Hamiltonian Laceability of Some Regular Product Graphs

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Abstract — *A simple connected graph G is Hamiltonian laceable if there is a Hamiltonian path connecting each pair of distinct vertices at an odd distance. In this, we discuss the Hamiltonian laceability of some regular product graphs.*

Keywords — *Hamiltonian laceable graph, OR product, AND product and EXOR product.*

I. INTRODUCTION

A Hamilton cycle of a graph is a cycle which passes through each vertex of the graph exactly once and the graph is Hamiltonian, if it contains a Hamilton cycle. A graph G is Hamiltonian laceable if there is a u-v Hamiltonian path for all pair of vertices u and v where d(u,v) is odd. Let G be a finite, simple, connected undirected graph. The graph G is Hamiltonian - k - laceable if there is a u-v Hamiltonian path for all pair of vertices u and v and Hamiltonian – k* - laceable, if there is a u-v Hamiltonian path for at least one pair of distinct vertices u and v with d(u,v) = k, $1 \leq k \leq$ diamG. Let $E^{'}$ be a set of minimum number of edges to be added to G such that $P \cup E^{'}$ is a u-v Hamiltonian path in G where P is a path in G with d(u,v)= k, then $|E^{'}|$ is denoted by $\lambda_{(k)}$ is called the laceability number of G and the edges belong to $E^{'}$ are called laceability edges with respect to u and v

Product Graph:

Definition: For two graphs G and H , if (a,b)∈G and $(a^{'}b^{'})$∈H such that there exists adjacency between
*(a∼a') and (b∼$b^{'}$), then it is AND product(∧)
*(a∼a') or (b∼$b^{'}$), then it is OR product (∨)
*(a∼a') or (b∼$b^{'}$), but not both ,then it is EX OR product ($\veebar$).

Lemma 1: Graph G= $K_2 \wedge C_n$ (n≥ 3), for odd n is Hamiltonian-1- laceable .

Proof: Let G=$K_2 \wedge C_n$ and U=$\{u_1, u_2, \dots . u_n\}$ and V=$\{v_1, v_2, \dots . v_n\}$ be the two parties of vertex set of G .By induction on $|V|$= p(say),

Step i: G is Hamiltonian laceable for p=6.

Step ii : For p = k where k is any arbitrary constant, following are the possible cases and there is a u-v Hamiltonian path for any pair of adjacent vertices u and v.

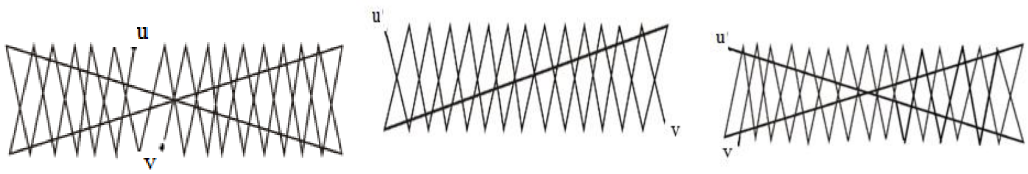


Fig.1.

Step iii: using step (ii) the path can be extended to the graph G for integer n≥ 3.

Lemma 2: Graph G= $K_2 \wedge C_n$ (n≥ 3) for an odd n, is Hamiltonian-d- laceable for distance $3 \leq d \leq (n-2)$ with $\leftthreetimes_{(d)} = 2$

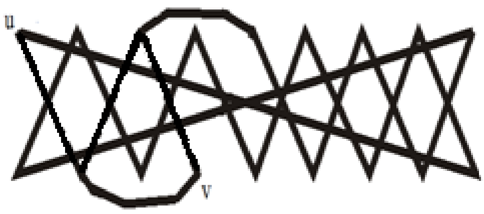
Proof: Let G= $K_2 \wedge C_n$, and U=$\{u_1, u_2, \ldots . u_n\}$ and V=$\{v_1, v_2, \ldots . v_n\}$ be the two parties of vertex set of G .

We prove the result by induction on $|V|$= p(say),

Case I : **G is Hamiltonian -3- laceable**.

Step i: G is Hamiltonian laceable for p=10

Step ii : For p = k where k is any arbitrary constant, following are the possible cases and there is a Hamiltonian path between every pair of vertices.



