Acceleration motion of a single vertically falling non-spherical particle in incompressible Newtonian Fluid by Different Methods

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Abstract- *In this paper, acceleration motion of a single vertically falling non-spherically particle in incompressible Newtonian fluid is investigated. The acceleration motion of vertically falling non-spherical particles in the fluid such as water can be described by the force balance equation (Basset-Boussinesq-Ossen equation).The main difficulty in the solution of this equation lies in the nonlinear term due to the nonlinearity nature of the drag coefficient. The settling velocity was calculated by using the Diagonal Pade' Approximant method and Varinational Iteration method (VIM). The results were also compared with fourth order Runge-Kutta method(R-K 4th order) to verify the accuracy of the above methods. It was shown that the velocity results are same when t∈[0,1] and from t>1, Diagonal Pade' approximation method can lead more accurate results as compared to VIM. Acceleration motion of single particle is shown in fig.4. In short time particle changing its velocity by a varying amount but t>3 sec. particle does not change its velocity (acceleration is zero)i.e. after 3 sec particle is not accelerating . it attains its highest velocity. To obtain the results for all different methods, the symbolic calculus software MATLAB is used.*

Keywords- *Acceleration motion, Diagonal Pade' approximation method, non-spherical particle and Varinational Iteration method (VIM).*

I. INTRODUCTION

The problem of acceleration motion of vertically falling spherical and non-spherical particle in Newtonian and Non-Newtonian fluids is relevant to many situations of practical interest. It is necessary to know the detailed trajectories of the accelerating particles for the purposes of design or improved operation. For example, the measurements of terminal velocity of raindrop in Newtonian fluids using the falling ball method. It is also necessary to know the time and distance required to reach the particle at terminal point to determine the reliable results for design models. In present, the non-spherical particle are considered. It is clear from previous literature the motion of particle is affected by particle shapes. Considerable attentions have been devoted to the study of the acceleration motion of spherical and non-spherical particles in fluids and an excellent account of theoretical development in this area has been given by Clift et al. [1] for spherical bodies. Less information is available in the previous literature for the case of motion of non-spherical particles. Many correlation for the drag coefficient in terms of the Reynolds number for motion of non-spherical particles were given in the literature [2],[3]. From all these, one of the well-known analytical correlation between Reynolds numbers and drag coefficient for non-spherical particles is presented by Chien [4]

$C_D = \frac{30}{Re} + 67.289e^{(-5.03\varphi)}$ Where Re= $\frac{\rho D u}{?}$ (1)

Where C_D drag coefficient and Re is Reynolds numbers. These are based on the equal volume sphere diameter [5]. Eq. (1) was stated to be valid in the ranges of 0.2<φ<1 and 0.001<Re<10000 for the different shapes of partical [4]. The analysis derived by Diagonal Pade' approximant and Varinational iteration method (VIM). The results of current methods are compared with the well-known R-K 4th order method in order to verify the accuracy of the proposed methods.

Nomenclature		Greek symbols	
Acc	acceleration, m/s^2	$\alpha_1, \alpha_2, \alpha_3, \alpha_4$	constants
C_D	Drag coefficient	φ	Sphericity
D	Particle diameter, m	μ	Dynamic viscosity,kg/ms
g	acc. due to gravity, m/s^2	ρ	Fluid density,kg/m^3
m	particle mass,kg	ρ_s	Spherical partical density, kg/m^3
Re	Reynolds number	∈	Element of
t	time,s		
u	Velocity,m/s		

II. PROBLEM STSTEMENT

Consider a rigid body, non-spherical particle with Sphericity φ, equivalent volume diameter D, mass m and particle density $\rho_s$is falling in an infinite extent of incompressible Newtonian fluid of density ρand viscosity μ , u represents the velocity of the non-spherical particle at any instant time t, and g is the acceleration due to gravity a [6],[19]. Thus, the equation of the partical motion is given by

$$m\frac{du}{dt}=mg\left(1-\frac{\rho}{\rho_s}\right)-\frac{1}{8}\pi D^2\rho C_D u^2-\frac{1}{12}\pi D^3\rho\frac{du}{dt} \quad (2)$$

Where $C_D$the drag coefficient. In right hand side of the eq.(2), the 1st term represent the buoyancy effect, the 2nd term corresponds to drag resistance, 3rd term is associated with the added mass effect which is due to acc. of fluid around the particle.

