

Existence of Best Proximity Points for Generalized $(\alpha-\eta)$ – Rational Proximal Contraction

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Abstract - In this work, we introduce the concept of generalized α - η - rational proximal contraction of first and second kind. Then we establish some best proximity theorems for such kind of contraction in the framework of metric spaces. The presented results generalize and improve several existing results in the best proximity theory.

Keywords - optimal approximate solution; best proximity point; generalized rational proximal contraction; fixed point.

1. INTRODUCTION

Fixed point theory is an important tool for solving equations $Tx = x$ for mappings T defined on subsets of metric spaces or normed spaces. Banach contraction principle is first fixed point theorem which guarantees that every contractive mapping in a complete metric space has a unique fixed point. Several authors have generalized this celebrated result in different directions and most of the generalization expatiates the existence of a fixed point for self-mappings. However, if T is a non-self mappings, then it is apparent that the equation $Tx = x$ has no solution. In such situations, it may be speculated to determine an element x that is in some sense closest to Tx . In fact, best approximation theorems and best proximity point theorems are suitable to be explored in this direction.

The classical best approximation theorem was introduced by Fan[1], and he ensure that if K is a nonempty, compact and convex subset of a normed space E , then for any continuous mapping $T: K \rightarrow E$ there exists an element x in K such that $d(x, Tx) = d(Tx, K)$. Afterward, several authors including Prolla [2], Reich [3], Shegal and Singh [4, 5] have derived the extensions of this theorem in various directions.

Let A and B be two nonempty subsets of a metric space (X, d) . A best proximity point of a non self mapping $T: A \rightarrow B$ is a point x in A such that $d(x, Tx) = d(A, B)$, where $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$.

Despite the fact that the best approximate theorems ensure the existence of approximate solution to the equation $Tx = x$, such results may not afford an approximate solution that is optimal. In other sense, best proximity point theorems offer an approximate solution that is optimal. Further, it can be observe that best proximity point theorems emerge as a natural generalization of fixed point theorems, as best proximity point theorem can be reduces to fixed point theorem when the mapping under consideration is a self- mapping. The works on the existence of best proximity point theorems for several variants of contraction can be found in [1-16]. Recently, Nashine et al. [11] ensure the existence and uniqueness of best proximity point for rational proximal contraction of first and second kind and generalize the several existing result of best proximity theory.

In this work, we introduce the notion of generalized α - η - rational proximal contraction of first and second kind. Then we establish certain best proximity theorems for such rational proximal contractions. The presented results generalize and improve various known results of best proximity theory. In particular, the presented results properly contains the results of Hussain et al [13], Nashine et al. [11] and Basha and Shahzad[12].

II. PRELIMINARIES

In this section, we give some basic notations and definitions that will be used in the sequel.

Let (X, d) be a metric space. Then for given nonempty subsets A and B , we define A_0 and B_0 as follows

$$A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},$$

$$B_0 = \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\},$$

where $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. If A and B are closed subsets of a normed linear space such that $d(A, B) > 0$, then A_0 and B_0 are contained in the boundaries of A and B respectively (see [16]). Also if $A \cap B \neq \emptyset$, then A_0 and B_0 are nonempty.

Definition 2.1 (see [13]) Let $T: A \rightarrow B$ and $\alpha, \eta : A \times A \rightarrow [0, \infty)$ be functions. Then T is α - proximal admissible with respect to η if the condition

$$\alpha(x, y) \geq \eta(x, y), \quad d(u, Tx) = d(A, B) \text{ and } d(v, Ty) = d(A, B)$$

imply that

$$\alpha(u, v) \geq \eta(u, v),$$

for all $x, y, u, v \in A$.

Definition 2.2 (see [13]) Let A and B be nonempty subsets of a metric space (X, d) . Then $T: A \rightarrow B$ is said to be generalized α - η - rational proximal contraction of first kind if there exist nonnegative real numbers a, b, c and d with $a + b + 2c + 2d < 1$, such that the condition

$$\alpha(x, y) \geq \eta(x, y), \quad d(u, Tx) = d(A, B) \text{ and } d(v, Ty) = d(A, B)$$

imply the inequality that

$$d(u, v) \leq ad(x, y) + b \frac{[1 + d(x, u)]d(y, v)}{1 + d(x, y)} + c[d(x, u) + d(y, v)] + d[d(x, v) + d(y, u)]$$

for all $x, y, u, v \in A$.

