Multisymplectic Structure of a Finite Difference Scheme for a Complex Nonlinear System

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Abstract: *A Numerical study for the complex nonlinear system (coupled 1D nonlinear Schrödinger system (CNLS)) is considered as a Model for Complex Nonlinear System which is also a model for wave-wave interaction in ionic media. A finite difference scheme is derived for the model equations. A new six point scheme, which is equivalent to the multi-symplectic integrator, is derived. We investigate the conservation property of the multi-symplectic integrator of the complex nonlinear system (CNLS). The numerical simulation is also presented for the model equations.*
Keywords: *Coupled 1D nonlinear Schrödinger system; Multi-symplectic scheme; Energy conservation; Six-point scheme.*

I. INTRODUCTION

Wave-wave interaction is an important problem for both physical and mathematical reasons. Physically, the wave-wave interaction or the wave collisions are common phenomena in science and engineering for both solitary and non-solitary waves. Mathematically solitary wave collision is a major branch of nonlinear wave interaction in ionic media. An example of the model for wave-wave interaction is the coupled 1D nonlinear Schrödinger system (CNLS)

$$\begin{aligned} &\mathrm{i}u_t + u_{xx} + \left(|u|^2 + \beta|v|^2\right)u = 0, \\ &\mathrm{i}v_t + v_{xx} + \left(|v|^2 + \beta|u|^2\right)v = 0, \end{aligned} \tag{1}$$

with initial conditions $u(x, 0) = u_0(x)$, $v(x, 0) = v_0(x)$, and β is a constant. In an integrable system, solitary waves collide elastically but if the system is nonintegrable, this interaction may be highly nontrivial. Its application can be found in many areas of mathematics and physics, including nonlinear optics and plasma physics [1,11,12]. Much work has been done on interactions in large array of physical systems. Various interaction scenarios such as transmission, reflection, annihilation, trapping, creation of solitary waves and even mutual spiraling have been reported. However in their numerical simulations, in order to keep the accuracy, there are many constraints.

Moreover they neglect many properties of the system, such as energy conservation, momentum conservation, etc. Several attempts were done to solve the above mentioned coupled 1D nonlinear Schrödinger system and it is solved both analytically and numerically.

Recently, specification has been paid to multi-symplectic geometry [2–4]. Bridge and Reich introduced the concept of multi-symplectic integrator in the form of finite difference scheme for some conservative PDEs [5,9,10]. The theoretical results indicated that it is a strictly local concept and can be formulated in the form of finite difference scheme [6-8]. Thus the multi-symplectic integrator has excellent local invariant conserving properties [13]. The CNLS system has multi-symplectic structure; therefore we can apply this approach to obtain multi-symplectic integrator in difference equations form. In the paper, we discretize the system with finite difference schemes to show the multi-symplectic structure of CNLS system. We prove the advantage of the multi-symplectic structure of CNLS system by numerical simulations. In Section 2, We derived a six point difference scheme which is equivalent to multi-symplectic integrator for coupled nonlinear Schrödinger system. In Section 3, we investigate the conservation property of coupled nonlinear Schrödinger system. In Section 4, numerical simulations are reported to coupled nonlinear Schrödinger system.

II. A DIFFERENCE SCHEME FOR CNLS SYSTEM

We consider the following generalized CNLS system

$$\begin{aligned} &\mathrm{i}u_t + u_{xx} + \left(|u|^2 + \beta|v|^2\right)u = 0, \\ &\mathrm{i}v_t + v_{xx} + \left(|v|^2 + \beta|u|^2\right)v = 0, \end{aligned} \tag{2}$$

Where

$$u(x,t) = p(x,t) + q(x,t)\mathrm{i},\ v(x,t) = \mu(x,t) + \zeta(x,t)\mathrm{i}. \tag{3}$$

we have

$$\mathrm{i}(p_t+q_t\mathrm{i})+p_{xx}+q_{xx}\mathrm{i}+\left((p^2+q^2)+\beta(\mu^2+\zeta^2)\right)(p+q\mathrm{i})=0,$$
$$\mathrm{i}(\mu_t+\zeta_t\mathrm{i})+\mu_{xx}+\zeta_{xx}\mathrm{i}+\left((\mu^2+\zeta^2)+\beta(p^2+q^2)\right)(\mu+\zeta\,\mathrm{i})=0. \tag{4}$$

