Common tripled fixed point results for hybrid pair of mappings under new condition

Bhavana Deshpande
Department of Mathematics
Govt. P. G. Arts & Science College
Ratlam (M. P.) India

Shamim Ahmad Thoker[1]
Department of Mathematics
Govt. P. G. Arts & Science College
Ratlam (M. P.) India

Riyaz Ahmad Shah
Department of Mathematics
Govt. Degree College
Kulgam (Jammu & Kashmir)

Abstract. We introduce the concept of (EA) property and occasionally $w-$compatibility for hybrid pair $g : X \to X$ and $F : X \times X \times X \to 2^X$. We establish some common tripled fixed point theorems for hybrid pair of mappings satisfying, (EA) property and occasionally $w-$compatibility conditions, under weak $\psi - \varphi$ contraction. It is to be noted that to find tripled coincidence point, we do not employ completeness on space and not partially orderdness. Also the condition of continuity is not necessary for any mapping involved therein. An example is also given to validate our results. We extend and generalize several known results.

Mathematics Subject Classification: 47H10, 54H25.

Key words and phrases: Tripled fixed point, tripled coincidence point, (EA) property, $w-$compatibility, occasionally $w-$compatibility, weak $\psi-\varphi$ contraction.

1. Introduction and Preliminaries

The study of fixed points for multivalued contraction mappings using the Hausdorff metric was initiated by Nadler [23]. The theory of multivalued mappings has wide range of applications. It is applied in control theory, convex optimization, differential inclusions, and economics.

[1] Corresponding Author

1

Let $(X,\ d)$ be a metric space and $CB(X)$ be the set of all nonempty closed bounded subsets of X. Let $D(x,\ A)$ denote the distance from x to $A \subset X$ and H denote the Hausdorff metric induced by d, that is,

$$D(x,\ A) = \inf_{a \in A} d(x,\ a)$$

and
$$H(A,\ B) = \max\left\{\sup_{a \in A} D(a,\ B),\ \sup_{b \in B} D(b,\ A)\right\}, \text{ for all } A,\ B \in CB(X).$$

The existence of fixed points for various multivalued contractive mappings has been studied by many authors under different conditions and a significant number of papers have been reported. For details, we refer to [4, 22, 23, 25, 27] and the references therein.

Samet and Vetro [26] introduced the concept of coupled fixed point for multivalued mapping and later several authors proved existence of coupled fixed points for multivalued mappings under different conditions. Subsequently, many results in this direction were given (see, e.g., [14, 15, 16, 18, 21, 26])

Berinde and Borcut [9] , introduced the concept of triple fixed points. In [9] ; Berinde and Borcut established the existence of tripled fixed point of single valued mappings in partially ordered metric spaces. For more details on tripled fixed point theory, we also refer the reader to [3, 5, 6, 7, 8, 10, 12, 24] .

Deshpande et al. in [17] introduced Triple fixed, Triple coincidence and Triple common fixed points for multivalued maps.

Definition 1.1. [17] Let X be a non empty set, $F : X \times X \times X \to 2^X$ (Collection of all non empty subsets of X). $g : X \to X$.

(i) The point $(x, y, z) \in X \times X \times X$ is called a tripled fixed point of F if $x \in F(x, y, z), y \in F(y, z, x)$ and $z \in F(z, x, y)$.

(ii) The point $(x, y, z) \in X \times X \times X$ is called a tripled coincident point of F and g if $gx \in F(x, y, z), gy \in F(y, z, x)$ and $gz \in F(z, x, y)$.

(iii) The point $(x, y, z) \in X \times X \times X$ is called a tripled common fixed point of F and g if $x = gx \in F(x, y, z), y = gy \in F(y, z, x)$ and $z = gz \in F(z, x, y)$.

We denote the set of tripled coincidence points of mappings F and g by $C(F, g)$.

Note that if $(x, y, z) \in C(F, g)$, then (y, z, x) and (z, x, y) are also in $C(F, g)$.

Definition 1.2. [17] Let $F : X \times X \times X \to 2^X$ be a multivalued map and g be a self map on X. The Hybrid pair $\{F, g\}$ is called w - compatible if $g(F(x, y, z)) \subseteq F(gx, gy, gz)$ whenever (x, y, z) is a tripled coincidence point of F and g.

Definition 1.3.[17] Let $F : X \times X \times X \to 2^X$ be a multivalued mapping and g be a self-map on X. The mapping g is called F-weakly commuting at some point $(x, y, z) \in X \times X \times X$ if $g^2x \in F(gx, gy, gz), g^2y \in F(gy, gz, gx)$ and $g^2z \in F(gz, gx, gy)$.

Aamri and ElMoutawakil [1] defined (EA) property for self-mappings which contained the class of non-compatible mappings. Kamran [20] extended the (EA) property for hybrid pair $g : X \to X$ and $F : X \to 2^X$. Abbas and Rhoades [2] extended the concept of occasionally weakly compatible mappings for hybrid pair $g : X \to X$ and $F : X \to 2^X$. Deshpande and Handa [15] introduced the concept of (EA) property and occasionally $w-$compatibility for hybrid pair $g : X \to X$ and $F : X \times X \to 2^X$.

In this paper, we introduce the concept of (EA) property and occasionally $w-$compatibility for hybrid pair $g : X \to X$ and $F : X \times X \times X \to 2^X$. We establish some common tripled fixed point theorems for hybrid pair of mappings satisfying, (EA) property and occasionally $w-$compatibility conditions, under weak $\psi - \varphi$ contraction. It is to be noted that to find tripled coincidence point, we do not employ completeness on space and not partially orderdness. Also the condition of continuity is not necessary for any mapping involved therein. An example is also given to validate our results. We improve, extend and generalize the results of Bhaskar and Lakshmikantham [11], Ciric et al. [13], Ding et al. [18], Gordji et al. [19], Deshpande and Handa [15] and Lakshmikantham and Ciric [21]. The effectiveness of our generalization demonstrated with the help of an example.

