Diameter and Travers ability of PAN Critical Graphs

J. Suresh Kumar
Post-Graduate Department of Mathematics,
N.S.S.Hindu College, Changanacherry, Kottayam Dist., Kerala, India-686102

ABSTRACT: A pseudo-complete coloring of a graph G is an assignment of colors to the vertices of G such that for any two distinct colors, there existadjacent vertices having those colors. The maximum number of colors used in a pseudo-complete coloring of G is called the pseudo-achromatic number of G and is denoted by$\psi_s(G)$. A graph G is called edge critical if $\psi_s(G-e)$<$\psi_s(G)$for any edge e of G. A graph G is called vertex critical if $\psi_s(G-v)<.\ \psi_s(G)$for every vertex v of G. These graphs are generally called as pseudo-achromatic number critical graphs (shortly as PAN Critical graphs). In this paper, we investigate the properties of these critical graphs.

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1. Introduction

By a graph we mean a finite undirected graph without loops, multiple edges and isolated vertices.

An assignment of colors to the vertices of a graph *G = (V, E)* is called a *proper coloring*, if any two adjacent vertices receive distinct colors and is called a *pseudo--complete coloring* if for any two distinct colors, there exist adjacent vertices having those colors. A pseudo-complete proper coloring of *G* is called a *complete coloring* of *G*.

The minimum number of colors used in a proper coloring of *G is* called the *chromatic number* of *G* and is denoted by x(G). The maximum number of colors used in a completecoloring of *G* is called the *achromatic number* of G and is denoted by*(ψG))* [6]. The maximum number of colors used in a pseudo-complete coloring of *G* is called the *achromatic number* of G and is denoted by $\psi_s(G)$.[4]. Several bounds for these coloring parameters were obtained in [4, 5, 6, 7]. A graph which admits a pseudo-complete coloring by *k* colors is called a *k-pseudo complete colorable graph*.

The concept of critical graphs with respect to chromatic number, was introduced by Dirac [2, 3] in a bid to settle the four color conjecture. In [1], Sureshkumar introduced the concepts of criticality in graphs with respect to pseudo-achromatic number and obtained characterizations of edge critical graphs, critical cycles and critical paths. In this paper, we further investigate theproperties of these critical graphs such as degrees, degree sequences, diameter and traversibility.

2. PAN Critical Graphs

The graphs which are critical with respect to Pseudo-achromatic number are generally called as PAN-critical graphs. Formal definitions are as follows:

Definition 2.1. A graph G is called k-edge critical if $\psi_s(G)=k$ and $\psi_s(G-e)<k$ for any edge e of G. A graph G is called k-vertex critical if $\psi_s(G)=k$ and $\psi_s(G-v)<k$ for any vertex v of G.

Definition 2.2. Let G be a graph and $v \in V(G)$be a vertex of degree d. Let n be a positive integer less than d. Then an n-splitting of v is the replacement of v by a set of n new pairwise independent vertices $\{u_i\}_{i=1}^n$ with$\deg u_i > 1$, for all i, 1<i<n, $\sum_{i=1}^n \deg(u_i) = d$ and $N(\{u_i\}_{i=1}^n) = N(v)$, where for any subset S of V(G), N(S) means the set of all neighbors of vertices in S.

The following simple observations, which are quite useful later, follow directly from the definitions of critical graphs.

Proposition 2.3. A graph G is k-edge critical if and only if G is k-pseudo-complete colorable and $|E(G)| = \binom{n}{2}$

Proposition 2.4. Any k-edge critical graph is k-vertex critical.

Proposition 2.5. If G is a k-edge critical graph and II is the graph obtained from G by n-splitting a vertex of G. Then H is k-edge critical.

Proposition 2.6. Let G be a k-edge critical graph and H be the graph obtained from G by identifying a pair of vertices, having same color with respect to a k-pseudo-complete coloring of G. Then H is k-edge critical.

Proposition 2.7. IfG is k-edge critical, then $G + K_n$, is (n+ k)-vertex critical

Proposition 2.8. Let k be an odd integer.
Then, the cycle of order$\binom{k}{2}$) is k--edge critical.

3. Diameter and Traversibility

Theorem 3.1. Let G be a k-edge critical graph with $G \neq K_k, k \geq 3$. Then, $3 \leq d(G) \leq \binom{k}{2}$and when k is even, $3 \leq d(G) \leq \binom{k}{2} -$

$(k/2)+1$, where d(G) denotes the diameter of G.