Definition 2.3 (see [13]) Let A and B be nonempty subsets of a metric space (X, d) . Then $T: A \rightarrow B$ is said to be generalized α - η - rational proximal contraction of second kind if there exist nonnegative real numbers a, b, c and d with $a + b + 2c + 2d < 1$, such that the condition

$$\alpha(x, y) \geq \eta(x, y), \quad d(u, Tx) = d(A, B) \text{ and } d(v, Ty) = d(A, B)$$

imply the inequality that

$$d(Tu, Tv) \leq ad(Tx, Ty) + b \frac{[1 + d(Tx, Tu)]d(Ty, Tv)}{1 + d(Tx, Ty)} + c[d(Tx, Tu) + d(Ty, Tv)] + d[d(Tx, Tv) + d(Ty, Tu)]$$

for all $x, y, u, v \in A$.

Definition 2.4 Let (X, d) be a metric space and let A and B be two nonempty subset of X . Then B is said to be approximatively compact with respect to A if every sequence $\{y_n\}$ of B satisfying the condition that $d(x, y_n) \rightarrow d(x, B)$ for some x in A has a convergent subsequence.

III. MAIN RESULTS

In this section, we introduce a new class of rational contraction, the so called generalized α - η - rational proximal contraction and prove the best proximity theorems for this class.

Definition 3.1 Let A and B be nonempty subsets of a metric space (X, d) . Then $T: A \rightarrow B$ is said to be generalized α - η - rational proximal contraction of first kind if there exist mappings $k_i: A \rightarrow [0,1)$ for $i = 1, 2, \dots, 6$ with $(k_1 + k_2 + k_3 + k_4 + k_5 + 2k_6)(x) < 1$, such that the conditions

$$\alpha(x, y) \geq \eta(x, y), \quad d(u, Tx) = d(A, B) \text{ and } d(v, Ty) = d(A, B) \text{ imply that}$$

$$d(u, v) \leq k_1(x)d(x, y) + k_2(x) \frac{[1 + d(x, u)]d(y, v)}{1 + d(x, y)} + k_3(x) \frac{[1 + d(x, v)]d(y, u)}{1 + d(x, y)} + k_4(x)d(x, u) + k_5(x)d(y, v) + k_6(x)[d(x, v) + d(y, u)]$$

for all $x, y, u, v \in A$.

Definition 3.2 Let A and B be nonempty subsets of a metric space (X, d) . Then $T: A \rightarrow B$ is said to be generalized α - η - rational proximal contraction of second kind if there exist mappings $k_i: A \rightarrow [0,1)$ for $i = 1, 2, \dots, 6$ with $(k_1 + k_2 + k_3 + k_4 + k_5 + 2k_6)(x) < 1$ such that the conditions

$$\alpha(x, y) \geq \eta(x, y), \quad d(u, Tx) = d(A, B) \text{ and } d(v, Ty) = d(A, B)$$

imply that

$$d(Tu, Tv) \leq k_1(x)d(Tx, Ty) + k_2(x) \frac{[1 + d(Tx, Tu)]d(Ty, Tv)}{1 + d(Tx, Ty)} + k_3(x) \frac{[1 + d(Tx, Tv)]d(Ty, Tu)}{1 + d(Tx, Ty)} + k_4(x)d(Tx, Tu) + k_5(x)d(Ty, Tv) + k_6(x)[d(Tx, Tv) + d(Ty, Tu)]$$

for all $x, y, u, v \in A$.

Remark 3.3: If we define $k_i: A \rightarrow [0,1)$ for $i = 1, 2, \dots, 6$ as

$$k_1(x) = a, \quad k_2(x) = b, \quad k_3(x) = 0, \quad k_4(x) = k_5(x) = c, \quad k_6(x) = d,$$

for all $x \in A$, where a, b, c, d are nonnegative real numbers with $a + b + 2c + 2d < 1$. Then definition 3.1 and definition 3.2 can be reduced to definition 2.2 and definition 2.3 respectively.