2. Main results

First we introduce the following

Definition 2.1. Mappings $g : X \to X$ and $F : X \times X \times X \to CB(X)$ are said to satisfy the (EA) property if there exist sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in X, some r, s, t in X and A, B, C in $CB(X)$ such that

$$\begin{aligned} \lim_{n\to\infty} gx_n &= r \in A = \lim_{n\to\infty} F(x_n, y_n, z_n), \\ \lim_{n\to\infty} gy_n &= s \in B = \lim_{n\to\infty} F(y_n, z_n, x_n), \\ \lim_{n\to\infty} gz_n &= t \in C = \lim_{n\to\infty} F(z_n, x_n, y_n). \end{aligned}$$

Example 2.1. Let $X = [1, +\infty)$ with the usual metric. Define $g : X \to X$ and $F : X \times X \times X \to CB(X)$ by

$$\begin{aligned} gx &= 2 + x \text{ and} \\ F(x, y, z) &= [2, 3 + 2x + y + z] \text{ for all } x, y, z \in X. \end{aligned}$$

Consider the sequences

$$\{x_n\} = \{2 + \frac{1}{n}\}, \{y_n\} = \{4 + \frac{1}{n}\} \text{ and } \{z_n\} = \{6 + \frac{1}{n}\}.$$

3

Clearly,

$$\begin{aligned}\lim_{n\to\infty} gx_n &= 4 \in A = \lim_{n\to\infty} F(x_n, y_n, z_n) = [2, 17],\\ \lim_{n\to\infty} gy_n &= 6 \in B = \lim_{n\to\infty} F(y_n, z_n, x_n) = [2, 19],\\ \lim_{n\to\infty} gz_n &= 8 \in C = \lim_{n\to\infty} F(z_n, x_n, y_n) = [2, 21].\end{aligned}$$

Hence g and F satisfy (EA) property.

Definition 2.2. Mappings $F : X \times X \times X \to 2^X$ and $g : X \to X$ are said to be occasionally $w-$compatible if and only if there exists some point $(x, y, z) \in X \times X \times X$ such that $gx \in F(x, y, z)$, $gy \in F(y, z, x), gz \in F(z, x, y)$, and $gF(x, y, z) \subseteq F(gx, gy, gz)$.

Following example shows that, occasionally $w-$compatibility is weaker condition than $w-$compatibility.

Example 2.2. Let $X = [0, +\infty)$ with usual metric. Define $g : X \to X$, $F : X \times X \times X \to CB(X)$, for all $x, y, z \in X$, by

$$gx = \begin{cases} 0, & 0 \le x \le 1,\\ 4x, & 1 \le x < \infty,\end{cases}$$

$$F(x, y, z) = \begin{cases} [0, 1 + 2x + y + z], & (x, y, z) \neq (0, 0, 0),\\ \{x\}, & (x, y, z) = (0, 0, 0).\end{cases}$$

It can be easily verified that $(0, 0, 0)$ and $(1, 1, 1)$ are tripled coincidence points of g and F, but $gF(0, 0, 0) \subseteq F(g0, g0, g0)$ and $gF(1, 1, 1) \subsetneq F(g1, g1, g1)$ So g and F are not w-compatible. However, the pair $\{F, g\}$ is occasionally w-compatible.

Let Ψ denote the set of all functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying

(i_ψ) ψ is continuous and non-decreasing,

(ii_ψ) $\psi(t) = 0 \Leftrightarrow t = 0$,

(iii_ψ) $\limsup_{s\to 0+} \frac{s}{\psi(s)} < \infty$,

and Φ denote the set of all functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying

(i_φ) φ is lower semi-continuous and non-decreasing,

(ii_φ) $\varphi(t) = 0 \Leftrightarrow t = 0$,

(iii_φ) for any sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = 0$, there exist $k \in (0, 1)$ and $n_0 \in \mathbb{N}$, such that $\varphi(t_n) \ge kt_n$ for each $n \ge n_0$,

and Θ denote the set of all functions $\theta : [0, +\infty) \to [0, +\infty)$ satisfying

(i_θ) θ is continuous,

(ii_θ) $\theta(t) = 0 \Leftrightarrow t = 0$.

For simplicity, we define

$$
\begin{aligned}
& (I)\ M(x,y,z,u,v,w) \\
= \quad & \max\left\{\begin{array}{c}
d(gx,gu), D(gx,F(x,y,z)), D(gu,F(u,v,w)), \\
d(gy,gv), D(gy,F(y,z,x)), D(gv,F(v,w,u)), \\
d(gz,gw), D(gz,F(z,x,y)), D(gw,F(w,u,v)), \\
\frac{D(gx,\ F(u,v,w))+D(gu,\ F(x,y,z))}{2}, \\
\frac{D(gy,F(v,w,u))+D(gv,F(y,z,x))}{2}, \\
\frac{D(gz,F(w,u,v))+D(gw,F(z,x,y))}{2}.
\end{array}\right\}, \\
& (II)\ N(x,y,z,u,v,w) \\
= \quad & \min\left\{\begin{array}{c}
D(gx,F(x,y,z)), D(gu,F(u,v,w)), \\
D(gx,F(u,v,w)), D(gu,F(x,y,z)), \\
D(gy,F(y,z,x)), D(gv,F(v,w,u)), \\
D(gy,F(v,w,u)), D(gv,F(y,z,x)), \\
D(gz,F(z,x,y)), D(gw,F(w,u,v)), \\
D(gz,F(w,u,v)), D(gw,F(z,x,y)).
\end{array}\right\}.
\end{aligned}
$$

Theorem 2.1. Let $(X,\ d)$ be a metric space, $F : X \times X \times X \to CB(X)$ and $g : X \to X$ be two mappings. Suppose that there exist some $\psi \in \Psi$, $\varphi \in \Phi$ and $\theta \in \Theta$ such that

$$
\begin{aligned}
& \psi\left(H(F(x,y,z),F(u,v,w))\right) \qquad (2.1) \\
\leq \quad & \psi\left(M(x,y,z,u,v,w)\right) - \varphi\left(\psi\left(M(x,y,z,u,v,w)\right)\right) + \theta\left(N(x,y,z,u,v,w)\right),
\end{aligned}
$$

for all $x,y,z,u,v,w \in X$. Furthermore, assume that $\{F,g\}$ satisfies the (EA) property. Then F and g have a tripled coincidence point. Moreover, F and g have a common tripled fixed point, if one of the following conditions holds:

(a) F and g are $w-$compatible. $\lim_{n\to\infty} g^n x = u, \lim_{n\to\infty} g^n y = v$ and $\lim_{n\to\infty} g^n z = w$ for some $(x,y,z) \in C\{F,\ g\}$ and for some $u,v,w \in X$ and g is continuous at u,v and w.