Proof. Suppose d(G) = 2. Then any two vertices of G are either adjacent or having a common neighbor. Since G is k-edge critical, it follows that any k-pseudo-complete coloring of G is a proper coloring of G. Thus, G = K_k which is a contradiction. Hence d(G) > 3. Also, $d(G) \leq |E(G)| \leq \binom{k}{2}$.

Suppose k is even. Since G is k-edge critical, it follows from Proposition 2.6 that the graph obtained from G by identifying all pairs of vertices having same color, with respect to any k-pseudo-complete coloring of G, is isomorphic to K_k and a path in G corresponds to a trail of same length in K_k. Since the maximum length of a trail in K_k is k—(k/2)+1, $d(G) \leq \binom{k}{2} - (k/2) + 1$

Theorem 3.2. Let m and n be two positive integers such that $3 \leq m \leq \binom{n}{2}$ when n is odd and $3 \leq m \leq \binom{n}{2} - (n/2) + 1$, when n is even. Then there exists n-edge critical graph G with d(G) = m.

Proof. Case 1. $3 \leq m \leq n$

Consider a path P_n= (u1, u2,…,u„) on n vertices. For $1 \leq i \leq n-2$, take a set of n—i—1 pairwise independent vertices, $(w_{i,j})_{j=1}^{n-i-1}$ and join each $w_{i,j}$ with u_j. Call the resulting graph as G_n. Then, $f: V(G_n) \to \{1,2,\dots n\}$ defined by $f(u_i) = i, f(w_{i,j}) = i+j+1$ assigns an n-pseudo-complete coloring for G_n. Since $|E(G_n)| = \binom{n}{2}$, G_n is n-edge critical and d(G_n)= n

Now, for $2 \leq i \leq n-2$ and $1 \leq j \leq n-i+1$, remove each pendant vertex of the form $w_{i,j}$ from G_n and join $w_{1,i-2}$ *with* $w_{1,i+j+1}$ by an edge, remove each pendant vertex of the form $w_{2,j}$ from G_n and join u_2 with $w_{1,j+1}$ by an edge and remove u_n and join $w_{1,n-3}$ with $w_{1,n-2}$ by an edge. Call the resulting graph as G_{n-1} Also, for $1 \leq k \leq n-4$, let G_{n-1-k} be the graph obtained from G_{n-k} by removing the vertex u_{n-k} and joining $w_{1,n-k-2}$ with $w_{1,n-k-3}$ by an edge. Clearly, G_{n-1-k} is n-edge critical and d(G_{n-1-k}) = n - **1** - k for each k, $0 \leq k \leq n-4$

Case 2. $n < m \leq \binom{n}{2}$ and n is odd.

Consider a path $P_{N+1} = (u_1, u_2, \dots u_{N+1})$, where $N = \binom{n}{2}$. Define a function $f: V(P_N) \to \{1,2,\dots n\}$ by

$$f(u_i) = \begin{cases} n & if\ i \equiv 1 (mod\ n) \\ \left(\left\{\frac{i}{n}\right\} + (-1)^{1+g(i)}\left[\frac{g(i)}{2}\right]\right)(mod\ n-1) & otherwise \end{cases}$$

Where g is a function $g: \{1,2,\dots N+1\} \to \{1,2,\dots n\}$ defined by $g(i) = (i-1)(mod\ n)$

It can be easily verified that f assigns an n-pseudo-complete coloring for P_{N+1} Hence, by Proposition 2.3 P_{N+1} is n-edge critical and $d(P_{N+1})$ = N.

Now, for each i with $N-1 \geq i \geq n+1$, consider the edges $\{u_j u_{j+1}\}_{j=i+1}^{N}$ and let $c_j = min\{f(u_j), f(u_{j+1})\}$ and $c_j' = max\{f(u_j), f(u_{j+1})\}$. Then, for each j, $i+1 \leq j \leq N$, remove the vertex u_{j+1} from P_{N+1} and add a vertex, with c_j' as its f-value, and join it to u_{c_j}. The resulting graph G_i is n-edge critical and d(G_i)=**i**.

Case 3. $n < m \leq \binom{n}{2} - (n/2) + 1$ and n is even.