Now we prove the following best proximity point result for generalized α - η - rational proximal contraction of first kind.

Theorem 3.4 Let A and B be nonempty closed subsets of a complete metric space (X, d) such that B is approximately compact with respect to A . Also, suppose that A_o and B_o are non-void and $\alpha, \eta: A \times A \rightarrow [0, \infty)$ are functions. Let $T: A \rightarrow B$ is a nonself- mapping satisfy the following conditions

- (a) T is generalized α - η - rational proximal contraction of first kind and also α - proximal admissible with respect to η
- (b) $T(A_o) \subseteq B_o$
- (c) there exist x_0 and x_1 in A_o such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$,
- (d) if $\{x_n\}$ is a sequence in A such that $x_n \rightarrow x$ in A with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$, then $\alpha(x_n, x) \geq \eta(x_n, x)$ for all $n \in \mathbb{N}$.
- (e) For any mapping $k: A \rightarrow [0,1)$, $k(x) \leq k(y)$ whenever $d(x, Ty) = d(A, B)$.

Then there exists $p \in A$ such that $d(p, Tp) = d(A, B)$. Also, if $\alpha(x, y) \geq \eta(x, y)$ for all best proximity point x and y of T , then p is unique best proximity point of T .

Proof From (c) there exist x_0 and x_1 in A_0 such that

$$d(x_1, Tx_0) = d(A, B) \text{ and } \alpha(x_0, x_1) \geq \eta(x_0, x_1)$$

With the fact that $T(A_0) \subseteq B_0$, there is an element x_2 in A_0 satisfying

$$d(x_2, Tx_1) = d(A, B)$$

As, T is α - proximal admissible with respect to η , therefore we have $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$.

Further, since $T(A_0) \subseteq B_0$ and Tx_2 is an element of $T(A_0)$, therefore there is an element x_3 in A_0 such that $d(x_3, Tx_2) = d(A, B)$

Thus,

$$d(x_2, Tx_1) = d(A, B),$$

$$d(x_3, Tx_2) = d(A, B),$$

$$\alpha(x_1, x_2) \geq \eta(x_1, x_2)$$

together with the assumption T is α - proximal admissible with respect to η imply $\alpha(x_2, x_3) \geq \eta(x_2, x_3)$.

Continuing this process, we can construct a sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1}, Tx_n) = d(A, B),$$

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}), \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{3.4.1}$$

Due to the fact that T is a generalized α - η - rational proximal contraction of first kind, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq k_1(x_{n-1})d(x_{n-1}, x_n) + k_2(x_{n-1}) \frac{[1 + d(x_{n-1}, x_n)]d(x_n, x_{n+1})}{1 + d(x_{n-1}, x_n)} \\ &\quad + k_3(x_{n-1}) \frac{[1 + d(x_{n-1}, x_{n+1})]d(x_n, x_n)}{1 + d(x_{n-1}, x_n)} + k_4(x_{n-1})d(x_{n-1}, x_n) \\ &\quad + k_5(x_{n-1})d(x_n, x_{n+1}) + k_6(x_{n-1})[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)] \\ &\leq k_1(x_{n-1})d(x_{n-1}, x_n) + k_2(x_{n-1})d(x_n, x_{n+1}) \\ &\quad + k_4(x_{n-1})d(x_{n-1}, x_n) + k_5(x_{n-1})d(x_n, x_{n+1}) \\ &\quad + k_6(x_{n-1})[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \end{aligned} \tag{3.4.2}$$

