EVEN VERTEX MAGIC TOTAL LABELING OF ISOMORPHIC AND NON ISOMORPHIC SUNS

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Abstract. A vertex magic total labeling of a graph $G(V, E)$ is defined as one - to - one mapping from the set of integers $\{1, 2, 3, ..., |V| + |E|\}$ to $V \cup E$ with the property that the sum of the label of a vertex and the labels of all edges adjacent to this vertex is the same constant for all vertices of the graph. Such a labeling is even if $f(V(G)) = \{2, 4, 6, ..., 2n\}$. In this paper, we present an even vertex magic total labeling of union of suns, in particular disjoint union of isomorphic and non isomorphic suns.

Key words: Even vertex magic total labeling, isomorphic suns, non isomorphic suns.

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1. Introduction

In this paper all graphs are finite, simple and undirected. The graph G has vertex set $V(G)$ and edge set $E(G)$ and we let $m = |E(G)|$ and $n = |V(G)|$. Macdougall et al [1] introduced the concept of a vertex magic total labeling. This is an assignment of the integers from 1 to $m+n$ to the vertices and edges of G so that at each vertex the vertex label and the labels on the edges incident at that vertex add to a constant. More formally, a one to one map f from $V \cup E$ onto the integers $\{1, 2, 3, ..., m+n\}$ is a vertex magic total labeling if there is a constant k so that for every vertex u, $f(u) + \sum f(uv) = k$ where the sum is over all edges uv, where v is adjacent to u. Let us call the sum of labels at vertex u the weight $w_f(u)$

of the vertex u, we require $w_f(u) = k$ for all u. The constant k is called the magic constant for f.

If a regular graph G possesses a vertex magic total labeling f, we can create a new labeling f' from f by setting $f'(u) = n + m + 1 - f(u)$ for every vertex u, and $f'(uv) = n + m + 1 - f(uv)$ for every edge uv. Clearly the new labeling f' is also a one to one map from the set $V \cup E$ to $\{1, 2, 3, ..., m + n\}$, and we call this new labeling as the dual of the previous labeling. If r is the degree of each vertex of G, then $k' = (r + 1)(n + m + 1) - k$ is the new magic constant.

MacDougall et al. [2] introduced the concept of super vertex magic total labeling. They call a vertex magic total labeling is super if $f(V(G)) = \{1, 2, 3, ..., n\}$. In this labeling, the smallest labels are assigned to the vertices.

Swaminathan and Jeyanthi [4] introduced a concept with the same name of super vertex magic labeling. They call a vertex magic total labeling is super if $f(E(G)) = \{1, 2, 3, ..., m\}$. Here the smallest labels are assigned to the edges.

Nagaraj et al. [3] introduced the concept of an Even vertex magic total labeling. They call vertex magic total labeling is even if $f(V(G)) = \{2, 4, 6, ..., 2n\}$. In this labeling, the smallest labels are assigned to the vertices.

Most of the known results are concerning on vertex magic total labeling of connected graphs. For the case of disconnected graph, wallis [5] proved the following theorem.

Theorem 1.1. *suppose G is regular graph of degree r which has a vertex magic total labeling*

(1) *If r is even, then tG is vertex magic whenever t is an odd positive integer.*
(2) *If r is odd, then tG is vertex magic for every positive integer t.*

The following definition and Theorems that will subsequently be very useful to prove some theorems.

Definition 1.2. A sun S_p is a cycle C_p with an edge terminating in a vertex of degree 1 attached to each vertex. The sun S_p consists of vertex set $V(S_p) = \{v_i/1 \le i \le p\} \cup \{a_i/1 \le i \le p\}$ and edge set $E(S_p) = \{v_iv_{i+1}/1 \le i \le p\} \cup \{v_ia_i/1 \le i \le p\}$. We note that if $i = p$ then $i + 1 = 1$. The t copies of suns, denoted by tS_p, has vertex set $V(tS_p) = \{v_i^j / 1 \leq i \leq p, 1 \leq j \leq t\} \cup \{a_i^j / 1 \leq i \leq p, 1 \leq j \leq t\}$ and edge set $E(tS_p) = \{v_i^j v_{i+1}^j / 1 \leq i \leq p, 1 \leq j \leq t\} \cup \{v_i^j a_i^j / 1 \leq i \leq p, 1 \leq j \leq t\}$. Thus tS_p has $2pt$ vertices and $2pt$ edges.