So (4) can be written as

$$p_t+q_{xx}+\left(p^2+q^2+\beta(\mu^2+\zeta^2)\right)q=0,$$
$$q_t-p_{xx}-\left(p^2+q^2+\beta(\mu^2+\zeta^2)\right)p=0,$$
$$\mu_t+\zeta_{xx}+\left(\mu^2+\zeta^2+\beta(p^2+q^2)\right)\zeta=0,$$
$$\zeta_t-\mu_{xx}-\left(\mu^2+\zeta^2+\beta(p^2+q^2)\right)\mu=0.$$

Introducing the canonical momenta

$$p_x=b,\ q_x=a,\ \mu_x=d,\ \zeta_x=c,$$

The above system can be written in the following form

$$Kz_t+Lz_x=\nabla_z S(z), \tag{5}$$

with independent variable $(t,x)\in R^2$ and state variable $z\in R^d$, $d\geq 2$. Here K, $L\in R^{dxd}$ are two skew-symmetric matrices and S: $R^d\rightarrow R$ is a scalar-valued smooth function. ∇_z is the standard gradient in R^d. For S(z) and $\nabla_z S(z)$, the system is multi-symplectic in the sense that K is a skew-symmetric matrix representative of the t direction and L is a skew-symmetric matrix representative of the x direction. S represents a Hamiltonian function [6,9,14].

The equation (5) is multi-symplectic in nature with the state variables $z=(p,q,b,a,\mu,\zeta,d,c)^{\mathrm{T}}\in R^8$ and the Hamiltonian is

$$S(z)=\tfrac{1}{2}\left(a^2+b^2+c^2+d^2+\tfrac{1}{2}(p^2+q^2)^2+\tfrac{1}{2}(\mu^2+\zeta^2)^2\right)+\tfrac{1}{2}\beta(\mu^2+\zeta^2)(p^2+q^2) \tag{6}$$

So the

$$\nabla_z S(z)=(kp,kq,b,a,sp,sq,d,c)^{\mathrm{T}},$$
$$kp=(p^2+q^2+\beta(\mu^2+\zeta^2))p$$
$$kq=(p^2+q^2+\beta(\mu^2+\zeta^2))q$$
$$sp=(\mu^2+\zeta^2+\beta(p^2+q^2))\mu$$
$$sq=(\mu^2+\zeta^2+\beta(p^2+q^2))\zeta$$

and the pair of skew symmetric matrix K and L are

$$K=\begin{pmatrix}0&1&0&0&0&0&0&0\\-1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&-1&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}\qquad L=\begin{pmatrix}0&0&-1&0&0&0&0&0\\0&0&0&-1&0&0&0&0\\1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&0&0&0&-1&0\\0&0&0&0&0&0&0&-1\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\end{pmatrix}$$

Using midpoint difference scheme to discretize multi-symplectic CNLS system, we can get

$$\frac{q_{l+1/2}^{n+1}-q_{l+1/2}^{n}}{\Delta t}-\frac{b_{l+1}^{n+1/2}-b_{l}^{n+1/2}}{\Delta x}=\left((\hat{p})^2+(\hat{q})^2+\beta\left((\hat{\mu})^2+(\hat{\zeta})^2\right)\right)\hat{p} \tag{7}$$

$$-\frac{p_{l+1/2}^{n+1}-p_{l+1/2}^{n}}{\Delta t}-\frac{a_{l+1}^{n+1/2}-a_{l}^{n+1/2}}{\Delta x}=\left((\hat{p})^2+(\hat{q})^2+\beta\left((\hat{\mu})^2+(\hat{\zeta})^2\right)\right)\hat{q}. \tag{8}$$

$$\frac{p_{l+1}^{n+1/2}-p_{l}^{n+1/2}}{\Delta x}=b_{l+1/2}^{n+1/2} \tag{9}$$

$$\frac{q_{l+1}^{n+1/2}-q_l^{n+1/2}}{\Delta x}=a_{l+1/2}^{n+1/2} \qquad (10)$$

$$\frac{\zeta_{l+1/2}^{n+1}-\zeta_{l+1/2}^{n}}{\Delta t}-\frac{d_{l+1}^{n+1/2}-d_l^{n+1/2}}{\Delta x}=\left((\hat{\mu})^2+(\hat{\zeta})^2+\beta\left((\hat{p})^2+(\hat{q})^2\right)\right)\hat{\mu} \qquad (11)$$