Now by using (3.4.2) and (e), we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq k_1(x_{n-2})d(x_{n-1}, x_n) + k_2(x_{n-2})d(x_n, x_{n+1}) \\ &\quad + k_4(x_{n-2})d(x_{n-1}, x_n) + k_5(x_{n-2})d(x_n, x_{n+1}) \\ &\quad + k_6(x_{n-2})[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ &\quad \dots\dots \\ &\leq k_1(x_0)d(x_{n-1}, x_n) + k_2(x_0)d(x_n, x_{n+1}) + k_3(x_0)d(x_{n-1}, x_n) \end{aligned}$$

$$+k_4(x_0)d(x_{n-1}, x_n) + k_5(x_0)d(x_n, x_{n+1})$$

$$+k_6(x_0)[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$$

Therefore, we get

$$d(x_n, x_{n+1}) \leq \frac{k_1(x_0)+k_3(x_0)+k_4(x_0)+k_6(x_0)}{1-[k_2(x_0)+k_5(x_0)+k_6(x_0)]} d(x_{n-1}, x_n)$$

$$= h(x_0)d(x_{n-1}, x_n)$$

$$\leq (h(x_0))^2 d(x_{n-2}, x_{n-1})$$

.....

$$\leq (h(x_0))^n d(x_0, x_1) = h^n d(x_0, x_1)$$

where $h = h(x_0) = \frac{k_1(x_0)+k_3(x_0)+k_4(x_0)+k_6(x_0)}{1-[k_2(x_0)+k_5(x_0)+k_6(x_0)]} \in [0,1)$. Let m and n be two positive integers such that $m \geq n$. Then we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{m-1}, x_m)$$

$$\leq h^n d(x_0, x_1) + h^{n+1} d(x_0, x_1) + \dots + h^{m-1} d(x_0, x_1)$$

$$\leq \frac{h^n}{1-h} d(x_0, x_1) \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ since } h \in [0,1).$$

Hence, $\{x_n\}$ is a Cauchy sequence and since (X, d) is a complete metric space and A is closed, the sequence $\{x_n\}$ converges to some $p \in A$. However, we have

$$d(p, B) \leq d(p, Tx_n)$$

$$\leq d(p, x_{n+1}) + d(x_{n+1}, Tx_n)$$

$$= d(p, x_{n+1}) + d(A, B)$$

$$\leq d(p, x_{n+1}) + d(p, B)$$

Now taking limit as $n \rightarrow \infty$, we get $d(p, Tx_n) \rightarrow d(p, B)$. As B is approximately compact with respect to A , therefore the sequence $\{Tx_n\}$ has a subsequence $\{Tx_{n_k}\}$ that converges to some $y \in B$. Hence,

$$d(p, y) = \lim_{k \rightarrow \infty} d(x_{n_{k+1}}, Tx_{n_k}) = d(A, B)$$

and so p lies in A_0 . Using the fact that $T(A_0) \subseteq B_0$, we have $d(w, Tp) = d(A, B)$ for some $w \in A$. Further (d) imply $\alpha(x_n, p) \geq \eta(x_n, p)$ for all $n \in \mathbb{N}$. Therefore, we proved that

$$\alpha(x_n, p) \geq \eta(x_n, p),$$

$$d(w, Tp) = d(A, B),$$

$$d(x_{n+1}, Tx_n) = d(A, B)$$

for all $n \in \mathbb{N}$. Again, with the fact that T is generalized α - η -rational proximal contraction of first kind, we have

$$d(w, x_{n+1}) \leq k_1(p)d(p, x_n) + k_2(p) \frac{[1 + d(p, w)]d(x_n, x_{n+1})}{1 + d(p, x_n)} + k_3(p) \frac{[1 + d(p, x_{n+1})]d(x_n, w)}{1 + d(p, x_n)}$$

$$+ k_4(p)d(p, w) + k_5(p)d(x_n, x_{n+1}) + k_6(p)[d(p, x_{n+1}) + d(x_n, w)]$$

After letting $n \rightarrow \infty$, we get

$$d(w, p) \leq [k_3(p) + k_4(p) + k_6(p)]d(w, p)$$

Since $k_3(p) + k_4(p) + k_6(p) < 1$ therefore $w = p$. Hence

$$d(p, Tp) = d(w, Tp) = d(A, B).$$

Now we prove the uniqueness of best proximity point, for this assume that p^* is another best proximity point of T such that $\alpha(p, p^*) \geq \eta(p, p^*)$. That is

$$\alpha(p, p^*) \geq \eta(p, p^*),$$

$$d(p, Tp) = d(A, B),$$

$$d(p^*, Tp^*) = d(A, B)$$

As T is generalized α - η - rational proximal contraction of first kind, so we have

$$d(p, p^*) \leq k_1(p)d(p, p^*) + k_2(p) \frac{[1 + d(p, p)]d(p^*, p^*)}{1 + d(p, p^*)} + k_3(p) \frac{[1 + d(p, p^*)]d(p^*, p)}{1 + d(p, p^*)} + k_4(p)d(p, p) + k_5(p)d(p^*, p^*) + k_6(p)[d(p, p^*) + d(p^*, p)]$$

That imply $d(p, p^*) \leq [k_1(p) + k_3(p) + 2k_6(p)]d(p, p^*)$, therefore $p = p^*$.