Consider the complete graph K_m with vertex set $\{v_1, v_2, \dots v_n\}$. Let $F = \{v_i v_{(n/2)+i} \in E(K_n): 1 \leq i \leq n/2\}$ and $C = \{v_i v_{(n/2)+i} \in E(K_n): 1 \leq i \leq n/2\} \cup \{v_n v_1\}$

Then $K_n = F \sqcup C$ is Eulerian and has an Euler tour say $T = (v_{k_1}, v_{k_2}, \dots v_{k_{N-n+1}})$, where $v_{k_1} = v_{k_{N-n+1}} = v_1$ and $N = \binom{n}{2} - n/2$. Now let G_N the graph obtained from the path $(v_1, v_2, \dots v_{N+1})$, by adding n/2 vertices

w_1, w_2,….$w_{n/2}$ and joining w_i with u_i. Now define $f: V(G_N) \to \{1,2,\dots n\}$ by

$$f(u_i) = i; \quad 1 \leq i \leq n$$
$$f(w_i) = (/2) + i; \quad 1 \leq i \leq n/2$$
$$f(u_{n+j}) = k_j; \quad 1 \leq j \leq N-n+1$$

Then f Is an n-pseudo-complete coloring of G_N. Since $|E(G_N)| = \binom{n}{2}$, G_N is n-edge critical and d(G_N) = N.

Now for each i with $n = 1 \leq I \leq N-1$, consider the edges $\{u_j u_{j+1}\}_{j=i+1}^{N}$ and let $c_j = min\{f(u_j), f(u_{j+1})\}$ and $c_j' = max\{f(u_j), f(u_{j+1})\}$. Then, for each j, $i+1 \leq j \leq N$, remove the vertex u_{j+1} from P_{N+1} and add a vertex, with c_j' as its f-value, and join it to u_{c_j}. The resulting graph G_i is n-edge critical and d(G_i)=**i**.

Theorem 3.3. Let k > 3 be any odd integer and let G be a k-edge critical Eulerian graph. Then, $k \leq |V(G)| \leq \binom{k}{2}$. Also, given any integer n such that $k \leq n \leq \binom{k}{2}$, there exists a k-edge critical Eulerian graph with exactly n vertices.

Proof. Since $|E(G)| = \binom{k}{2}$ and G is not a tree, it follows that $k \leq |V(G)| \leq \binom{k}{2}$. Now, the cycle $C_{\binom{k}{2}}$ is k-edge critical, by Proposition 2.8. If n is any integer such that $k \leq n \leq \binom{k}{2}$, then by Proposition 2.6, a k-edge critical Eulerian graph on n vertices can be obtained from $C_{\binom{k}{2}}$, by a sequence of identifications of vertices of same color, with respect to any k-pseudo-complete coloring of $C_{\binom{k}{2}}$

Remark 3.4. Since in any n-pseudo-complete coloring of an n-edge critical graph, the sum of the degrees of all the vertices having same color is n—1, it follows thatthere is no n-edge critical, Eulerian graph when n is an even integer.

Theorem 3.5. Let G be a k-edge critical Hamiltonian graph where k ≥ 3.Then (i) $k \le |V(G)| \le \binom{k}{2}$ and (ii) when k is even, $k \le |V(G)| \le \binom{k}{2} - (k/2)$.Moreover, |V(G)|=k if and only if G=K_k and |V (G)| = $\binom{k}{2}$ with odd k if and only if $G = C_{\binom{k}{2}}$. Also, when k is even, $|V(G)| = \binom{k}{2} - (k/2)$ iff G is a cycle,$C_{\binom{k}{2}-(k/2)} = \left(v_1, v_2, \ldots v_{\binom{k}{2}-(k/2)}, v_1\right)$withk/2 chords $\{v_{n+i}v_{n+i+(k/2)}\}_i$ where n is some integer such that $0 < n < \binom{k}{2} - (k/2)$.

Proof.Since $|E(G)| = \binom{k}{2}$and G is not a tree, it follows that $k \le |V(G)| \le \binom{k}{2}$ and |V(G)|=k if and only if G=K_k. Also, when k is odd, |V (G)| = $\binom{k}{2}$implies if $G = C_{\binom{k}{2}}$ and the converse follows from Proposition 2.8.

Now, suppose k is even. Let G be a k-edge critical Hamiltonian graph. Then it follows from Proposition 2.6 that the graph obtained from G by identifying all pairs of vertices having same color, with respect to any k-pseudo-complete coloring of G, is isomorphic to K_k and a cycle in G corresponds to a closed trail of the same length in K_k. Since the maximum length ofa closed trail in K_k is $\binom{k}{2} - (k/2)$, we have $|V(G)| \le \binom{k}{2} - (k/2)$ Also, if$|V(G)| = \binom{k}{2} - (k/2)$ then G is a cycle on $\binom{k}{2} - (k/2)$ verticeswith exactly k/2 chords and this cycle must correspond to a maximal Eulerian subgraph of K_k so that the chords are as required. Converse is obvious.

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