Theorem 1.3. [3] *Let G be a nontrivial graph. If G is an even vertex magic, then the magic constant k is given by*

$$k = \frac{m^2 + 2mn + m}{n}$$

Theorem 1.4. [3] *For $p \geq 3$, sun S_p has an even vertex magic.*

2. Main Results

Theorem 2.1. *For $p \geq 3$ and $q \geq 3$, the disjoint union of two suns $S_p \cup S_q$ has even vertex magic total labeling with the magic constant $k = 6(p+q)+1$.*

Proof. For any positive integer $p, q \geq 3$, label the vertices and edges of the first sun S_p in the following way

$$f(v_i^1) = 2i$$

for $i = 1, 2, 3, ..., p$

$$f(v_i^1 v_{i+1}^1) = \begin{cases} 2(p+q-i)+1 & \text{for} \quad i = 1, 2, ..., p-1 \\ 2q+1 & \text{for} \quad i = p \end{cases}$$

$$f(a_i^1) = \begin{cases} 2(p+2q+1) & \text{for} \quad i = 1 \\ 4(p+q+1)-2i & \text{for} \quad i = 2, 3, ..., p \end{cases}$$

$$f(v_i^1 a_i^1) = \begin{cases} 2(2p+q)-1 & \text{for} \quad i = 1 \\ 2(p+q+i)-3 & \text{for} \quad i = 2, 3, ..., p \end{cases}$$

Then label the vertices and edges of the second sun S_q for $q \geq 3$ in the following way:

$$f(v_j^2) = 2(p+j)$$

for $\quad j = 1, 2, ..., q$

$$f(v_j^2 v_{j+1}^2) = \begin{cases} 2(q-j)+1 & \text{for} \quad j = 1, 2, ..., q-1 \\ 1 & \text{for} \quad j = q \end{cases}$$

$$f(a_j^2) = \begin{cases} 2(p+q+1) & \text{for} \quad j = 1 \\ 2(p+2q-j+2) & \text{for} \quad j = 2, 3, ..., q \end{cases}$$

$$f(v_j^2 a_j^2) = \begin{cases} 4(p+q)-1 & \text{for} \quad j = 1 \\ 2(2p+q+j)-3 & \text{for} \quad j = 2, 3, ..., q \end{cases}$$

It is easy to verify that the labeling f is a bijection from the set $V(S_p \cup S_q) \cup E(S_p \cup S_q)$ onto the set $\{1, 2, ..., 4(p+q)\}$

Let us denote the weights (Under labeling f) of vertices v_i^1 of $S_p \cup S_q$ by

$$w_f(v_i^1) = f(v_i^1) + f(v_i^1 v^{i+1}) + f(v_i^1 a_i^1) + f(v_{i-1}^1 v_i^1)$$

the weights of vertices a_i^1 by

$$w_f(a_i^1) = f(a_i^1) + f(v_i^1 a_i^1)$$

the weight of vertices v_j^2 by

$$w_f(v_j^2) = f(v_j^2) + f(v_j^2 v_{j+1}^2 + f(v_j^2 a_j^2) + f(v_{j-1}^2 v_j^2)$$

and the weights of vertices a_j^2 by

$$w_f(a_j^2) = f(a_j^2) + f(v_j^2 a_j^2)$$

It is clearly true that

$$w_f(v_i^1) = 6(p+q)+1$$
$$w_f(a_i^1) = 6(p+q)+1$$
$$w_f(v_j^2) = 6(p+q)+1$$

and

$$w_f(a_j^2) = 6(p+q)+1$$

for all $i = 1, 2, ..., p$ and $j = 1, 2, ..., q$. Thus f is the even vertex magic total labeling of $S_p \cup S_q$ for any positive integer $p, q \geq 3$ with the magic constant $k = 6(p+q)+1$ □

