto [2].

**Results 1.1:** Laurent and Poljak [8] introduced a set of inequalities, called **gapinequalities**, that can be stronger than the k-konal inequalities when $s \ge 2$. The gap g = g(b) of an integer sequence $b_1 , . . . , b_n$ is defined by

$$g = g(b) = \min_{S\subseteq\{1,2,\dots,n\}} \left|\sum_{i\in S} b_i - \sum_{i\notin S} b_i\right| \quad \text{------------(2)}$$

It is easy to see that for a negative type inequality, the b vector has gap zero and for a hypermetric inequality it has gap one. A gap inequality if formed by modifying the right hand side of (1):

$$Q(b, x) = \sum_{1\le i<j\le n} b_i b_j x_{ij} \le \frac{1}{4}(s^2 - g^2) \quad \text{------------(3)}$$

Since s and g have the same parity, the right hand side is always integral.

We call a gap inequality k-konal If k = $\sum_{i=1}^{n} |bi|$

we call (3) a k − konal gap inequality.

**Result 1.2:** Let G be an undirected graph with vertices V = V (G) and edges E = E(G).
Construct the edge weights

$$x_{ij}^{i}(G) = \begin{cases} 1 & if\ (ij) \in E \\ 1+t & if\ (ij) \notin E \end{cases} \quad \text{------------(4)}$$

**Result 1.3:** Let b be an integer vector of length n and define

$$k = k(b) = \sum_{i=1}^{n} |b_i| \quad , \quad s = s(b) = \sum_{i=1}^{n} |b_i| \quad \text{------------(5)}$$

Which implies that

$$\sum_{i \in V_+(b)} b_i = \frac{(s+k)}{2}, \quad \sum_{i \in V_-(b)} b_i = \frac{(s-k)}{2} \quad \text{------------(6)}$$

Let $Q_G$ (b, $x^t$) be the left hand side of the inequality (1) with $x_t = x^t(G)$.

We calculate $Q_G$ (b, $x^t$) .

We set

$V_+(b) = \{i : b_i > 0\}$, $V_-(b) = \{i : b_i < 0\}$, $V(b) = V_+(b) \cup V_-(b)$

$n_+ = |V_+(b)|$ , $n_- = |V_-(b)|$, $n_b = n_+ + n_-$ .

**Result 1.4:** We denote by G(b) the subgraph of G induced on the set V(b). Let $K_{n_+, n_-}$ be the complete bipartite graph on the set V(b) with the partition

(V+ (b), V− (b)). Let $E_b(G) = E(G(b))\ E(Kn_+, n_-)$,

**Theorem 1.5** : G be a graph and let b and $E_b(G)$ be defined as above. Then

$$Q_G(b, x^t) = \frac{s^2}{2} + \frac{t}{4}(s^2 + k^2) - \frac{1+t}{2}\sum_{i=1}^{n} b_i^2 - t \sum_{(i,j) \in E_b(G)} |b_i|\,|b_j| \quad \text{------------(7)}$$

**Proof**: Suppose at first that $E_b(G) = \emptyset$, i.e. the set V(b) induces a complete bipartite graph K(b). From the definitions we have $Q_G(b,x^t) = Q_{K_{n+,n-}}(b, x^t) = \sum_{i \in V_+, j \in V_-} b_i b_j + (1+t)\left(\sum_{i,j \in V_+, i<j} b_i b_j + \sum_{i,j \in V_-, i<j} b_i b_j\right)$

Since for any set X

$$\sum_{i,j \in X, i<j} b_i b_j = \frac{1}{2}\left( \left(\sum_{i \in X} b_i\right)^2 - \sum_{i \in X} b_i^2 \right)$$

Using (6) we obtain

$$Q_{K_{n+,n-}}(b, x^t) = \frac{s^2}{2} + \frac{1+t}{4}\left(\frac{(s+k)^2}{4} + \frac{(s-k)^2}{4} - \sum_{i=1}^{n} b_i^2\right)$$

After simplification, we obtain

$$Q_{K_{n+,n-}}(b, x^t) = \frac{s^2}{2} + \frac{t}{4}(s^2 + k^2) - \frac{(1+t)}{2}\sum_{i=1}^{n} b_i^2$$

which are the first 3 terms of (7).