Case i:

Case ii:

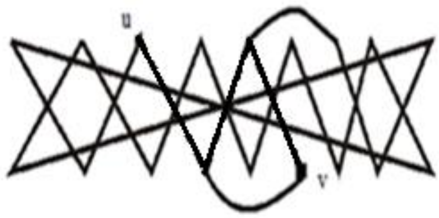


Fig.2.

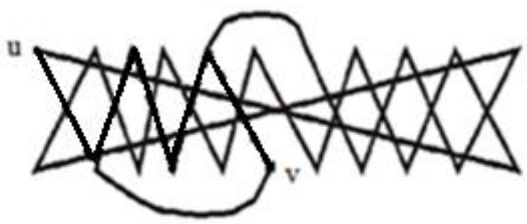
Step iii: using step (ii) the path can be extended to a graph G for n≥ 3.Therefore G is Hamiltonian -3- laceable.

Case II : G is Ham iltonian -5- laceable.

Step i: G is Hamiltonian laceable for p=14.

Step ii : For p = k where k is any arbitrary costant, following are the possible cases and there is a Hamiltonian path between every pair of vertices.

Case i:



Case ii:

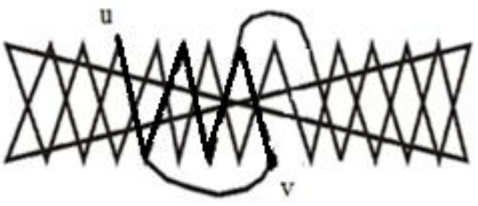


Fig.3.

Step iii: using step (ii) the path can be extended to a graph G for (n≥ 3).Therefore G is Hamiltonian -5-laceable.

Hence for $(u_i, v_{i+(n-2)})$, the Hamiltonian path exists with laceabilty edges $(u_{i+(n-3)}, u_{i+(n-1)})$ $and(v_{i+1}, v_{i+(n-2)})$.

Lemma 3: Graph G= $K_2 \wedge C_n$ (n≥ 3) for an odd n, is Hamiltonian-n- laceable with $\curlywedge_{(n)} = 1$

Proof: Let G= $K_2 \wedge C_n$, and U=$\{u_1, u_2, \dots . u_n\}$ and V=$\{v_1, v_2, \dots . v_n\}$ be the two parties of vertex set of G . For any two vertices (u,v) at odd distance n, there **is** a Hamiltonian path with a laceability edge. Following are the two possible cases.

Case i:

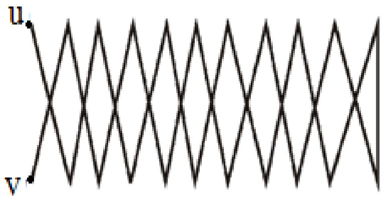


Fig.4.

Case ii:

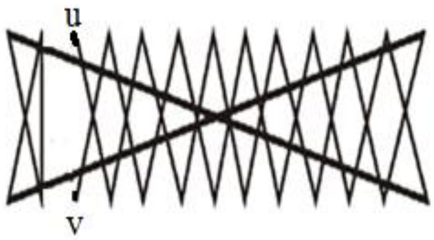


Fig.5.

Hence the result.

Theorem 1: The graph $K_2 \wedge C_n$(n ≥ 3) for odd n is

i. Hamiltonian-1- laceable,

ii. Hamiltonian-d- laceable with $\curlywedge_{(d)} = 2$

iii. Hamiltonian-n- laceable with $\curlywedge_{(n)} = 1$

The proof follows from lemma 1,2 and 3.

Lemma 4: A graph G= $K_2 \wedge C_n$ (n≥ 4), for an even n, is Hamiltonian-1- laceable with $\leftthreetimes_{(1)} = 2$.

Proof: Let G= $K_2 \wedge C_n$, and U=$\{u_1, u_2, \dots . u_n\}$ and V=$\{v_1, v_2, \dots . v_n\}$ be the two parties of vertex set of G . By induction on $|V|$= p(say),

Step i: G is Hamiltonian laceable for p=8.

Step ii : For p = k where k is any arbitrary constant, following are the possible cases and there is a u-v Hamiltonian path for any pair of adjacent vertices u and v.