The non-linear terms due to non-linearity nature of the drag coefficient C_D is the main difficulty in solving eq. (2) could be re-written as follows:

$$\left(m+\frac{1}{12}\pi D^3\rho\right)\frac{du}{dt}=mg\left(1-\frac{\rho}{\rho_s}\right)-\frac{1}{8}\pi D^2\rho C_D u^2$$

$$\alpha_1\frac{du}{dt}+\alpha_2 u+\alpha_3 u^2-\alpha_4=0 \quad (3)$$

Where

$$\alpha_1=\left(m+\frac{1}{12}\pi D^3\rho\right) \quad (3a)$$

$$\alpha_2=3.75\pi D\mu \quad (3b)$$

$$\alpha_3=\frac{67.289e^{(-5.03\varphi)}}{8}\pi D^2\rho \quad (3c)$$

$$\alpha_4=mg\left(1-\frac{\rho}{\rho_s}\right) \quad (3d)$$

III. DIAGONAL PADE' APPROXIMANTS

A Pade' approximant is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function u (t). The technique was developed around 1890 by Henri Pade', but goes back to Georg Frobenius who introduced the idea and investigated the features of rational approximations of power series. The Padé approximant often gives better approximation of the function and it may still work where the Taylor series does not converge. For these reasons Padé approximants are used extensively in computer calculation.. They have also been used as in Diophantine approximation and transcendental number theory. The [L/M] Pade' approximants to a function u (t) are given by [6],[7],[20].

$$\left[\frac{L}{M}\right]=\frac{P_L(t)}{q_M(t)},\text{ where L=M (for diagonal pade' approximants)} \quad (4)$$

Where$P_L(t)$ is a polynomial of the degree of at most L and q_M(t) is a polynomial of the degree of at most M. The formal power series is given

$$u(t)=\sum_{i=1}^{\infty}a_i t^i$$

$$\text{i.e. } u(t)=a_0+a_1t^1+a_2t^2+a_3t^3+\ldots \quad (5)$$

Find the coefficients $a_0, a_1, a_2, a_3, \ldots$with help of Taylor's expansion for $\alpha_1=\alpha_2=\alpha_3=\alpha_4=1$

The given equation (3) becomes

$u'(t)=1-u^2-u$, with initial condition u(0)=0

Solve the above equation by Taylor's series about zero is given by

$$u(t)=u_0+tu'_0+\frac{t^2}{2!}u''_0+\frac{t^3}{3!}u'''_0+\frac{t^4}{4!}u^{iv}_0+\frac{t^5}{5!}u^{v}_0+\frac{t^6}{6!}u^{vi}_0+\frac{t^7}{7!}u^{vii}_0+\frac{t^8}{8!}u^{viii}_0+\ldots \quad (6)$$

$$\text{Solution is } u(t)=t-\frac{t^2}{2!}-\frac{t^3}{3!}+7\frac{t^4}{4!}-5\frac{t^5}{5!}-85\frac{t^6}{6!}+335\frac{t^7}{7!}+1135\frac{t^8}{8!}+\ldots \quad (6a)$$

Now compare equ. (6a) with equ. (5), Soa_0 =0, a_1 =1, $a_2 = -\frac{1}{2!}$, $a_3 = -\frac{1}{3!}$, $a_4 = \frac{7}{4!}$,

$a_5 = -\frac{5}{5!}$, $a_6 = -\frac{85}{6!}$, $a_7 = \frac{335}{7!}$, $a_8 = \frac{1135}{8!}$and so on (6b)

From eq.(4), u (t)-$\frac{P_L(t)}{q_M(t)}$=0(t^{L+M+1}) (7)

Determine the coefficient of p_L(t) and q_M(t) by the help of eq.(6a) and take normalization condition q_M (0)=1 (8)

From eq.(7), u(t)-$\frac{P_L(t)}{Q_M(t)}$=0 (9)

$p_L(t)=p_0+p_1t^1 + p_2t^2 + p_3t^3+\ldots\ldots\ldots\ldots + p_Lt^L$ (9a)

$q_M(t)=q_0+q_1t^1 + q_2t^2 + q_3t^3+\ldots\ldots\ldots\ldots + q_Mt^M$ (9b)

To obtain a diagonal pade' approximants of a different order such as [1/1], [2/2], [3/3], the symbolic calculus software MATLAB is used.