Now we prove the following best proximity theorem using generalized α - η - rational proximal contraction of second kind.

Theorem 3.5 Let A and B be nonempty closed subsets of a complete metric space (X, d) such that A is approximately compact with respect to B . Also, suppose that A_o and B_o are non-void and $\alpha, \eta: A \times A \rightarrow [0, \infty)$ are functions. Let $T: A \rightarrow B$ is a nonself- mapping satisfy the following conditions

- (a) T is continuous generalized α - η - rational proximal contraction of second kind and also α -proximal admissible with respect to η .
- (b) $T(A_o) \subseteq B_o$
- (c) there exist x_0 and x_1 in A_o such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$,
- (d) For any mapping $k: A \rightarrow [0, 1)$, $k(x) \leq k(y)$ whenever $d(x, Ty) = d(A, B)$.

Then there exists $x \in A$ such that

$$d(x, Tx) = d(A, B)$$

and the sequence $\{x_n\}$ defined by

$$d(x_{n+1}, Tx_n) = d(A, B), \quad n \geq 0$$

converges to the best proximity point x , where x_0 is any fixed element in A_o and also, if $\alpha(x, y) \geq \eta(x, y)$ for all best proximity point x and y of T , then $Tx = Tx^*$ for all best proximity point of x^* of T .

Proof As we have done in theorem 3.4, there exists a sequence $\{x_n\}$ in A_o such that

$$d(x_{n+1}, Tx_n) = d(A, B),$$

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}), \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{3.5.1}$$

Using the fact that T is generalized α - η - rational proximal contraction of second kind, we get

$$\begin{aligned}
 d(Tx_n, Tx_{n+1}) &\leq k_1(x_{n-1})d(Tx_{n-1}, Tx_n) + k_2(x_{n-1}) \frac{[1 + d(Tx_{n-1}, Tx_n)]d(Tx_n, Tx_{n+1})}{1 + d(Tx_{n-1}, Tx_n)} \\
 &\quad + k_3(x_{n-1}) \frac{[1 + d(Tx_{n-1}, Tx_{n+1})]d(Tx_n, Tx_n)}{1 + d(Tx_{n-1}, Tx_n)} + k_4(x_{n-1})d(Tx_{n-1}, Tx_n) \\
 &\quad + k_5(x_{n-1})d(Tx_n, Tx_{n+1}) + k_6(x_{n-1})[d(Tx_{n-1}, Tx_{n+1}) + d(Tx_n, Tx_n)] \\
 &\leq k_1(x_{n-1})d(Tx_{n-1}, Tx_n) + k_2(x_{n-1})d(Tx_n, Tx_{n+1}) \\
 &\quad + k_4(x_{n-1})d(Tx_{n-1}, Tx_n) + k_5(x_{n-1})d(Tx_n, Tx_{n+1}) \\
 &\quad + k_6(x_{n-1})[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})] \tag{3.5.2}
 \end{aligned}$$

Now by using (3.5.2) and (d), we have

$$\begin{aligned}
 d(Tx_n, Tx_{n+1}) &\leq k_1(x_{n-2})d(Tx_{n-1}, Tx_n) + k_2(x_{n-2})d(Tx_n, Tx_{n+1}) \\
 &\quad + k_4(x_{n-2})d(Tx_{n-1}, Tx_n) + k_5(x_{n-2})d(Tx_n, Tx_{n+1}) \\
 &\quad + k_6(x_{n-2})[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})] \\
 &\quad \dots\dots \\
 &\leq k_1(x_0)d(Tx_{n-1}, Tx_n) + k_2(x_0)d(Tx_n, Tx_{n+1}) \\
 &\quad + k_3(x_0)d(Tx_{n-1}, Tx_n) + k_4(x_0)d(Tx_{n-1}, Tx_n) \\
 &\quad + k_5(x_0)d(Tx_n, Tx_{n+1}) \\
 &\quad + k_6(x_0)[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})]
 \end{aligned}$$