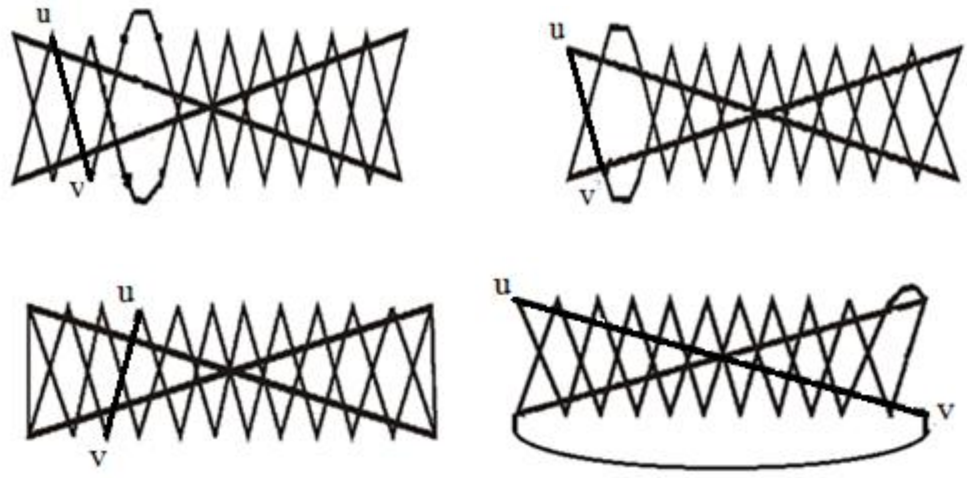


Fig.6.

Step iii: using step (ii) the path can be extended to a graph G for n≥ 4.

Lemma 5: Graph G= $K_2 \wedge C_n$ (n≥ 4) for an even n, is Hamiltonian-d- laceable for distance 3 ≤ d ≤ n/2 with $\leftthreetimes_{(d)} = 2$.

Proof: Let G= $K_2 \wedge C_n$ and U=$\{u_1, u_2, \dots . u_n\}$ and V=$\{v_1, v_2, \dots . v_n\}$ be the two parties of vertex set of G .We prove the result by induction on $|V|$= p(say),

Case I : G is Hamiltonian -3- laceable.

Step i: G is Hamiltonian laceable for p=12.

Step ii : For p = k where k is any arbitrary constant, following are the possible cases and there is a Hamiltonian path between every pair of vertices .

Case i:

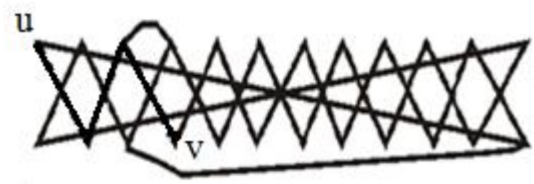


Fig.7.

Case ii:

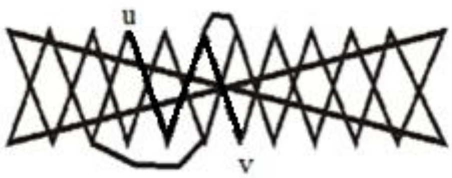


Fig.8.

Step iii: using step (ii) the path can be extended to a graph G for n$\geq$ 3.

Case II : G is Hamiltonian -5- laceable.

Step i: G is Hamiltonian laceable for p=16.

Step ii : For p = k the following are the possible cases and there is a Hamiltonian path between every pair of vertices.

Case i:

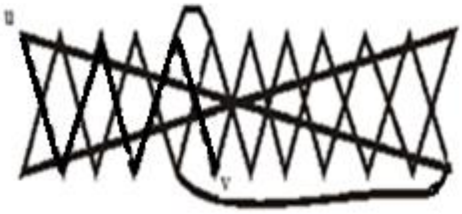


Fig .9.

case ii:

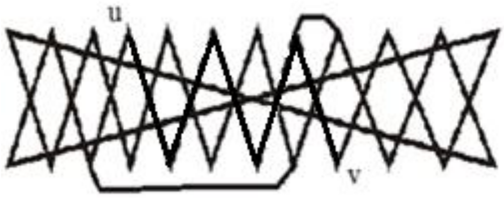


Fig.10.

Step iii: using step (ii) the path can be extended to a graph G for n$\geq$ 4.

Hence for $(u_1, v_{1+n/2})$, the Hamiltonian path exists with laceabilty edges $(u_{i+n/2-1}, u_{i+n/2})$ $and(v_{i+n/2-1}, v_n)$ and for $(u_i, v_{i+n/2})$, where $i \geq 2$, the Hamiltonian path exists with laceabilty edges $(u_{i+n/2-1}, u_{i+n/2})$ $and(v_{i+n/2-1}, v_{i-1})$.