(b) g is $F-$weakly commuting for some $(x,y,z) \in C\{F,\ g\}$ and gx, gy and gz are fixed points of g, that is, $g^2x = gx, g^2y = gy$ and $g^2z = gz$.

(c) g is continuous at x,y and z. $\lim_{n\to\infty} g^n u = x, \lim_{n\to\infty} g^n v = y$ and $\lim_{n\to\infty} g^n w = z$ for some $(x,y,z) \in C\{F,\ g\}$ and for some $u,v,w \in X$.

(d) $g(C\{F,\ g\})$ is a singleton subset of $C\{F,\ g\}$.

Proof. Since $\{F,\ g\}$ satisfies the (EA) property, therefore there exist sequences $\{x_n\}$, $\{y_n\}$and $\{z_n\}$ in X, some $r,s,t,$ in X and A,B,C in $CB(X)$ such that

$$
\begin{aligned}
\lim_{n\to\infty} gx_n &= r \in A = \lim_{n\to\infty} F(x_n,y_n,z_n), \\
\lim_{n\to\infty} gy_n &= s \in B = \lim_{n\to\infty} F(y_n,z_n,x_n), \\
\lim_{n\to\infty} gz_n &= t \in B = \lim_{n\to\infty} F(z_n,x_n,y_n). \qquad (2.2)
\end{aligned}
$$

Since $g(X)$ is a subset of X, then there exist $x, y, z \in X$, we have

$$r = gx, s = gy \text{ and } t = gz \tag{2.3}$$

Now, by using condition (2.1) and (i_ψ), we get

$$\begin{aligned} & \psi\left(H(F(x_n, y_n, z_n), F(x, y, z))\right) \\ \leq\ & \psi\left(M(x_n, y_n, z_n, x, y, z)\right) - \varphi\left(\psi\left(M(x_n, y_n, z_n, x, y, z)\right)\right) \\ & +\theta\left(N(x_n, y_n, z_n, x, y, z)\right), \end{aligned}$$

where

$$\begin{aligned} & M(x_n, y_n, z_n, x, y, z) \\ = \ & \max\left\{\begin{array}{c} d(gx_n, gx), D(gx_n, F(x_n, y_n, z_n)), D(gx, F(x, y, z)), \\ d(gy_n, gy), D(gy_n, F(y_n, z_n, x_n)), D(gy, F(y, z, x)), \\ d(gz_n, gz), D(gz_n, F(z_n, x_n, y_n)), D(gz, F(z, x, y)), \\ \frac{D(gx_n, F(x,y,z)) + D(gx, F(x_n, y_n, z_n))}{2}, \\ \frac{D(gy_n, F(y,z,x)) + D(gy, F(y_n, z_n, x_n))}{2}, \\ \frac{D(gz_n, F(z,x,y)) + D(gz, F(z_n, x_n, y_n))}{2}. \end{array}\right\} \end{aligned}$$

and

$$\begin{aligned} & N(x_n, y_n, z_n, x, y, z) \\ = \ & \min\left\{\begin{array}{l} D(gx_n, F(x_n, y_n, z_n)), D(gx, F(x, y, z)), \\ D(gx_n, F(x, y, z)), D(gx, F(x_n, y_n, z_n)), \\ D(gy_n, F(y_n, z_n, x_n)), D(gy, F(y, z, x)), \\ D(gy_n, F(y, z, x)), D(gy, F(y_n, z_n, x_n)), \\ D(gz_n, F(z_n, x_n, y_n)), D(gz, F(z, x, y)), \\ D(gz_n, F(z, x, y)), D(gz, F(z_n, x_n, y_n)). \end{array}\right\}. \end{aligned}$$

Letting $n \to \infty$ in the above inequality, by using (i_ψ), (i_φ), (i_θ), (ii_θ), (2.2), (2.3), $gx \in A$, $gy \in B$ and $gz \in C$,we get

$$\begin{aligned} & \psi\left(D(gx, F(x, y, z))\right) \\ \leq\ & \psi\left(\max\left\{D(gx, F(x, y, z)), D(gy, F(y, z, x)), D(gz, F(z, x, y)\right\}\right) \\ & -\varphi\left(\psi\left(\max\left\{D(gx, F(x, y, z)), D(gy, F(y, z, x)), D(gz, F(z, x, y)\right\}\right)\right). \end{aligned}$$

Similarly, we can obtain that

$$\begin{aligned} & \psi\left(D(gy, F(y, z, x))\right) \\ \leq\ & \psi\left(\max\left\{D(gx, F(x, y, z)), D(gy, F(y, z, x)), D(gz, F(z, x, y)\right\}\right) \\ & -\varphi\left(\psi\left(\max\left\{D(gx, F(x, y, z)), D(gy, F(y, z, x)), D(gz, F(z, x, y)\right\}\right)\right), \end{aligned}$$

and

$$\begin{aligned} & \psi\left(D(gz, F(z, x, y))\right) \\ \leq\ & \psi\left(\max\left\{D(gx, F(x, y, z)), D(gy, F(y, z, x)), D(gz, F(z, x, y)\right\}\right) \\ & -\varphi\left(\psi\left(\max\left\{D(gx, F(x, y, z)), D(gy, F(y, z, x)), D(gz, F(z, x, y)\right\}\right)\right). \end{aligned}$$