$$-\frac{\mu_{l+1/2}^{n+1}-\mu_{l+1/2}^{n}}{\Delta t}-\frac{c_{l+1}^{n+1/2}-c_l^{n+1/2}}{\Delta x}=\left((\hat{\mu})^2+(\hat{\zeta})^2+\beta\left((\hat{p})^2+(\hat{q})^2\right)\right)\hat{\zeta} \qquad (12)$$

$$\frac{\mu_{l+1}^{n+1/2}-\mu_l^{n+1/2}}{\Delta x}=d_{l+1/2}^{n+1/2} \qquad (13)$$

$$\frac{\zeta_{l+1}^{n+1/2}-\zeta_l^{n+1/2}}{\Delta x}=c_{l+1/2}^{n+1/2} \qquad (14)$$

where

$$\hat{p}=p_{l+1/2}^{n+1/2},\ \hat{q}=q_{l+1/2}^{n+1/2},\ \hat{\mu}=\mu_{l+1/2}^{n+1/2},\ \hat{\zeta}=\zeta_{l+1/2}^{n+1/2}$$

we eliminate a ,b, c and d So we can get

$$\mathrm{i}\frac{(u_{l-1}^{n+1}+2u_l^{n+1}+u_{l+1}^{n+1})-(u_{l-1}^{n}+2u_l^{n}+u_{l+1}^{n})}{2\Delta t}+\frac{u_{l+1}^{n}+u_{l+1}^{n+1}-2(u_l^n+u_l^{n+1})+u_{l-1}^{n}+u_{l-1}^{n+1}}{\Delta^2 x}$$

$$+\left(|u_{l-1/2}^{n+1/2}|^2+\beta|v_{l-1/2}^{n+1/2}|^2\right)u_{l-1/2}^{n+1/2}+\left(|u_{l+1/2}^{n+1/2}|^2+\beta|v_{l+1/2}^{n+1/2}|^2\right)u_{l+1/2}^{n+1/2}=0 \qquad (15)$$

and

$$\mathrm{i}\frac{(v_{l-1}^{n+1}+2v_l^{n+1}+v_{l+1}^{n+1})-(v_{l-1}^{n}+2v_l^{n}+v_{l+1}^{n})}{2\Delta t}+\frac{v_{l+1}^{n}+v_{l+1}^{n+1}-2(v_l^n+v_l^{n+1})+v_{l-1}^{n}+v_{l-1}^{n+1}}{\Delta^2 x}$$

$$+\left(|v_{l-1/2}^{n+1/2}|^2+\beta|u_{l-1/2}^{n+1/2}|^2\right)v_{l-1/2}^{n+1/2}+\left(|v_{l+1/2}^{n+1/2}|^2+\beta|u_{l+1/2}^{n+1/2}|^2\right)v_{l+1/2}^{n+1/2}=0 \qquad (16)$$

So in (15) and (16), we get a six-point difference scheme for u & v and this can be treated as multi-symplectic integrator.

III. CONSERVATION PROPERTY OF MULTI-SYMPLECTIC INTEGRATORS

In this section, we investigate the conservation property of the multi-symplectic integrator of the coupled 1D nonlinear Schrödinger system. Moreover we will check average norm conservation property to the coupled 1D nonlinear Schrödinger system.

The multi-symplectic integrator (15) can be rewritten with the following discretization form

$$z_{n+1}^{m+1/2}=\frac{1}{2}(z_{n+1}^{m+1}+z_{n+1}^{m}),\quad z_{n+1/2}^{m}=\frac{1}{2}(z_{n+1}^{m}+z_{n}^{m}),\quad z_{n+1/2}^{m+1/2}=\frac{1}{4}(z_n^m+z_{n+1}^m+z_n^{m+1}+z_{n+1}^{m+1})$$

where z_n^m an approximation of $z(m\Delta t, n\Delta x)$.We will get the following form

$$\frac{\mathrm{i}}{2\Delta t}\left((u_{l-1/2}^{n+1}-u_{l-1/2}^{n})+(u_{l+1/2}^{n+1}-u_{l+1/2}^{n})\right)+\frac{1}{\Delta x^2}(u_{l+1}^{n+1/2}-2u_l^{n+1/2}+u_{l-1}^{n+1/2})$$

$$+a\left(|u_{l-1/2}^{n+1/2}|^2+\beta|v_{l-1/2}^{n+1/2}|^2\right)u_{l-1/2}^{n+1/2}+a\left(|u_{l+1/2}^{n+1/2}|^2+\beta|v_{l+1/2}^{n+1/2}|^2\right)u_{l+1/2}^{n+1/2}=0. \qquad (17)$$