Example 2.2. *Figure 1 shows an example of even vertex magic total labeling of $S_4 \cup S_6$ with the magic constant $k = 61$.*

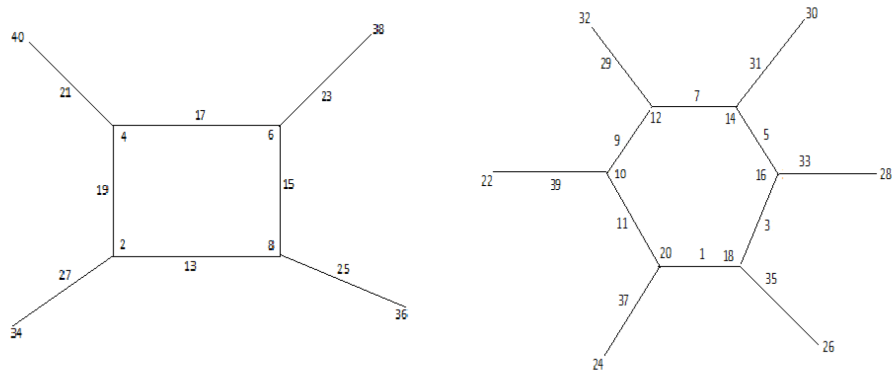

FIGURE 1. Even vertex magic total labeling of $S_4 \cup S_6$ with k=61

We note that if $p = q$, then S_p is isomorphic to S_q. In this case Theorem 2.1 shows that the vertex magic total labeling of 2 copies of isomorphic suns. General result on the even vertex magic total labeling of t copies of isomorphic suns, for any integers $t \geq 1$ is given below.

Theorem 2.3. *For $p \geq 3$ and $t \geq 1$, the t copies of sun tS_p has even vertex magic total labeling with the magic constant $k = 6pt + 1$.*

Proof. For all $j = 1, 2, ..., t$ label vertices and edges of tS_p in the following way:

$$f(v_i^j) = 2n(j-1) + 2i$$

for $i = 1, 2, ..., p$

$$f(v_i^j v_{i+1}^j) = \begin{cases} 2pt - 2p(j-1) - 2i + 1 & \text{for} \quad i = 1, 2, ..., p-1 \\ 2pt - 2pj + 1 & \text{for} \quad i = n \end{cases}$$

$$f(a_i^j) = \begin{cases} 4pt - 2p(j-1) - 2(p-1) & \text{for} \quad i = 1 \\ 4pt - 2p(j-1) - 2(i-2) & \text{for} \quad i = 2, 3, ..., p \end{cases}$$

$$f(v_i^j a_i^j) = \begin{cases} 2pt + 2pj - 1 & \text{for} \quad i = 1 \\ 2pt + 2p(j-1) + 2i - 3 & \text{for} \quad i = 2, 3, ..., p \end{cases}$$

It is easy to verify that the labeling f is a bijection from the set $V(tS_p) \cup E(tS_p)$ onto the set $\{1, 2, 3, ..., 4nt\}$

Let us denote the weights (under labeling f) of vertices v_i^j of tS_p by

$$w_f(v_i^j) = f(v_i^j) + f(v_i^j v_{i+1}^j) + f(v_i^j a_i^j) + f(v_{i-1}^j v_i^j)$$

and the weights of vertices a_i^j by

$$w_f(a_i^j) = f(a_i^j) + f(v_i^j a_i^j)$$

Then , for all $j = 1, 2, ..., t$, the weights of vertices v_i^j can be determined as follows: for $i = 1$

$$\begin{aligned} w_f(v_1^j) &= (2p(j-1)+2) + (2pt - 2p(j-1) - 2 + 1) + (2pt + 2pj - 1) \\ &+ (2pt - 2pj + 1) \\ &= (2pj - 2p + 2) + (2pt - 2pj + 2p - 1) + (2pt + 2pj - 1) + (2pt - 2pj + 1) \\ &= 6pt + 1 \end{aligned}$$