III.A. For pade' [1/1]

u (t)-$\frac{P_1(t)}{Q_1(t)}$=0, where u (t)=$a_0+a_1t^1 + a_2t^2 + a_3t^3+\ldots\ldots$ (10)

from eq.(10) $a_0+a_1t^1 + a_2t^2 + a_3t^3+\ldots\ldots=\frac{p_0+p_1t^1}{q_0+q_1t^1}$ (10a)

So diagonal pade' [1/1]=$\frac{t}{1+\frac{1}{2}t}$ (10c)

III.B. For pade' [2/2]

u (t)-$\frac{P_2(t)}{Q_2(t)}$=0

(11$a_0+a_1t^1 + a_2t^2 + a_3t^3+\ldots\ldots=\frac{p_0+p_1t^1+p_2t^2}{q_0+q_1t^1+q_2t^2}$ (11a)

So diagonal Pade' [2/2] =$\frac{t}{1+\frac{1}{2}t+\frac{5}{12}t^2}$ (11b)

III.C. For pade' [3/3]

u (t)-$\frac{P_3(t)}{Q_3(t)}$=0 where u (t)=$a_0+a_1t^1 + a_2t^2 + a_3t^3+\ldots\ldots$ (12)

$a_0+a_1t^1 + a_2t^2 + a_3t^3+\ldots\ldots=\frac{p_0+p_1t^1+p_2t^2+p_3t^3}{q_0+q_1t^1+q_2t^2+q_3t^3}$ (12a)

Generally, $p_L = a_{L+} \sum_{i=1}^{min.(L,M)} q_i a_{L-i}$ (12b)

So, diagonal Pade' [3/3]=$\frac{t+\frac{1}{12}t^3}{1+\frac{1}{2}t+\frac{1}{2}t^2+\frac{1}{24}t^3}$ (12c)

III.**Table 1.The results of diagonal Pade' [1/1], diagonal Pade' [2/2] and diagonal Pade' [3/3] for $\boldsymbol{\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1}$**

t	pade' [1/1]	pade' [2/2]	pade'[3/3]
0	0	0	0
0.2	0.181818181818182	0.179104477611940	0.179113359119310
0.4	0.333333333333333	0.315789473684211	0.316008316008316
0.6	0.461538461538462	0.413793103448276	0.415043653458697
0.8	0.571428571428571	0.480000000000000	0.483920367534456
1.0	0.666666666666667	0.521739130434783	0.530612244897959
1.2	0.750000000000000	0.545454545454545	0.561872909698997
1.4	0.823529411764706	0.556291390728477	0.582846236430872

In fig.1, u (vertically) denotes the velocity of particle w.r.t. time t (horizontally).Solution for velocity of the non-spherical particle during the acceleration motion is obtain by diagonal pade' approximants of a different order such as [1/1],[2/2], [3/3], the symbolic calculus software MATLAB is used. From Fig.1.the diagonal pade' [3/3] gives the more accurate results as compared to other. i.e. the graph of pade' [3/3] and R-K 4th order (numerical method) are approximately coincide. So from pade' diagonal ([1/1],[2/2],[3/3]), we will chose diagonal pade'[3/3] for further investigation which is more accurate as compared to other order.

IV. VARINATIONAL ITERATION METHOD (VIM)

Jihuan He in 1997, was introduced Varinational iteration Method (VIM)[8] to solve the several nonlinear ordinary and partial differential equations. He's Varinational iteration method (VIM) has been extensively applied as a power tool for solving various kinds of problems [9],[10],[11]. Using VIM, Liu and Gurram have solved the problems of free vibration involving an Euler-Bernoulli beam and obtained accurate results and compare the result with ADM [12]. Slota obtained results for the Heat equation by VIM which were same as the exact solution [13],[14].To clarify the VIM, we consider the following differential

Lu (t)+Nu(t)=g(t) (13)