Theorem 2: The graph $K_2 \wedge C_n$(n $\geq$ 4), for even n is Hamiltonian laceable with $\curlywedge_{(d)} = 2$ for $1 \leq d \leq diam\, G.$

The proof follows from lemma 4 and 5.

Theorem 3 : The graph $K_2 \vee C_n$(n$\geq$ 3) is Hamiltonian laceable.

Proof: Let G= $K_2 \vee C_n$ (n $\geq$ 3) , with vertex set V={ $u_1, u_2, \dots . u_n, v_1, v_2, \dots . v_n$ } and an edge set E={$(u_1 v_2), (u_2 v_3)$, $(u_3 v_4), \dots . (u_1 v_n)$, $(v_1 u_n)$ ... . }. We prove the result by induction on $|V|$=p(say).

Step i: G is Hamiltonian laceable for p=6.

Step ii : For p = k**,** the following are the possible cases and there is a u-v Hamiltonian path for any pair of vertices at odd distance.

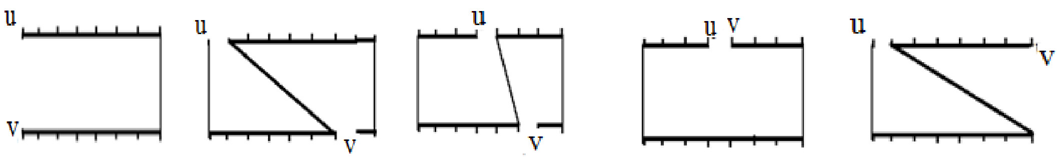


Fig.11.

Step iii: using step (ii) the path can be extended to a graph G for n$\geq$ *3.*

Hence the theorem.

Theorem 4: The graph $K_n \vee C_n (n \geq 3)$ is Hamiltonian laceable .

Proof: Let G = $K_n \vee C_n$ (n≥ 3). The graph G (V,E) has a Vertex Set

V= $\{u_1, u_2, \ldots u_n, v_1, v_2, \ldots . v_n, w_1, w_2, \ldots . w_n, \ldots \ldots . n_1, n_2, \ldots . n_n\}$

And an Edgeset E=$\{(u_1v_2), (u_2v_3), (u_3v_4), \ldots . (u_1v_n), (v_1u_n) \ldots \ldots (n_1n_2)(n_2n_3)..\}$

We prove the result by induction on $|V|$=p(say).

Step i: G is Hamiltonian laceable for p=9.

Step ii : For p = k where k is any arbitrary constant, following are the possible cases and there is a u-v Hamiltonian path for any pair of vertices u and v.

Case I: For odd n

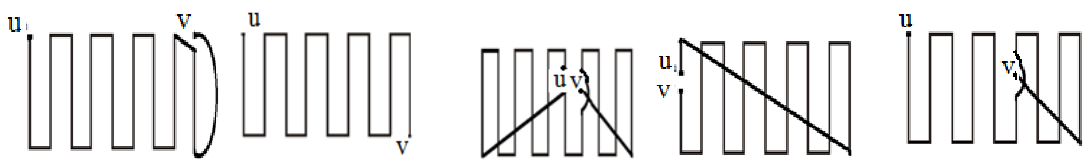


Fig.12.

Case II: For an even n.

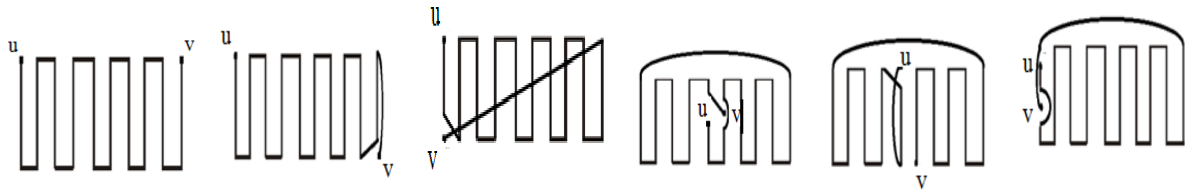


Fig.13.

Step iii: using step (ii) the path can be extended to a graph G for n≥ 3.

Hence the theorem.

Theorem 5: The graph $K_2 \underline{\vee} C_n$(n≥ 3) is Hamiltonian laceable.