for $i = 2, 3, ..., p-1$

$$\begin{aligned} w_f(v_i^j) &= (2p(j-1)+2i) + (2pt - 2p(j-1) - 2i + 1) + (2pt + 2p(j-1) + 2i - 3) \\ &+ (2pt - 2p(j-1) - 2(i-1) + 1) \\ &= (2pj - 2p + 2i) + (2pt - 2pj + 2p - 2i + 1) + (2pt + 2pj - 2p + 2i - 3) \\ &+ (2pt - 2pj + 2p - 2i + 2 + 1) \\ &= 6pt + 1 \end{aligned}$$

for $i = p$

$$\begin{aligned} w_f(v_p^j) &= (2p(j-1)+2p) + (2pt - 2pj + 1) + (2pt + 2p(j-1) + 2p - 3) \\ &+ (2pt - 2p(j-1) - 2(p-1) + 1) \\ &= (2pj - 2p + 2p) + (2pt - 2pj + 1) + (2pt + 2pj - 2p + 2p - 3) \\ &+ (2pt - 2pj + 2p - 2p + 2 + 1) \\ &= 6pt + 1 \end{aligned}$$

and the weights of vertices a_i^j can be determined as follows:
for $i = 1$

$$\begin{aligned} w_f(a_i^j) &= (4pt - 2p(j-1) - 2(p-1)) + (2pt + 2pj - 1) \\ &= (4pt - 2pj + 2p - 2p + 2)) + (2pt + 2pj - 1) \\ &= 6pt + 1 \end{aligned}$$

for $i = 2, 3, ..., p$

$$\begin{aligned} w_f(a_i^j) &= (4pt - 2p(j-1) - 2(i-2)) + (2pt - 2p(j-1) + 2i - 3) \\ &= (4pt - 2pj + 2p - 2i + 4)) + (2pt + 2pj - 2p + 2i - 3) \\ &= 6pt + 1 \end{aligned}$$

Since $w_f(v_i^j) = 6pt + 1$ and $w_f(a_i^j) = 6pt + 1$ for all $i = 1, 2, 3, ..., p$ and $j = 1, 2, 3, ..., t$ then f is the even vertex magic total labeling of tS_p with the magic constant $k = 6pt + 1$ $\square$

Example 2.4. *Figure 2 shows an example of even vertex magic total labeling of 3 copies of S_5 with the magic constant $k = 91$*

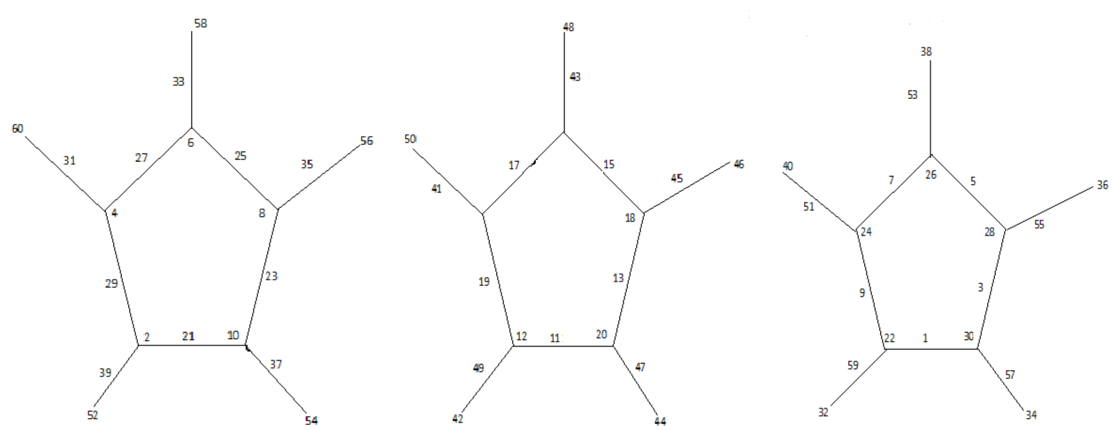

FIGURE 2. Even vertex magic labeling of $3S_5$ with $k = 91$

Theorem 2.5. *If $t_i \geq 3$ for every $i = 1, 2, ..., p$ and $p \geq 1$ the p disjoint copies of suns $S_{t_1} \cup S_{t_2} \cup ... \cup S_{t_p}$ has an even vertex magic total labeling with magic constant $6\sum_{k=1}^{p} t_k + 1$.*