Where L is a linear operator, N is a nonlinear operator and g(t) is a non-homogeneous term. By using the Varinational iteration method, a correction functional can be constructed as

u_{n+1}(t)=$u_n(t) + \int_0^t ʎ\{Lu_n(\zeta) + N\tilde{u}(\zeta) - g(\zeta)\}d\zeta$ (13a)

Where ʎ is a general Lagrange multiplier, which can be determined by the help of Varinational theory., the subscript $_n$ means the nth approximation; u_n is restricted variation and $\delta\tilde{u}_n$=0.[16],[17]

According to VIM , firstly we will find Lagrange multiplier and then trial function u_0 to get the successive iterations u_{n+1}, n≥0 which converge to the exact solution . The solution is u=$\lim_{n\to\infty} u_n$

To solve eq. (3) using VIM [15], the correction functional can be constructed as follows:

u_{n+1}(t)=$u_n(t) + \int_0^t ʎ\{\alpha_1 \frac{du_n(s)}{ds} + \alpha_2\, u_n(s) + \alpha_3 u^2{}_n(s) - \alpha_4\}ds$ (14)

For $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$, eq.(14) becomes

u_{n+1}(t)=$u_n(t) + \int_0^t ʎ\{\frac{du_n(s)}{ds} + u_n(s) + u^2{}_n(s) - 1\}ds$ (14a)

The stationary condition can be obtained as follows:

$ʎ_{s=t} - ʎ'_{s=t}$=0

1+$ʎ(t)_{s=t}$=0 (14b)

Subsequently, the Lagrangian multiplier is obtained as:

ʎ= -e^{s-t} [9] (14c)

Substituting eq. (14c) in eq. (14a) and assuming$u_0(t) = 0$, solution will be gained for velocity variation w.r.t. time and for $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$

u_{n+1}(t)=$u_n(t) - \int_0^t e^{s-t}\{\frac{du_n(s)}{ds} + u_n(s) + u^2{}_n(s) - 1\}ds$, with condition$u_0(t) = 0$, (14d)

Solve (14d) for different iterations with the help of MATLAB softerware

u_0=0

u_1=1 - 1/exp (t)

u_2= (2*t*exp(t) - exp(t) + 1)/exp(2*t)

u_3=5/exp(2*t) - 19/(3*exp(t)) + 1/(3*exp(4*t)) + (4*t)/exp(2*t) + (2*t)/exp(3*t) + (4*t^2)/exp(2*t) + 1

u_4=667/(9*exp(2*t)) - 4354267/(113400*exp(t)) - 56/exp(3*t) + 1481/(81*exp(4*t)) + 77/(72*exp(5*t)) + 22/(25*exp(6*t)) + 1/(27*exp(7*t)) + 1/(63*exp(8*t)) + (38*t)/(3*exp(t)) + (24*t)/exp(2*t) - (146*t)/(3*exp(3*t)) + (788*t)/(27*exp(4*t)) + (17*t)/(2*exp(5*t)) + (16*t)/(15*exp(6*t)) + (2*t)/(9*exp(7*t))

+ (8*t^2)/exp(2*t) - (76*t^2)/(3*exp(3*t)) + (328*t^2)/(9*exp(4*t)) + (7*t^2)/exp(5*t) + (160*t^3)/(9*exp(4*t)) + (4*t^2)/(3*exp(6*t)) + (4*t^3)/exp(5*t) + (16*t^4)/(3*exp(4*t)) (15)

VI. Table 2. The results of VIM 1st, 2nd, 3rd and 4th iteration for $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$

t	VIM 1st iteration	VIM 2nd iteration	VIM 3rd iteration	VIM 4th iteration
0	0	0	0	0
0.2	0.181269246922018	0.179081594188850	0.179113680688544	0.179113343756190
0.4	0.329679953964361	0.315264954910094	0.316035617510949	0.316006232438869
0.6	0.451188363905974	0.410956539131007	0.415347788337407	0.415008507960582
0.8	0.550671035882778	0.471493896464988	0.485392554135938	0.483681024791490
1.0	0.632120558828558	0.503214724408055	0.535134904355470	0.529638091578952
1.2	0.698805788087798	0.512389849966495	0.572328193962487	0.559019698064430
1.4	0.753403036058394	0.504684597720110	0.602777019515010	0.576176937572276

In Table 2, Solution for velocity of the non-spherical particle during the acceleration motion is obtain by Varinational Iteration Method (VIM) of different iterations and compare with R-K 4th order(numerical solution of the same problem) to choose accurate iteration which gives the more accurate results. From fig.2. , we obvers that 4th iteration of Varinational Iteration Method (VIM) is more close to numerical method as comparative to other iterations. So we will choose 4th iteration of VIM for comparison with other proposed method (Diagonal pade 'method).