Conjugating Eq. (17), we can get

$$-\frac{\mathrm{i}}{2\Delta t}\left(\overline{(u_{l-1/2}^{n+1}-u_{l-1/2}^{n})}+\overline{(u_{l+1/2}^{n+1}-u_{l+1/2}^{n})}\right)+\frac{1}{\Delta x^2}\left(\overline{u_{l+1}^{n+1/2}}-2\overline{u_{l}^{n+1/2}}+\overline{u_{l-1}^{n+1/2}}\right)$$

$$+a\left(|u_{l-1/2}^{n+1/2}|^2+\beta|v_{l-1/2}^{n+1/2}|^2\right)\overline{u_{l-1/2}^{n+1/2}}+a\left(|u_{l+1/2}^{n+1/2}|^2+\beta|v_{l+1/2}^{n+1/2}|^2\right)\overline{u_{l+1/2}^{n+1/2}}=0. \qquad (18)$$

In Eq. (17), take l = 1, 2. . . N, we can get the sum equation

$$\frac{\mathrm{i}}{2\Delta t}\left(\sum_{l=1}^{N}(u_{l-1/2}^{n+1}-u_{l-1/2}^{n})+\sum_{l=1}^{N}(u_{l+1/2}^{n+1}-u_{l+1/2}^{n})\right)+\frac{1}{\Delta x^2}\sum_{l=1}^{N}(u_{l+1}^{n+1/2}-2u_{l}^{n+1/2}+u_{l-1}^{n+1/2})$$

$$+\sum_{l=1}^{N}a\left((|u_{l-1/2}^{n+1/2}|^2+\beta|v_{l-1/2}^{n+1/2}|^2)u_{l-1/2}^{n+1/2}+(|u_{l+1/2}^{n+1/2}|^2+\beta|v_{l+1/2}^{n+1/2}|^2)u_{l+1/2}^{n+1/2}\right)=0. \qquad (19)$$

We multiply Eq. (19) using $\overline{u_l^{n+1/2}}$ We have

$$\frac{\mathrm{i}}{2\Delta t}\left(\sum_{l=1}^{N}(u_{l-1/2}^{n+1}-u_{l-1/2}^{n})\overline{u_{l}^{n+1/2}}+\sum_{l=1}^{N}(u_{l+1/2}^{n+1}-u_{l+1/2}^{n})\overline{u_{l}^{n+1/2}}\right)$$

$$+\frac{1}{\Delta x^2}\sum_{l=1}^{N}(u_{l+1}^{n+1/2}-2u_{l}^{n+1/2}+u_{l-1}^{n+1/2})\overline{u_{l}^{n+1/2}}$$

$$+\sum_{l=1}^{N}a\left((|u_{l-1/2}^{n+1/2}|^2+\beta|v_{l-1/2}^{n+1/2}|^2)u_{l-1/2}^{n+1/2}\overline{u_{l}^{n+1/2}}+(|u_{l+1/2}^{n+1/2}|^2+\beta|v_{l+1/2}^{n+1/2}|^2)u_{l+1/2}^{n+1/2}\overline{u_{l}^{n+1/2}}\right)=0 \qquad (20)$$

In Eq. (20), we have

$$\sum_{l=1}^{N}(u_{l-1/2}^{n+1}-u_{l-1/2}^{n})\overline{u_{l}^{n+1/2}}=\sum_{l=1}^{N}(u_{l+1/2}^{n+1}-u_{l+1/2}^{n})\overline{u_{l+1}^{n+1/2}}+(u_{1/2}^{n+1}-u_{1/2}^{n})\overline{u_{1}^{n+1/2}}-(u_{N+1/2}^{n+1}-u_{N+1/2}^{n})\overline{u_{N+1}^{n+1/2}},$$

$$\sum_{l=1}^{N}a\left(|u_{l-1/2}^{n+1/2}|^2+\beta|v_{l-1/2}^{n+1/2}|^2\right)u_{l-1/2}^{n+1/2}\overline{u_{l}^{n+1/2}}$$

$$=\sum_{l=1}^{N}a\left(|u_{l+1/2}^{n+1/2}|^2+\beta|v_{l+1/2}^{n+1/2}|^2\right)u_{l+1/2}^{n+1/2}\overline{u_{l+1}^{n+1/2}}$$