6

Combining them, we get

$$\begin{aligned} & \max\left(\psi\left(D(gx,F(x,y,z))\right),\ \psi\left(D(gy,F(y,z,x))\right),\psi\left(D(gz,F(z,x,y))\right)\right) \\ \leq\ & \psi\left(\max\left\{D(gx,F(x,y,z)),D(gy,F(y,z,x)),D(gz,F(z,x,y)\right\}\right) \\ & -\varphi\left(\psi\left(\max\left\{D(gx,F(x,y,z)),D(gy,F(y,z,x)),D(gz,F(z,x,y)\right\}\right)\right). \end{aligned}$$

Since ψ is non-decreasing, therefore,

$$\begin{aligned} & \psi\left(\max\left\{D(gx,F(x,y,z)),\left(D(gy,F(y,z,x))\right),\left(D(gz,F(z,x,y))\right)\right\}\right) \\ \leq\ & \psi\left(\max\left\{D(gx,F(x,y,z)),D(gy,F(y,z,x)),D(gz,F(z,x,y)\right\}\right) \\ & -\varphi\left(\psi\left(\max\left\{D(gx,F(x,y,z)),D(gy,F(y,z,x)),D(gz,F(z,x,y)\right\}\right)\right). \end{aligned}$$

which, by (ii_φ) and (ii_ψ), implies that

$$\max\left\{D(gx,F(x,y,z)),D(gy,F(y,z,x)),D(gz,F(z,x,y))\right\}=0,$$

it follows that

$$gx\in F(x,y,z), gy\in F(y,z,x) \text{ and } gz\in F\left(z,x,y\right)$$

that is, (x,y,z) is a tripled coincidence point of F and g. That is $C\{F,\ g\}$ is non empty.

Suppose now that (a) holds. Assume that for some $(x,y,z)\in C\{F,\ g\}$,

$$\lim_{n\to\infty} g^n x=u,\ \lim_{n\to\infty} g^n y=v \text{ and } \lim_{n\to\infty} g^n z=w, \tag{2.4}$$

where $u,v,w\in X$. Since g is continuous at u,v and w. We have, by (2.4), that u,v and w are fixed points of g, that is,

$$gu=u, gv=v \text{ and } gw=w. \tag{2.5}$$

As F and g are $w-$compatible, so

$$(g^n x,g^n y,g^n z)\in C\{F,\ g\},\ \text{for all } n\geq 1,$$

that is,

$$\begin{aligned} g^n x &\in F(g^{n-1}x,g^{n-1}y,g^{n-1}z), \\ g^n y &\in F(g^{n-1}y,\ g^{n-1}z,g^{n-1}x) \text{ and} \\ g^n z &\in F(g^{n-1}z,\ g^{n-1}x,g^{n-1}y),\ \text{for all } n\geq 1. \end{aligned} \tag{2.6}$$

Now, by using (2.1), (2.6) and (i_ψ), we obtain

$$\begin{aligned} & \psi\left(D(g^n x,F(u,v,w))\right) \\ \leq\ & \psi\left(H(F(g^{n-1}x,g^{n-1}y,g^{n-1}z),F(u,v,w))\right) \\ \leq\ & \psi\left(M(g^{n-1}x,g^{n-1}y,g^{n-1}z,\ u,v,w)\right)-\varphi\left(\psi\left(M(g^{n-1}x,g^{n-1}y,g^{n-1}z,u,v,w)\right)\right) \\ & +\theta\left(N(g^{n-1}x,g^{n-1}y,g^{n-1}z,u,v,w)\right), \end{aligned}$$

where

$$
\begin{aligned}
& M(g^{n-1}x, g^{n-1}y, g^{n-1}z, u, v, w) \\
= \quad & \max \left\{ \begin{array}{c} d(g^n x, gu), D(g^n x, F(g^{n-1}x, g^{n-1}y, g^{n-1}z)), D(gu, F(u,v,w)), \\ d(g^n y, gv), D(g^n y, F(g^{n-1}y, g^{n-1}z, g^{n-1}x)), D(gv, F(v,w,u)), \\ d(g^n z, gw), D(g^n z, F(g^{n-1}z, g^{n-1}x, g^{n-1}y)), D(gw, F(w,u,v)), \\ \frac{D(g^n x, F(u,v,w)) + D(gu, F(g^{n-1}x, g^{n-1}y, g^{n-1}z))}{2}, \\ \frac{D(g^n y, F(v,w,u)) + D(gv, F(g^{n-1}y, g^{n-1}z, g^{n-1}x))}{2}, \\ \frac{D(g^n z, F(w,u,v)) + D(gw, F(g^{n-1}z, g^{n-1}x, g^{n-1}y))}{2}. \end{array} \right\} \\
\leq \quad & \max \left\{ \begin{array}{c} d(g^n x, gu), d(g^n x, g^n x), D(gu, F(u,v,w)), \\ d(g^n y, gv), d(g^n y, g^n y), D(gv, F(v,w,u)), \\ d(g^n z, gw), d(g^n z, g^n z), D(gw, F(w,u,v)), \\ \frac{D(g^n x, F(u,v,w)) + d(gu, g^n x)}{2}, \\ \frac{D(g^n y, F(v,w,u)) + D(gv, g^n y)}{2}, \\ \frac{D(g^n z, F(w,u,v)) + D(gw, g^n z)}{2}. \end{array} \right\},
\end{aligned}
$$

and

$$
\begin{aligned}
& N(g^{n-1}x, g^{n-1}y, g^{n-1}z, u, v, w) \\
= \quad & \min \left\{ \begin{array}{c} D(g^n x, F(g^{n-1}x, g^{n-1}y, g^{n-1}z)), D(gu, F(u,v,w)), \\ D(g^n x, F(u,v,w)), D(gu, F(g^{n-1}x, g^{n-1}y, g^{n-1}z)), \\ D(g^n y, F(g^{n-1}y, g^{n-1}z, g^{n-1}x)), D(gv, F(v,w,u)), \\ D(g^n y, F(v,w,u)), D(gv, F(g^{n-1}y, g^{n-1}z, g^{n-1}x)), \\ D(g^n z, F(g^{n-1}z, g^{n-1}x, g^{n-1}y)), D(gw, F(w,u,v)), \\ D(g^n z, F(w,u,v)), D(gw, F(g^{n-1}z, g^{n-1}x, g^{n-1}y)). \end{array} \right\} = 0.
\end{aligned}
$$