Proof. We label the vertices and edges of the graph in the following way
$f(v_i^{t_j}) = 2\sum_{k=1}^{j-1} t_k + 2i$; $i = 1, 2, ..., t_j$ and $j = 1, 2, ..., p$

$$f(a_i^{t_j}) = \begin{cases} 4\sum_{k=1}^{p} t_k - 2\sum_{k=1}^{j} t_k + 2 & for \quad i = 1 \\ 2\sum_{k=j}^{p} t_k + 2\sum_{k=1}^{p} t_k - 2(i-2) & for \quad i = 2, 3, 4, ..., t_j. \end{cases}$$

$$f(v_i^{t_j} v_{i+1}^{t_j}) = \begin{cases} 2\sum_{k=1}^{p} t_k - 2\sum_{k=1}^{j} t_k + 1 & for \quad i = t_j \\ 2\sum_{k=1}^{p} t_k - 2\sum_{k=1}^{j-1} t_k - 2i + 1 & for \quad i = 1, 2, 3, 4, ..., t_j - 1. \end{cases}$$

$$f(v_i^{t_j} a_i^{t_j}) = \begin{cases} 2\sum_{k=1}^{p} t_k + 2\sum_{k=1}^{j} t_{k-1} & for \quad i = 1 \\ 2\sum_{k=1}^{p} t_k + 2\sum_{k=1}^{j-1} t_k + 2i - 3 & for \quad i = 2, 3, 4, ..., t_j. \end{cases}$$

It is easy to verify that the labeling f is a bijection from the set $V(S_{t_1} \cup S_{t_2} \cup S_{t_3} \cup ... \cup S_{t_p})$ onto the set $\{1, 2, ..., 4\sum_{k=1}^{p} t_k\}$.
Let us denote the weights of the vertices $v_i^{t_j}$ of S_{t_i} under the labeling f by

$$w_f(v_i^{t_j}) = f(v_i^{t_j}) + f(v_i^{t_j} v_{i+1}^{t_j}) + f(v_i^{t_j} a_i^{t_j}) + f(v_{i-1}^{t_j} v_i^{t_j})$$

and the weights of the vertices $a_i^{t_j}$ by

$$w_f(a_i^{t_j}) = f(a_i^{t_j}) + f(v_i^{tj} a_i^{t_j})$$

Then for all $j = 1, 2, 3, ..., p$ and $i = 1, 2, ..., t_j$ the weights of the vertices $v_i^{t_j}$ can be determined as follows:
For $i = 1$ we have

$$\begin{aligned} w_f(v_i^{t_j}) &= f(v_1^{t_j}) + f(v_1^{t_j} v_2^{t_j}) + f(v_1^{t_j} a_1^{t_j}) \\ &+ f(v_{t_j}^{t_j} v_1^{t_j}) \\ &= 2\sum_{k=1}^{j-1} t_k + 2 + 2\sum_{k=1}^{p} t_k - 2\sum_{k=1}^{j-1} t_k - 2 + 1 + 2\sum_{k=1}^{p} t_k \\ &+ 2\sum_{k=1}^{j} t_k - 1 + 2\sum_{k=1}^{p} t_k - 2\sum_{k=1}^{j} t_k + 1 \\ &= 6\sum_{k=1}^{p} t_k + 1 \end{aligned}$$

For $i = 2, 3, ..., t_j - 1$

$$\begin{aligned}
w_f(v_i^{t_j}) &= f(v_i^{t_j}) + f(v_{i-1}^{t_j} v_i^{t_j}) + f(v_i^{t_j} v_{i+1}^{t_j}) + f(v_i^{t_j} a_i^{t_j}) \\
&= 2\sum_{k=1}^{j-1} t_k + 2i + 2\sum_{k=1}^{p} t_k - 2\sum_{k=1}^{j-1} t_k - 2i + 1 \\
&+ 2\sum_{k=1}^{p} t_k - 2\sum_{k=1}^{j-1} t_k - 2(i-1) \\
&+ 1 + 2\sum_{k=1}^{p} t_k + 2\sum_{k=1}^{j-1} t_k + 2i - 3. \\
&= 6\sum_{k=1}^{p} t_k + 1
\end{aligned}$$