$$+a\left(|u_{1/2}^{n+1/2}|^2+\beta|v_{1/2}^{n+1/2}|^2\right)u_{1/2}^{n+1/2}\overline{u_{1}^{n+1/2}}-a\left(|u_{N+1/2}^{n+1/2}|^2+\beta|v_{N+1/2}^{n+1/2}|^2\right)u_{N+1/2}^{n+1/2}\overline{u_{N+1}^{n+1/2}},$$

$$\sum_{l=1}^{N}(|u_{l-1/2}^{n+1/2}|^2+\beta|v_{l-1/2}^{n+1/2}|^2)u_{l-1/2}^{n+1/2}\overline{u_{l}^{n+1/2}}$$

$$=\sum_{l=1}^{N}(|u_{l+1/2}^{n+1/2}|^2+\beta|v_{l+1/2}^{n+1/2}|^2)u_{l+1/2}^{n+1/2}\overline{u_{l+1}^{n+1/2}}$$

$$+(|u_{1/2}^{n+1/2}|^2+\beta|v_{1/2}^{n+1/2}|^2)u_{1/2}^{n+1/2}\overline{u_{1}^{n+1/2}}-(|u_{N+1/2}^{n+1/2}|^2+\beta|v_{N+1/2}^{n+1/2}|^2)u_{N+1/2}^{n+1/2}\overline{u_{N+1}^{n+1/2}}$$

Applying the 0 boundary points or periodical boundary conditions, so we can have

$$\frac{i}{2\Delta t}\sum_{l=1}^{N}(u_{l+1/2}^{n+1}-u_{l+1/2}^{n})\overline{(u_l^{n+1/2}+u_{l+1}^{n+1/2})}+\frac{-2}{\Delta x^2}\sum_{l=1}^{N}(|u_l^n+u_l^{n+1}|^2)\ +\frac{1}{\Delta x^2}\sum_{l=1}^{N}(u_{l+1}^{n+1/2}\overline{u_l^{n+1/2}}+u_l^{n+1/2}\overline{u_{l+1}^{n+1/2}})$$
$$+a\sum_{l=1}^{N}(|u_{l+1/2}^{n+1/2}|^2+\beta|v_{l+1/2}^{n+1/2}|^2)u_{l+1/2}^{n+1/2}\overline{(u_l^{n+1/2}+u_{l+1}^{n+1/2})}=0 \tag{21}$$

In the same way for Eq. (17) using $u_l^{n+1/2}$ and applying the 0 boundary points or periodic boundary condition, we can have

$$-\frac{i}{2\Delta t}\sum_{l=1}^{N}\overline{(u_{l+1/2}^{n+1}-u_{l+1/2}^{n})}(u_l^{n+1/2}+u_{l+1}^{n+1/2})+\frac{-2}{\Delta x^2}\sum_{l=1}^{N}(|u_l^n+u_l^{n+1}|^2)$$
$$+\frac{1}{\Delta x^2}\sum_{l=1}^{N}(\overline{u_{l+1}^{n+1/2}}u_l^{n+1/2}+\overline{u_l^{n+1/2}}u_{l+1}^{n+1/2})$$
$$+a\sum_{l=1}^{N}(|u_{l+1/2}^{n+1/2}|^2+\beta|v_{l+1/2}^{n+1/2}|^2)\overline{u_{l+1/2}^{n+1/2}}(u_l^{n+1/2}+u_{l+1}^{n+1/2})=0.$$

(22)

Using (21), (22), we can get the following conservation formula

$$\sum_{l=1}^{N}(|u_l^n+u_{l+1}^n|^2)=\sum_{l=1}^{N}(|u_l^{n+1}+u_{l+1}^{n+1}|^2)$$