On taking limit as $n \to \infty$ in the above inequality, by using (i_ψ), (i_φ), (ii_θ), (2.4), (2.5) and (2.6), we get

$$
\begin{aligned}
& \psi\left(D(gu, F(u,v,w))\right) \\
\leq \quad & \psi\left(\max\left\{D(gu, F(u,v,w)), D(gv, F(v,w,u)), D(gw, F(w,u,v))\right\}\right) \\
& -\varphi\left(\psi\left(\max\left\{D(gu, F(u,v,w)), D(gv, F(v,w,u)), D(gw, F(w,u,v))\right\}\right)\right).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
& \psi\left(D(gv, F(v,w,u))\right) \\
\leq \quad & \psi\left(\max\left\{D(gu, F(u,v,w)), D(gv, F(v,w,u)), D(gw, F(w,u,v))\right\}\right) \\
& -\varphi\left(\psi\left(\max\left\{D(gu, F(u,v,w)), D(gv, F(v,w,u)), D(gw, F(w,u,v))\right\}\right)\right).
\end{aligned}
$$

and

$$
\begin{aligned}
& \psi\left(D(gw, F(w,u,v))\right) \\
\leq \quad & \psi\left(\max\left\{D(gu, F(u,v,w)), D(gv, F(v,w,u)), D(gw, F(w,u,v))\right\}\right) \\
& -\varphi\left(\psi\left(\max\left\{D(gu, F(u,v,w)), D(gv, F(v,w,u)), D(gw, F(w,u,v))\right\}\right)\right).
\end{aligned}
$$

Combining them, we get

$$\begin{aligned} & \max\left\{\psi\left(D(gu, F(u,v,w))\right), \psi\left(D(gv, F(v,w,u))\right), \psi\left(D(gw, F(w,u,v))\right)\right\} \\ \leq\ & \psi\left(\max\left\{D(gu, F(u,v,w)), D(gv, F(v,w,u)), D(gw, F(w,u,v))\right\}\right) \\ & -\varphi\left(\psi\left(\max\left\{D(gu, F(u,v,w)), D(gv, F(v,w,u)), D(gw, F(w,u,v))\right\}\right)\right). \end{aligned}$$

Since ψ is non-decreasing, therefore

$$\begin{aligned} & \psi\left(\max\left\{D(gu, F(u,v,w)), \left(D(gv, F(v,w,u))\right), \left(D(gw, F(w,u,v))\right)\right\}\right) \\ \leq\ & \psi\left(\max\left\{D(gu, F(u,v,w)), D(gv, F(v,w,u)), D(gw, F(w,u,v))\right\}\right) \\ & -\varphi\left(\psi\left(\max\left\{D(gu, F(u,v,w)), D(gv, F(v,w,u)), D(gw, F(w,u,v))\right\}\right)\right), \end{aligned}$$

which, by (ii_φ) and (ii_ψ), implies that

$$\max\left\{D(gu, F(u,v,w)), D(gv, F(v,w,u)), D(gw, F(w,u,v))\right\} = 0,$$

it follows that

$$gu \in F(u,v,w), gv \in F(v,w,u) \text{ and } gw \in F(w,u,v). \tag{2.7}$$

Now, from (2.5) and (2.7), we have

$$u = gu \in F(u,v,w), v = gv \in F(v,w,u) \text{ and } w = gw \in F(w,u,v),$$

that is, (u,v,w) is a common tripled fixed point of F and g.

Suppose now that (b) holds. Assume that for some $(x,y,z) \in C\{F,g\}$, g is $F-$weakly commuting, that is $g^2x \in F(gx,gy,gz), g^2y \in F(gy,gz,gx), g^2z \in F(gz,gx,gy)$ and $g^2x = gx, g^2y = gy, g^2z = gz$. Thus $gx = g^2x \in F(gx,gy,gz), gy = g^2y \in F(gy,gz,gx)$ and $gz = g^2z \in F(gz,gy,gx)$, that is, (gx,gy,gz) is a common tripled fixed point of F and g.

Suppose now that (c) holds. Assume that for some $(x,y,z) \in C\{F,\ g\}$ and for some $u,v,w \in X$,

$$\lim_{n\to\infty} g^n u = x, \lim_{n\to\infty} g^n v = y \text{ and } \lim_{n\to\infty} g^n w = z. \tag{2.8}$$

Since g is continuous at x, y and z. Therefore, by (2.8), we obtain that x, y and z are fixed points of g, that is,

$$gx = x, gy = y \text{ and } gz = z. \tag{2.9}$$

Since $(x,y,z) \in C\{F,\ g\}$. Therefore, by (2.9), we obtain

$$x = gx \in F(x,y,z), y = gy \in F(y,z,x) \text{ and } z = gz \in F(z,x,y),$$

that is, (x,y,z) is a common tripled fixed point of F and g.

Finally, suppose that (d) holds. Let $g(C\{F,\ g\}) = \{(x,x,x)\}$. Then $\{x\} = \{gx\} = F(x,x,x)$. Hence (x,x,x) is a common tripled fixed point of F and g.

If we put $\theta(t) = 0$ in the Theorem 2.1, we get the following result:

Corollary 2.2. Let (X, d) be a metric space, $F : X \times X \times X \to CB(X)$ and $g : X \to X$ be two mappings. Suppose that there exist some $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\begin{aligned} & \psi\left(H(F(x,y,z),F(u,v,w))\right) \\ \leq\ & \psi\left(M(x,y,z,u,v,w)\right) - \varphi\left(\psi\left(M(x,y,z,u,v,w)\right)\right), \end{aligned} \tag{2.10}$$

for all $x, y, z, u, v, w \in X$. Furthermore, assume that $\{F, g\}$ satisfies the (EA) property. Then F and g have a tripled coincidence point. Moreover, F and g have a common tripled fixed point, if one of the conditions (a) to (d) of Theorem 2.1 holds.