Proof: Let G = $K_2 \underline{\vee} C$, with vertex set V={ $u_1, u_2, \ldots . u_n, v_1, v_2, \ldots . v_n$ } and an edge set E=$\{(u_1v_2), (u_2v_3), (u_3v_4), \ldots . (u_1v_n), (v_1u_n) \ldots \ldots ..\}$. We prove the result by induction on $|V|$=p(say).

Step i: G is Hamiltonian laceable for p=6.

Step ii : For p = k where k is any arbitrary constant, following are the possible cases and there is a u-v Hamiltonian path for any pair of vertices u and v.

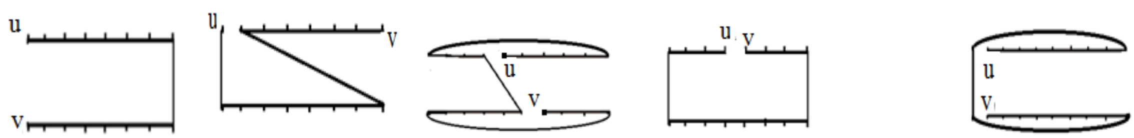


Fig.14.

Step iii: using step (ii) the path can be extended to a graph G for n≥ 3.

Hence the thorem.

Theorem 6: The graph $K_n \underline{\vee} C_n (n \geq 3)$ is Hamiltonian laceable .

Proof: Consider a graph G =$K_n \underline{\vee} C_n$ $(n \geq 3)$, here the graph G (V,E) has a vertex set V= $\{u_1, u_2, \ldots u_n, v_1, v_2, \ldots . v_n, w_1, w_2, \ldots . w_n, \ldots \ldots . n_1, n_2, \ldots . n_n\}$ and an edge set E=$\{(u_1v_2), (u_2v_3), (u_3v_4), \ldots . (u_1v_n), (v_1u_n) \ldots \ldots (n_1n_2)(n_2n_3)..\}$

We prove the result by induction on $|V|$=p(say).
Step i: G is Hamiltonian laceable for p=9.
Step ii : For p = k where k is any arbitrary constant, following are the possible cases and there is a u-v Hamiltonian path for any pair of vertices.

Case I: For odd n

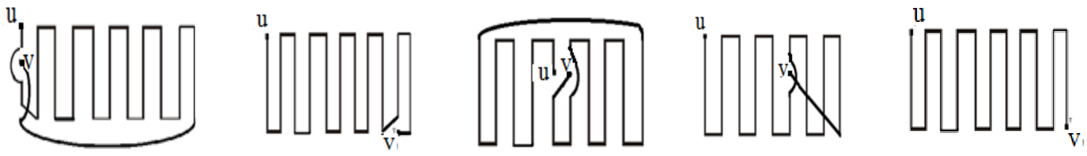


Fig.15.

Case II: For even n

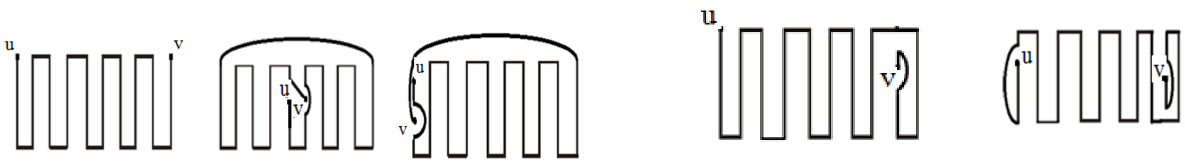


Fig.16.

Step iii: using step (ii) the path can be extended to a graph G for $n \geq 3$.

Hence the thorem.

CONCLUSIONS:

In this paper we have proved that the graph $K_2 \wedge C_n$($n \geq 3$) is Hamiltonian-k- laceable, where $1 \leq k \leq$ diamG with $\lambda_{(k)}$ =1,2 for odd n. The graph $K_2 \wedge C_n$($n \geq 4$) is Hamiltonian laceable for even n with $\lambda_{(d)} = 2$ for $1 \leq d \leq diam\, G$. The graph**s** $K_2 \vee C_n$($n \geq 3$), $K_n \vee C_n$($n \geq 3$), $K_2 \underline{\vee} C_n$($n \geq 3$) and $K_n \underline{\vee} C_n$($n \geq 3$) are Hamiltonian laceable.

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