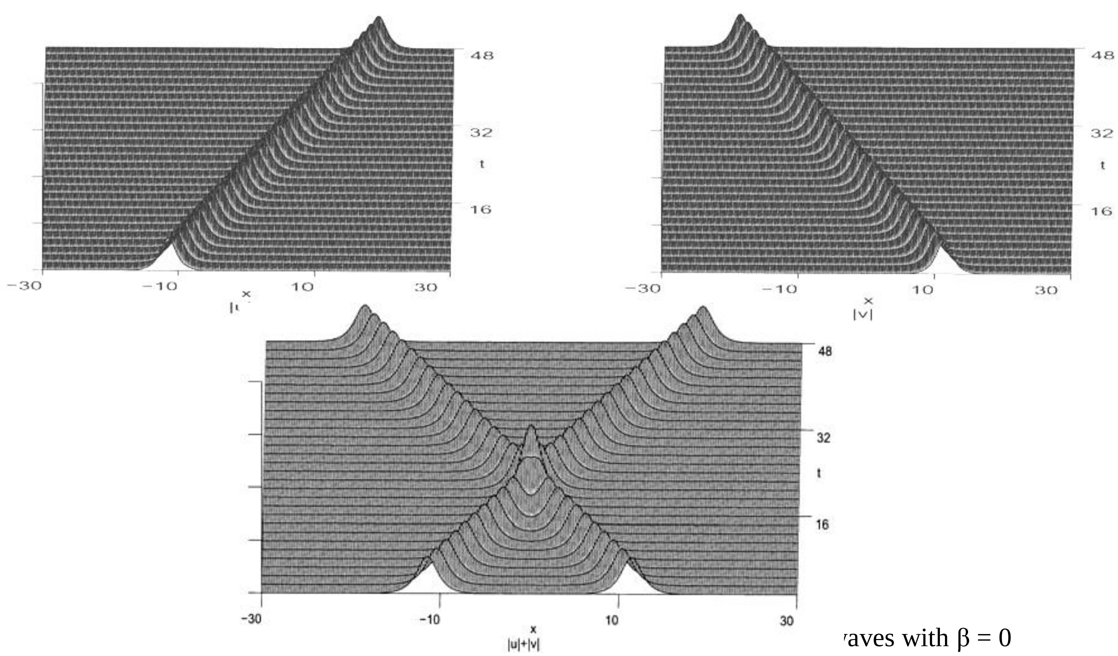
(23)

In the same way as above, we can prove $$\sum_{l=1}^{N}(|v_l^n+v_{l+1}^n|^2)=\sum_{l=1}^{N}(|v_l^{n+1}+v_{l+1}^{n+1}|^2)$$

(24)

Numerical simulation

In this section, we present the numerical result of the CNLS system using the multi-symplectic integrator.



aves with β = 0

As for conserving quantities, we focus on monitoring the energy conserving properties of the multi-symplectic integrator.

Now we consider the CNLS system

$$iu_t + u_{xx} + \left(|u|^2 + \beta|v|^2\right)u = 0$$
$$iv_t + v_{xx} + \left(|v|^2 + \beta|u|^2\right)v = 0.$$

with the initial value

$$u(x,0) = \sqrt{2}r_1 \operatorname{sech}\left(r_1 x + \tfrac{1}{2}D_0\right)e^{tV_0x/4},$$
$$v(x,0) = \sqrt{2}r_2 \operatorname{sech}\left(r_2 x - \tfrac{1}{2}D_0\right)e^{-tV_0x/4} \quad (25)$$

From [11,12], we know, when β = 1 and β = 0, the CNLS system is the integrable system. Here we consider the interaction of two waves with the initial condition (25). We take the time step $\Delta t = 0.02$ and a space step $\Delta x = 0.2$, $-30 \le x \le 30$, $D_0 = 25$, $r_1 = r_2 = 1$ and $V_0=1$. In Fig. 1, the computation is done for $0 \le t \le 48$. We can see after the colliding of the two soliton waves, they move forward in the same direction and with the same velocity as before

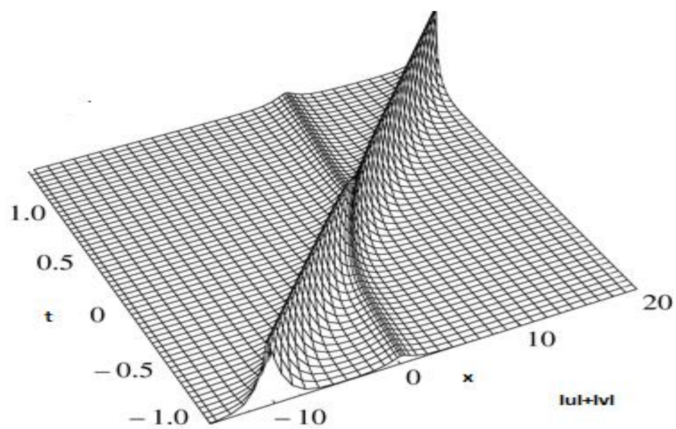


Fig. 2. Simulation results of the interaction of the two waves with β = 1

IV. CONCLUSIONS

In this paper, the multi-symplectic formulation for the coupled 1D nonlinear Schrödinger system is presented. Numerical experiments are also reported. We observe that the multi-symplectic scheme well simulates the evolution of the solitons and preserves energy conservation well. It has advantage for the long time computing accuracy and preserving the energy conservation property.

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