If we put $\varphi(t) = t - t\widetilde{\varphi}(t)$ for all $t \geq 0$ in Corollary 2.2, then we get the following result:

Corollary 2.3. Let (X, d) be a metric space, $F : X \times X \times X \to CB(X)$ and $g : X \to X$ be two mappings. Suppose that there exist some $\psi \in \Psi$ and $\widetilde{\varphi} \in \Phi$ such that

$$\psi\left(H(F(x,y,z),F(u,v,w))\right) \leq \widetilde{\varphi}\left(\psi\left(M(x,y,z,u,v,w)\right)\right)\psi\left(M(x,y,z,u,v,w)\right), \tag{2.11}$$

for all $x, y, z, u, v, w \in X$. Furthermore, assume that $\{F, g\}$ satisfies the (EA) property. Then F and g have a tripled coincidence point. Moreover, F and g have a common tripled fixed point, if one of the conditions (a) to (d) of Theorem 2.1 holds.

If we put $\psi(t) = 2t$ for all $t \geq 0$ in Corollary 2.3, then we get the following result:

Corollary 2.4. Let (X, d) be a metric space, $F : X \times X \times X \to CB(X)$ and $g : X \to X$ be two mappings. Suppose that there exists some $\widetilde{\varphi} \in \Phi$ such that

$$H(F(x,y,z),F(u,v,w)) \leq \widetilde{\varphi}\left(2M(x,y,z,u,v,w)\right)2M(x,y,z,u,v,w), \tag{2.12}$$

for all $x, y, z, u, v, w \in X$. Furthermore, assume that $\{F, g\}$ satisfies the (EA) property. Then F and g have a tripled coincidence point. Moreover, F and g have a common tripled fixed point, if one of the conditions (a) to (d) of Theorem 2.1 holds.

If we put $\widetilde{\varphi}(t) = \frac{k}{2}$ where $0 < k < 1$, for all $t \geq 0$ in Corollary 2.4, then we get the following result:

Corollary 2.5. Let (X, d) be a metric space. Assume $F : X \times X \times X \to CB(X)$ and $g : X \to X$ be two mappings satisfying

$$H(F(x,y,z),F(u,v,w)) \leq kM(x,y,z,u,v,w), \tag{2.13}$$

for all $x, y, z, u, v, w \in X$, where $0 < k < 1$. Furthermore, assume that $\{F, g\}$ satisfies the (EA) property. Then F and g have a tripled coincidence point.

Moreover, F and g have a common tripled fixed point, if one of the conditions (a) to (d) of Theorem 2.1 holds.

Theorem 2.6. Let $(X,\, d)$ be a metric space, $F : X \times X \times X \to CB(X)$ and $g : X \to X$ be two mappings. Suppose that there exist some $\psi \in \Psi$, $\varphi \in \Phi$ and $\theta \in \Theta$ satisfying (2.1) and $\{F,\, g\}$ is occasionally $w-$compatible. Then F and g have a common tripled fixed point.

Proof. Since the pair $\{F,\, g\}$ is occasionally $w-$compatible, therefore there exists some point $(x, y, z) \in X \times X \times X$ such that

$$gx \in F(x, y, z),\;\; gy \in F(y, z, x), gz \in F(z, x, y) \text{ and } gF(x, y, z) \subseteq F(gx, gy, gz). \tag{2.14}$$

It follows that

$$g^2x \in F(gx, gy, gz), g^2y \in F(gy, gz, gx) \text{ and } g^2z \in F(gz, gx, gy). \tag{2.15}$$

Now, suppose $u = gx, v = gy$ and $w = gz$, then by (2.15), we get

$$gu \in F(u, v, w), gv \in F(v, w, u) \text{ and } gw \in F(w, u, v). \tag{2.16}$$

Thus, by condition (2.1), we have

$$\begin{aligned} & \psi\left(H(F(x, y, z), F(u, v, w))\right) \\ \leq\;& \psi\left(M(x, y, z, u, v, w)\right) - \varphi\left(\psi\left(M(x, y, z, u, v, w)\right)\right) + \theta\left(N(x, y, z, u, v, w)\right). \end{aligned}$$

which, by (2.14), (2.16), (i_ψ), (i_φ), (i_θ), (ii_θ), and triangle inequality, implies

$$\begin{aligned} & \psi\left(d(gx, gu)\right) \\ \leq\;& \psi\left(\max\left\{d(gx, gu), d(gy, gv), d\left(gz, gw\right)\right\}\right) - \varphi\left(\psi\left(\max\left\{d(gx, gu), d(gy, gv), d\left(gz, gw\right)\right\}\right)\right). \end{aligned}$$

Similarly, we can obtain that

$$\begin{aligned} & \psi\left(d(gy, gv)\right) \\ \leq\;& \psi\left(\max\left\{d(gx, gu), d(gy, gv), d\left(gz, gw\right)\right\}\right) - \varphi\left(\psi\left(\max\left\{d(gx, gu), d(gy, gv), d\left(gz, gw\right)\right\}\right)\right). \end{aligned}$$

and

$$\begin{aligned} & \psi\left(d(gz, gw)\right) \\ \leq\;& \psi\left(\max\left\{d(gx, gu), d(gy, gv), d\left(gz, gw\right)\right\}\right) - \varphi\left(\psi\left(\max\left\{d(gx, gu), d(gy, gv), d\left(gz, gw\right)\right\}\right)\right). \end{aligned}$$

Combining them, we get

$$\begin{aligned} & \max\left\{\psi\left(d(gx, gu)\right), \psi\left(d(gy,\;\; gv)\right), \psi\left(d(gz,\;\; gw)\right)\right\} \\ \leq\;& \psi\left(\max\left\{d(gx, gu), d(gy, gv), d\left(gz, gw\right)\right\}\right) - \varphi\left(\psi\left(\max\left\{d(gx, gu), d(gy, gv), d\left(gz, gw\right)\right\}\right)\right). \end{aligned}$$

Since ψ is non-decreasing, therefore

$$\begin{aligned} & \psi\left(\max\left\{d(gx, gu), (gy, gv), (gz, gw)\right\}\right) \\ \leq\;& \psi\left(\max\left\{d(gx, gu), d(gy, gv), d\left(gz, gw\right)\right\}\right) - \varphi\left(\psi\left(\max\left\{d(gx, gu), d(gy, gv), d\left(gz, gw\right)\right\}\right)\right), \end{aligned}$$

which, by (ii_{φ}) and (ii_{ψ}), implies that

$$\max\{d(gx,gu),d(gy,gv),d(gz,gw)\}=0,$$

it follows that $d(gx,gu)=d(gy,gv)=d(gz,gw)=0$. Hence

$$u=gx=gu, v=gy=gv \text{ and } w=gz=gw. \quad (2.17)$$

Thus, by (2.16) and (2.17), we get

$$u=gu\in F(u,v,w), v=gv\in F(v,w,u) \text{ and } w=gw\in F(w,u,v),$$

that is, (u,v,w) is a common tripled fixed point of F and g.