V. RUNGE-KUTTA 4thORDER METHOD (NUMERICALMETHOD)

It is clear that the type of current problem is initial value problem (IVP) of 1st order. So far a solution, we can apply numerical methods like trapezoidal method, Euler's method (1st order R-K method), and mid-point method. Trapezoidal method is generally used for typical problems. Mid-point method is the modification of Euler's method. Thus the mid-point method is as a suitable numerical technique in present problem which is also called R-K 4th order method (numerical method)[18]

u' (t)=1-u^2 –u ,is a 1st order differential equation with initial condition u(0)=0

so f is a function of time and velocity

i.e. f (t,u)= 1-u^2 –u ,u(0)=0 (16)

Matlab code for R-K 4th order

```
f=@(t,u)(1-u^2-u);
t=0;
u=0;
h=0.2;
t= 0:h:1.4 ;
for i=1:(length(t)-1)
   k1=f(t(i),u(i));
   k2=f(t(i)+0.5*h,u(i)+0.5*h*k1);
   k3=f(t(i)+0.5*h,u(i)+0.5*h*k2);
   k4=f(t(i)+h,u(i)+h*k3);
u(i+1)=u(i)+1/6*(k1+2*k2+2*k3+k4)*h;
end
u(:)
```

VI. RESULTS AND DISCUSSION

Table 3. Compare results of Pade' [3/3], VIM (4^th^ iteration) with Runge-kutta 4^th^ order method for $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$

t	$u(t)_{pade'[3/3]}$	$u(t)_{VIM}$	$u(t)_{R\text{-}K}$	$Error_{(pade'[3/3])}\%$	$Error_{(VIM)}\%$
0	0	0	0	0	0
0.2	0.179113359119310	0.179113343756190	0.179097337759679	0.008945414934921	0.008937039601045
0.4	0.316008316008316	0.316006232438869	0.315975074554605	0.010520133604480	0.316008316008316
0.6	0.415043653458697	0.415008507960582	0.414985258839976	0.014071584172227	0.005602367675659
0.8	0.483920367534456	0.483681024791490	0.483790347421919	0.026875277833232	0.022596564934079
1.0	0.530612244897959	0.529638091578952	0.530284478652893	0.061810219456668	0.121894949978935
1.2	0.561872909698997	0.559019698064430	0.561110909911462	0.135802028893040	0.374257559941467
1.4	0.582846236430872	0.576176937572276	0.581292773696110	0.275248561895327	0.877354791041355

In table 3, the velocity results of vertically falling non-spherical particle in Newtonian fluid was investigated by different methods at different time. So we select the more accurate results from every methods. i.e. Pade'[3/3], VIM(4^th^ iteration) and compared these methods with Numerical Method(R-k 4^th^ order method). The absolute error % of Pade' [3/3] is less as compared to VIM (4^th^ iteration).

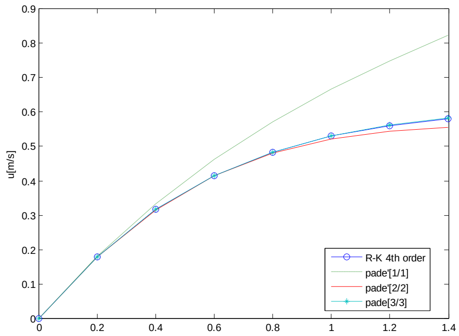

Fig. 1. Comparison between Pade' [1/1], Pade' [2/2], Pade' [3/3] with R-K 4^th^ order method