Therefore, we get

$$\begin{aligned}
 d(Tx_n, Tx_{n+1}) &\leq \frac{k_1(x_0)+k_3(x_0)+k_4(x_0)+k_6(x_0)}{1-[k_2(x_0)+k_5(x_0)+k_6(x_0)]} d(Tx_{n-1}, Tx_n) \\
 &= h(x_0)d(Tx_{n-1}, Tx_n) \\
 &\leq (h(x_0))^2 d(Tx_{n-2}, Tx_{n-1}) \\
 &\quad \dots\dots\dots \\
 &\leq (h(x_0))^n d(Tx_0, Tx_1) = h^n d(Tx_0, Tx_1)
 \end{aligned}$$

where $h = h(x_0) = \frac{k_1(x_0)+k_3(x_0)+k_4(x_0)+k_6(x_0)}{1-[k_2(x_0)+k_5(x_0)+k_6(x_0)]} \in [0,1]$. Let m and n be two positive integers such that $m \geq n$. Then we have

$$\begin{aligned}
 d(Tx_n, Tx_m) &\leq d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2}) + \dots\dots + d(Tx_{m-1}, Tx_m) \\
 &\leq h^n d(Tx_0, Tx_1) + h^{n+1} d(Tx_0, Tx_1) + \dots\dots h^{m-1} d(Tx_0, Tx_1) \\
 &\leq \frac{h^n}{1-h} d(Tx_0, Tx_1) \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ since } h \in [0,1].
 \end{aligned}$$

Eventually, $\{Tx_n\}$ is a Cauchy sequence and since (X, d) is a complete metric space and B is closed, then the sequence $\{Tx_n\}$ converges to some $y \in B$. Also, we have

$$\begin{aligned} d(y, A) &\leq d(y, x_{n+1}) \\ &\leq d(y, Tx_n) + d(Tx_n, x_{n+1}) \\ &= d(y, Tx_n) + d(A, B) \\ &\leq d(y, Tx_n) + d(y, A) \end{aligned}$$

Thus, by taking $n \rightarrow \infty$, we get

$$d(y, x_n) \rightarrow d(y, A)$$

As A is approximately compact with respect to B , therefore the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ that converges to some $x \in A$. Hence,

$$d(x, Tx) = \lim_{k \rightarrow \infty} d(x_{n_{k+1}}, Tx_{n_k}) = d(A, B)$$

Hence x is a best proximity point of T . Assume that x^* is another best proximity point of T , therefore

$$d(x^*, Tx^*) = d(A, B)$$

Since T is generalized α - η -rational proximal contraction of second kind, then by using the hypothesis $\alpha(x, y) \geq \eta(x, y)$ for all best proximity point x and y of T , we have

$$\begin{aligned} d(Tx, Tx^*) &\leq k_1(x)d(Tx, Tx^*) + k_2(x) \frac{[1 + d(Tx, Tx)]d(Tx^*, Tx^*)}{1 + d(Tx, Tx^*)} \\ &\quad + k_3(x) \frac{[1 + d(Tx, Tx^*)]d(Tx^*, Tx)}{1 + d(Tx, Tx^*)} + k_4(x)d(Tx, Tx) + k_5(x)d(Tx^*, Tx^*) \\ &\quad + k_6(x)[d(Tx, Tx^*) + d(Tx^*, Tx)] \\ &\leq [k_1(x) + k_3(x) + 2k_6(x)]d(Tx, Tx^*) \end{aligned}$$

which imply that $Tx = Tx^*$.

Remark: From remark 3.3, it can be concluded that Theorem 11 and Theorem 21 of [13], Theorem 3.1 and Theorem 3.2 of [11] and Theorem 3.1 and Theorem 3.4 of [12] are particular case of Theorem 3.4 and Theorem 3.5 respectively.

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