Example 2.3. Suppose that $X=[0,\ 1]$, equipped with the metric $d: X\times X\to[0,\ +\infty)$ defined as $d(x,y)=\max\{x,y\}$ and $d(x,x)=0$ for all $x,y,\in X$. Let $F: X\times X\times X\to CB(X)$ be defined as

$$F(x,y,z)=\begin{cases}\{0\}, & \text{for } x,y,z=1,\\ \left[0,\ \frac{x+y+z}{6}\right], & \text{for } x,y,z\in[0,\ 1).\end{cases}$$

and $g: X\to X$ be defined as

$$gx=\frac{x}{2} \text{ for all } x\in X.$$

Define $\psi:[0,\ +\infty)\to[0,\ +\infty)$ by

$$\psi(t)=\frac{t}{2}, \text{ for all } t\geq 0,$$

and $\varphi:[0,\ +\infty)\to[0,\ +\infty)$ by

$$\varphi(t)=\frac{t}{3}, \text{ for all } t\geq 0,$$

and $\theta:[0,\ +\infty)\to[0,\ +\infty)$ by

$$\theta(t)=\frac{t}{4}, \text{ for all } t\geq 0.$$

Now, for all $x,y,z,u,v,w\in X$ with $x,y,z,u,v,w\in[0,\ 1)$, we have

Case (a). If $x+y+z=u+v+w$, then

$$\begin{aligned}
&\psi(H(F(x,y,z),F(u,v,w)))\\
=\ &\frac{1}{2}H(F(x,y,z),F(u,v,w))\\
=\ &\frac{1}{12}(u+v+w)\\
\leq\ &\frac{1}{6}\max\left\{\frac{x}{2},\frac{u}{2}\right\}+\frac{1}{6}\max\left\{\frac{y}{2},\frac{v}{2}\right\}+\frac{1}{6}\max\left\{\frac{z}{2},\frac{w}{2}\right\}\\
\leq\ &\frac{1}{6}d(gx,gu)+\frac{1}{6}d(gy,gv)+\frac{1}{6}d(gz,gw)\\
\leq\ &\frac{1}{3}M(x,y,z,u,v,w)\\
\leq\ &\psi(M(x,y,z,u,v,w))-\varphi(\psi(M(x,y,z,u,v,w)))\\
\leq\ &\psi(M(x,y,z,u,v,w))-\varphi(\psi(M(x,y,z,u,v,w)))+\theta(N(x,y,z,u,v,w)).
\end{aligned}$$

Case (b). If $x+y+z \neq u+v+w$ with $x+y+z < u+v+w$, then

$$
\begin{aligned}
& \psi\left(H(F(x,y,z), F(u,v,w))\right) \\
= \; & \frac{1}{2} H(F(x,y,z), F(u,v,w)) \\
= \; & \frac{1}{12}(u+v+w) \\
\leq \; & \frac{1}{6}\max\left\{\frac{x}{2}, \frac{u}{2}\right\} + \frac{1}{6}\max\left\{\frac{y}{2}, \frac{v}{2}\right\} + \frac{1}{6}\max\left\{\frac{z}{2}, \frac{w}{2}\right\} \\
\leq \; & \frac{1}{6} d(gx, gu) + \frac{1}{6} d(gy, gv) + \frac{1}{6} d(gz, gw) \\
\leq \; & \frac{1}{3} M(x,y,z,u,v,w) \\
\leq \; & \psi\left(M(x,y,z,u,v,w)\right) - \varphi\left(\psi\left(M(x,y,z,u,v,w)\right)\right) \\
\leq \; & \psi\left(M(x,y,z,u,v,w)\right) - \varphi\left(\psi\left(M(x,y,z,u,v,w)\right)\right) + \theta\left(N(x,y,z,u,v,w)\right).
\end{aligned}
$$

Similarly, we obtain the same result for $u+v+w < x+y+z$. Thus the contractive condition (2.1) is satisfied for all $x,y,z,u,v,w \in X$ with $x,y,z,u,v,w \in [0,\ 1)$. Again, for all $x,y,z,u,v,w \in X$ with $x,y,z \in [0,\ 1)$ and $u,v,w = 1$, we have

$$
\begin{aligned}
& \psi\left(H(F(x,y,z), F(u,v,w))\right) \\
= \; & \frac{1}{2} H(F(x,y,z), F(u,v,w)) \\
= \; & \frac{1}{12}(x+y+z) \\
\leq \; & \frac{1}{6}\max\left\{\frac{x}{2}, \frac{u}{2}\right\} + \frac{1}{6}\max\left\{\frac{y}{2}, \frac{v}{2}\right\} + \frac{1}{6}\max\left\{\frac{z}{2}, \frac{w}{2}\right\} \\
\leq \; & \frac{1}{6} d(gx, gu) + \frac{1}{6} d(gy, gv) + \frac{1}{6} d(gz, gw) \\
\leq \; & \frac{1}{3} M(x,y,z,u,v,w) \\
\leq \; & \psi\left(M(x,y,z,u,v,w)\right) - \varphi\left(\psi\left(M(x,y,z,u,v,w)\right)\right) \\
\leq \; & \psi\left(M(x,y,z,u,v,w)\right) - \varphi\left(\psi\left(M(x,y,z,u,v,w)\right)\right) + \theta\left(N(x,y,z,u,v,w)\right).
\end{aligned}
$$

Thus the contractive condition (2.1) is satisfied for all $x,y,z,u,v,w \in X$ with $x,y,z \in [0,\ 1)$ and $u,v,w = 1$. Similarly, we can see that the contractive condition (2.1) is satisfied for all $x,y,z,u,v,w \in X$ with $x,y,z,u,v,w = 1$. Hence, the hybrid pair $\{F,\ g\}$ satisfies the contractive condition (2.1), for all $x,y,z,u,v,w \in X$. In addition, all the other conditions of Theorem 2.1 and Theorem 2.6 are satisfied and $z = (0,0,0)$ is a common tripled fixed point of hybrid pair $\{F,\ g\}$. The function $F : X \times X \to CB(X)$ involved in this example is not continuous at the point $(1,1,1) \in X \times X \times X$.