If $E_b(G) \neq \emptyset$, then it may contain two types of edges: those in G between vertices both in either V+ or V− , or edges in Kn+ ,n− not in G. In both cases, the right hand side of the equality (2) obtains additional negative summand

$$-t \sum_{(i,j) \in E_b(G)} |b_i||b_j|$$

This completes the proof of the theorem

**Remark 1.6:** Let vector b , with k ≤ n , the equality(7) takes the simple form

$$Q_G(b, x^t) = \frac{s^2}{2} + \frac{t}{4}(s^2 + k^2) - \frac{(1+t)}{2}k - t\,|E_b(G)| \quad \text{------------(8)}$$

**Result 1.7:** Le b be an integer vector of length n , and let k and s be defined as in (5).

Then $Q_G(b, x^t) = \frac{s^2}{2} + \frac{t}{4}(s^2 + k^2) - \frac{(1+t)}{2}k$ ------------(9)

**Theorem 1.8:** Let n ≥ 2m ≥ 6. Set $x^t = x^t(G)$, where $t = \frac{m^2+1}{m^2(m-1)}$. Then $Q_G\left(b, \frac{x^t}{n^2}\right) \leq \left\lfloor \frac{s^2}{4} \right\rfloor$

**Proof :** Let $x^t$ satisfies all k-konal negative type inequalities with $k < 2m$. Let $x^t$ satisfies all 2m – konal negative type inequalities, except when G contains $K_{m,m}$ as an induced subgraph. In this case only the pure 2m-konal negative type inequality is violated.

Let $\frac{x^t}{n^2}$ satisfies all k-konal inequalities with $k \leq n$ and $s \geq 2$.

If $k < 2m$, then it is easy to verify that $k \leq 2$ , hence $\frac{k^2 t}{4} \leq \frac{k}{2}$. Using Result 1.7 with s = 0, we see that $Q_G(b, x^t) \leq 0$. First consider the case where $G = G(b) = K_{m,m}$ , k = 2m, s = 0,and b is a pure 2m-konal inequality. From Result 1.7

we have that

$$Q_G(b,x^t) = m^2 t - mt = (m^2 - m)\frac{m^2+1}{m^2(m-1)} - m = \frac{1}{m} > 0 \qquad \text{------------(10)}$$

Therefore this pure negative type inequality is violated by $x^t$ . It is easy to check

that $t > \frac{1}{m}$ . If G is not $K_{m,m}$ , then $|E(b)| \geq 1$, and as we saw in the proof of theorem 1.1, $Q_G(b, x^t)$ will be reduced by at least t so it becomes negative. Therefore the inequality holds in this case.

Now let $G = K_{m,m}$ , and assume b is 2m-konal but not pure. In this case $|b_i| \geq 2$ for some i, and so $\sum_{i=1}^{n} b_i^2 \geq 2m + 2$.

Now from (7) with s = 0 and k = 2m

we obtain

$$Q_G(b,x^t) \leq m^2 t - \frac{1+t}{2}(2m+2) = m^2 t - m - t - 1 < 0$$

Therefore $x^t$ satisfies all 2m-konal negative type inequalities that are not pure.

we observe that when $k \leq n$

$$\sum_{1 \leq i < j \leq n} |b_i||b_j| \leq \frac{1}{2}\left(\sum_{i=1}^{n} |b_i|\right)^2 = \frac{k^2}{2} \leq \frac{n^2}{2},$$

Now $s \geq 2$, and since $m \geq 3$, $x^t \leq 2$, So,

$$Q_G\left(b, \frac{x^t}{n^2}\right) = \sum_{1 \leq i < j \leq n} b_i b_j \frac{x_{ij}^t}{n^2} \leq \sum_{1 \leq i < j \leq n} |b_i||b_j| \frac{2}{n^2} \leq 1 \leq \left\lfloor \frac{s^2}{4} \right\rfloor$$

This completes the proof of the theorem.

**Theorem 1.9:**. Let b, k and s be defined as in (5), such that $g = s \geq 2$. Then $\sum_{i=1}^{n} b_i^2 \geq k + s^2$

**Proof :** Assume $b_1 \geq b_2$ are the two largest integers in b. We may also assume without loss in generality that no integer $b_i$ is zero.

Since g = s we observe that the set S = {1, 2, ..., n} realizes the minimum gap. Assume first that $b_2 > 0$. It is easily seen that

$b_1 \geq b_2 \geq g = s \geq 2$, since otherwise a smaller gap could be

formed by removing $b_2$ from S. Therefore

$$\sum_{i=1}^{n} b_i^2 \geq b_1^2 + b_2^2 + \sum_{i=1}^{n} |b_i| = b_1^2 + b_2^2 + k - b_1 - b_2 \geq 2s^2 - 2s + k \geq k + s^2$$

Otherwise, suppose $b_2 < 0$. From (1), $b_1 = (k + s)/2$. Furthermore, since b has positive and negative components, $k = s + u$, for some $u \geq 2$. Therefore

$$\sum_{i=1}^{n} b_i^2 \geq \frac{(k+s)^2}{4} + \frac{k-s}{2} \geq \frac{(2s+u)^2}{4} + \frac{u}{2} = s^2 + su + \frac{u^2}{4} + \frac{u}{2} \geq s^2 + s + u = k + s^2$$

## Conclusion

We proved several results for finding hypermetric topology in graph theory. Here we have of course assumed that solutions to the subsidiary problems discussed at the beginning of the paper require constant time. The challenge is to find a nice solution and improve to the result of k-konal inequalities.

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