For $i = t_j$ we have

$$\begin{aligned}
w_f(v_i^{t_j}) &= 2\sum_{k=1}^{j-1} t_k + 2t_j + 2\sum_{k=1}^{p} t_k - 2\sum_{k=1}^{j} t_k \\
&+ 1 + 2\sum_{k=1}^{p} t_k - 2\sum_{k=1}^{j-1} t_k - 2(t_j - 1) + 1 \\
&+ 2\sum_{k=1}^{p} t_k + 2\sum_{k=1}^{j-1} t_k + 2t_j - 3. \\
&= 6\sum_{k=1}^{p} t_k + 1
\end{aligned}$$

For $i = 1$ the weights of the vertices $a_i^{t_j}$ are given by

$$\begin{aligned}
w_f(a_1^{t_j}) &= f(a_i^{t_j}) + f(v_1^{t_j} a_1^{t_j}) \\
&= 4\sum_{k=1}^{p} t_k - 2\sum_{k=1}^{j} t_j + 2 + 2\sum_{k=1}^{p} t_k + 2\sum_{k=1}^{j} t_k - 1 \\
&= 6\sum_{k=1}^{p} t_k + 1
\end{aligned}$$

For $i = 2, 3, ..., t_j$ the weights, of the vertices a_i^{tj} are given by

$$\begin{aligned} w_f(a_i^{t_j}) &= f(a_i^{t_j}) + f(v_i^{t_j} a_i^{t_j}) \\ &= 2\sum_{k=1}^{p} t_k + 2\sum_{k=j}^{p} t_k - 2i + 4 + 2\sum_{k=j}^{p} t_k + 2\sum_{k=1}^{j-1} t_k + 2i - 3 \\ &= 6\sum_{k=1}^{p} t_k + 1 \end{aligned}$$

$\square$

Example 2.6. *Figure 3 shows an example of even vertex magic total labeling of* $S_4 \cup S_5 \cup S_6$ *with the magic constant* $k = 91$

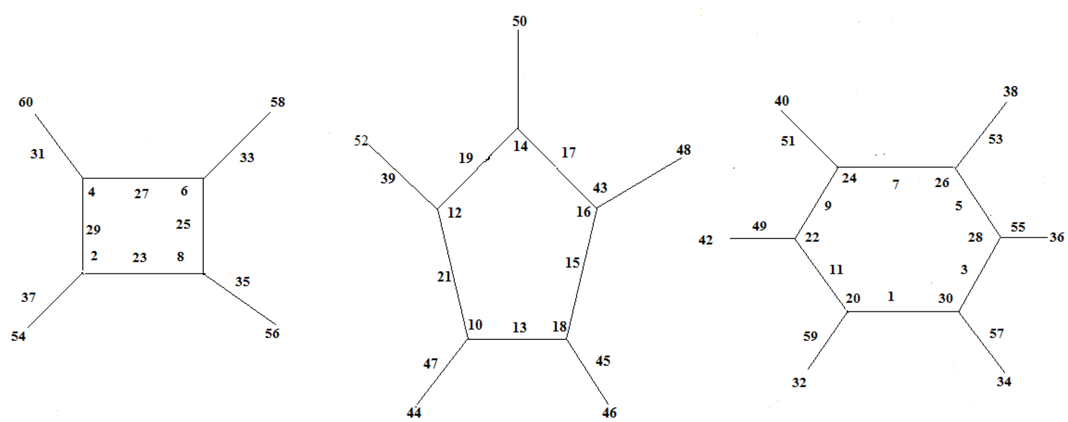

FIGURE 3. Even vertex magic total labeling of $S_4 \cup S_5 \cup S_6$ with the magic constant $k = 91$

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