References

[1] M. Aamri and D. ElMoutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl. 270 (1) (2002), 181–188.

[2] M. Abbas and B. E. Rhoades, Common fixed point theorems for hybrid pairs of occasionally weakly compatible mappings satisfying generalized contractive condition of integral type, Fixed Point Theory Appl. Volume 2007, Article ID 54101, 9 pages.

[3] S. M. Alsulami & A. Alotaibi, Tripled coincidence points for monotone operators in partially ordered metric spaces. International Mathematical Forum 7 (2012), no. 37, 1811-1824.

[4] I. Altun, A common fixed point theorem for multivalued Círíc type mappings with new type compatibility, An. St. Univ. Ovidius Constanta., 17(2), (2009), 19 - 26.

[5] H. Aydi, E. Karapinar & M. Postolache, Tripled coincidence point theorems for weak φ-contractions in partially ordered metric spaces. Fixed Point Theory Appl. doi:10.1186/1687-1812-2012-44 (2012).

[6] H. Aydi & E. Karapinar, Triple fixed points in ordered metric spaces. Bulletin of Mathematical Analysis and Applications 4 (2012), no. 1, 197-207.

[7] ——, New Meir-Keeler type tripled fixed point theorems on partially ordered metric spaces. Hindawi publishing corporation Mathematical Problems in Engineering Volume 2012, Article ID 409872, 17 pages.

[8] H. Aydi, E. Karapinar & C. Vetro, Meir-Keeler type contractions for tripled fixed points. Acta Mathematica Scientia 6 (2012), 2119-2130.

[9] V. Berinde and M. Borcut, Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces, Nonlinear Analysis, 74(15), (2011), 4889 - 4897.

[10] V. Berinde & M. Borcut, Tripled coincidence theorems of contractive type mappings in partially ordered metric spaces. Applied Mathematics and Computation 218 (2012), no. 10, 5929-5936.

[11] T. G. Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65 (7) (2006), 1379-1393.

[12] P. Charoensawan, Tripled fixed points theorems of φ-contractive mixed monotone operators on partially ordered metric spaces. Applied Mathematical Sciences 6 (2012), no. 105, 5229 - 5239.

[13] L. Ciric, B. Damjanovic, M. Jleli and B. Samet, Coupled fixed point theorems for generalized Mizoguchi-Takahashi contractions with applications, Fixed Point Theory Appl. 2012, 2012:51.

[14] B. Deshpande and S. Chouhan, Fixed points for two hybrid pairs of mappings satisfying some weaker conditions on noncomplete metric spaces, Southeast Asian Bull. Math. 35 (2011), 851-858.

[15] B. Deshpande and A. Handa, Common coupled fixed point theorems for two hybrid pairs of mappings under $\psi - \varphi$ contraction, Hindawi Publishing Corporation International Scholarly Research Notices Volume 2014, Article ID 608725, 10 pages.

[16] B. Deshpande, S. Sharma, and A. Handa, Common coupled fixed point theorems for nonlinear contractive condition on intuitionistic fuzzy metric spaces with application to integral equations, Journal of the Korean Society of Mathematical Education. Series B. The Pure and Applied Mathematics, (20)(3)(2013), 159–180.

[17] B. Deshpande, S. Sharma and A. Handa, Tripled fixed point theorem for hybrid pair of mappings under generalized nonlinear contraction, J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. Volume 21, Number 1 (February 2014), Pages 23-38.

[18] H. S. Ding, L. Li and S. Radenovic, Coupled coincidence point theorems for generalized nonlinear contraction in partially ordered metric spaces, Fixed Point Theory Appl. 2012, 2012:96.

[19] M. E. Gordji, H. Baghani and G. H. Kim, Common fixed point theorems for $(\psi, \varphi)-$weak nonlinear contraction in partially ordered sets, Fixed Point Theory Appl. 2012, 2012:62.

[20] T. Kamran, Coincidence and fixed points for hybrid strict contractions, J. Math. Anal. Appl. 299 (1) (2004), 235–241.

[21] V. Lakshmikantham and L. Ciric, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. 70 (12) (2009), 4341-4349.

[22] N. Mizoguchi and W. Takahashi, Fixed point theorems for multivalued mappings on complete metric spaces, J. Math. Anal. Appl., 141 (1989), 177 - 188.

[23] S. B. Jr. Nadler, Multivalued contraction mappings, Pacific J. Math., XXX (1969), 475 - 488.

[24] K. P. R. Rao, G. N. V. Kishore and K. Tas, A unique common triple fixed point theorem for hybrid pair of mappings, Abstract and Applied Analysis, Volume 2012, Article ID 750403, 9 pages, doi:10.1155/2012/750403.

[25] B. E. Rhoades, A fixed point theorem for a multivalued non - self mapping, Comment. Math. Univ. Carolin., 37 (1996), 401 - 404.

[26] B. Samet and C. Vetro, Coupled fixed point theorems for multivalued nonlinear contraction mappings in partially ordered metric spaces, Nonlinear Analysis, 74 (2011), 4260 - 4268.

[27] Wei - Shih Du, Some generalizations of Mizoguchi - Takahashi's fixed point theorem, Int. J. Contemp. Math. Sci., 3 (2008), 1283 - 1288.