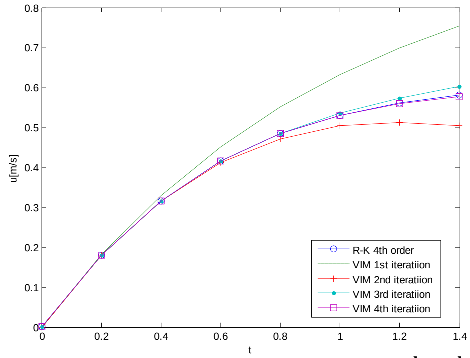

Fig.2. Comparison between VIM (1^st^, 2^nd^, 3^rd^ and 4^th^ iterations) with R-K 4^th^ order

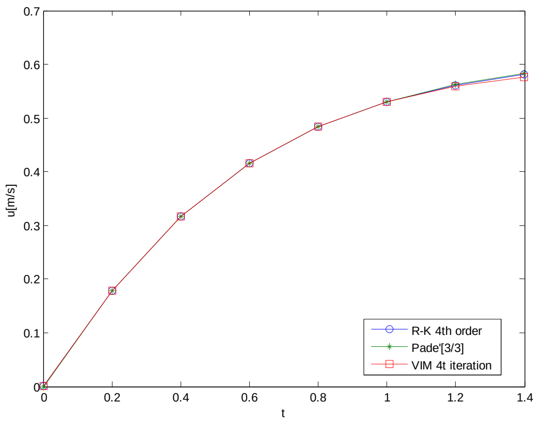

Fig.3.Pade'[3/3], VIM (4^th^ iteration) and R-K 4^th^ order (numerical solution) of eq.(3)

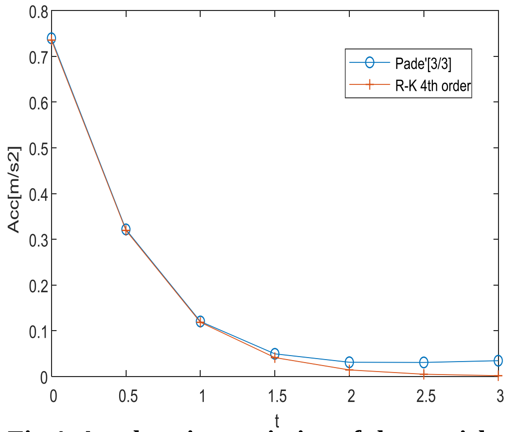

Fig.4. Acceleration variation of the particle obtained by Diagonal Pade' [3/3] and R-K 4^th^ order of eq.(3

Fig.3. depict the velocity versus time for the three methods. It observe that diagonal Pade' approximate of order [3/3] gives more accurate results as compared to VIM. In this study, a reliable combination of diagonal pade' approximate and VIM was applied to obtain approximate solution of the acceleration motion of single non-spherical particle moving in a continuous Newtonian fluid phase. Velocity was obtained at different time interval [0, 1.4] and was compared the results of present method (diagonal pade' approximate [3/3] and VIM (4^th^ iteration)) with numerical solutions of Runge-Kutta 4^th^ order method. From the above fig., it is clear that the results of VIM, diagonal Pade' and R-K method are almost same from time t=0 to t=1.0. Fromtime t>1.0 the graphs of all three methods are different but pade' [3/3] gives more closer results to numerical solution than VIM. So Pade' [3/3] ismore accurate than VIM. In all above discussion, it is clear velocity of particle increasing until it reaches at terminal velocity.

Fig.4. In this study, R-K 4th order and Diagonal Pade'[3/3] was applied for solution of acceleration motion of vertically falling non-spherical particle in incompressible Newtonian fluid. Acceleration decreasing as time increasing. After time 3 seconds particle is not accelerating. Results obtained with pade' approximant method and compared with numerical method. In short time Diagonal Pade' approximant method gives the accurate results

VII. CONCLUSION

The achievement of this work is to apply the current methods diagonal pade' and VIM in order to study the nonlinear differential equation of 1st order with initial condition that governed from the acceleration motion of vertically falling non-spherically particle in incompressible Newtonian fluid. The current methods are applied without using any linearization, discretization, restrictions or transformations. From above discussion, it is clear that the diagonal pade' method [3/3] has a good agreement with numerical method and provides highly reliable results. In addition, this method does not require many iterations like VIM to reach accurate results. Both methods gives the accurate results in short time, but pade' methods is also suitable for long time. Also, the current method (diagonal pade') can be used to develop the valid solution of other nonlinear differential equation of order more than one.

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