Numerical Solution of Fuzzy Differential Equation (FDE)

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Abstract: *In this paper we define the fuzzy differential equation of the first order and solve this equation by numerical solution in Runge-Kutta Method. We introduce an example to solve the problem by using this method, and applied in matlab computer software.*

Keywords: *Fuzzy differential equation, fuzzy numbers, Runge-Kutta Method.*

I. Introduction

The concept of fuzzy set was first introduced by Zadeh [1]. Since then, the theory has been developed and it is now emerged as an independent branch of Applied Mathematics. Theory of fuzzy differential equations plays an important role in modelling of science and engineering problems because this theory represents a natural way to model dynamical systems under uncertainty. The fuzzy differential equation and fuzzy initial value problems are studied by Kaleva [2], [3] and Seikkala [9]. In the last few years, many researchers have worked on theoretical and numericalSolution of FDEs [5–18]. In this paper we introduced the solve of fuzzy differential equation and using the Runge-Kutta method to solve some examples by using computer software to find the approximation solution.

II. Preliminaries

A general definition of fuzzy numbers may be found in [4]. Fuzzy numbers will be always triangular or triangular shaped fuzzy numbers.

A triangular fuzzy number N is defined by three real numbers a < b < c, where the base of the triangle is the interval [a, c] and its vertex is at x = b.

Triangular fuzzy numbers will be written as N= (a/b/c). The membership function for the triangular fuzzy number N = (a/b/c) is defined as the following:

$$N(x) = \begin{cases} \dfrac{x-a}{b-a} & a \le x \le b \\ \dfrac{x-c}{b-c} & b \le x \le c \end{cases}$$

For triangular shaped fuzzy number P, we write P ≈ (a/b/c) which is only partially specified by the three numbers a, b, c since the graph on [a, b] and [b, c] is not a straight line segment. To be a triangular shaped fuzzy number, we require the graph of the corresponding membership function to be continuous and:

1. Monotonically increasing on [a, b].
2. Monotonically decreasing on [b, c].

The core of a fuzzy number is the set of values where the membership value equals one.

If N = (a/b/c) or N ≈ (a/b/c) then the core of N is the single point b. Let T be the set of all triangular or triangular shaped fuzzy numbers and u ∈ T.

We define the r-level set

$$[u]_r = \{x|u(x) \ge r\}, \quad 0 \le r \le 1$$

Which is a closed bounded interval and denoted by

$$[u]_r = [\underline{u}(r), \overline{u}(r)]$$

It is clear that the following statements are true.

1. $\underline{u}(r)$ Is a bounded left continuous non decreasing function over [0, 1].
2. $\overline{u}(r)$ Is a bounded right continuous non increasing function over [0,1].
3. $\underline{u}(r) \le \overline{u}(r)$ for all $r \in [0, 1]$. For more details, see [4].

III. First Order Fuzzy Differential Equation:

A first order fuzzy initial value differential equation is given by:

$$\begin{cases} \acute{y} = f(t, y(t)) & t \in [t_o, T] \\ y(t_o) = y_o \end{cases}$$

Such that y is a fuzzy function of t, f (t, y) is a fuzzy function of the scrips variable t and the fuzzy variable y. y_o is the fuzzy derivative of y and $y(t_o) = y_o$ is a triangular or a triangular shaped fuzzy number. We denote the fuzzy function y by y = $[\underline{y}, \overline{y}]$. It means that the r-level set of y(t) for $t \in [t_o, T]$is

$$[\, y(t)]_r = [\underline{y}(t; r), \overline{y}(t; r)]$$

Also:

$$[\acute{y}(t)]_r = [\underline{\acute{y}}(t; r), \overline{\acute{y}}(t; r)]$$

$$[\, f(t, y(t))]_r = [\underline{f}(t, y(t); r), \overline{f}(t, y(t); r)]$$

We write:

$$f(t, y) = [\underline{f}(t, y), \overline{f}(t, y)]$$

We have:

$$\underline{\acute{y}}(t; r) = \underline{f}(t, y(t); r) = F[t, \underline{y}(t; r), \overline{y}(t; r)]$$

$$\overline{\acute{y}}(t; r) = \bar{f}(t, y(t); r) = G[t, \underline{y}(t; r), \overline{y}(t; r)]$$

Also we write

$$[\, y(t_o)]_r = [\underline{y}(t_o; r), \overline{y}(t_o; r)]$$

$$[y_o]_r = [\left(\underline{y_o}(r)\right), (\overline{y}_o(r)]$$

$$\underline{y}(t_o; r) = \underline{y_o}(\mathrm{r}) \;\; ,\overline{y}(t_o; r) = \overline{y_o}$$

By using the extension principle, we have the membership function:

$$f(t, y(t))(s) = \sup\{y(t)(\tau) | s = f(t, \tau)\}, \quad s \in R$$

So Fuzzy number$f\big(t, y(t)\big)$

$$[f(t, y(t))]_r = \left[\underline{f}(t, y(t); r), \overline{f}(t, y(t); r)\right], \qquad r \in [0,1]$$

Where

$$\underline{f}(t, y(t); r) = \min\{f(t, u) | u \in [y(t)]_r\}$$

$$\overline{f}(t, y(t); r) = \max\{f(t, u) | u \in [y(t)]_r\}$$

IV. Fourth Order Runge-Kutta Method in fuzzy differential equation

The form first order fuzzy differential equation as

$$\acute{y}(t) = f(t, y)$$

$$y(t_o) = y_o$$

The exact solution be

$$[Y(t_n)]_r = [\underline{Y}(t_n; r), \overline{Y}(t_n; r)]$$

The approximation solution is given by

$$[y(t_n)]_r = [\underline{y}(t_n; r), \overline{y}(t_n; r)]$$

By using fourth order Runge-Kutta method have

$$[y(t_n)]_r = [\underline{y}(t_n; r), \overline{y}(t_n; r)$$

$$\underline{y}(t_{n+1}; r) = \underline{y}(t_n; r) + \sum_{j=1}^{4} w_j\, k_{j,1}(t_n, y(t_n, r))$$

$$\overline{y}(t_{n+1};r) = \overline{y}(t_n;r) + \sum_{j=1}^{4} w_j\, k_{j,2}(t_n, y(t_n,r))$$

Where $k_{j,1}, k_{j,2}$ define as follow:

$$k_{1,1}(t_n, y(t_n;r)) = \min h\,\{y(t_n\,,u) | u \in (\underline{y}(t_n;r), \overline{y}(t_n;r))\}$$
$$k_{1,2}(t_n, y(t_n;r)) = \max h\,\{y(t_n\,,u) | u \in (\underline{y}(t_n;r), \overline{y}(t_n;r))\}$$
$$k_{2,1}(t_n, y(t_n;r)) = \min h\,\{y(t_n + \tfrac{h}{2}, u) | u \in (q_{1,1}(t_n; y(t_n,r)), q_{1,2}(t_n; y(t_n,r)))\}$$
$$k_{2,2}(t_n, y(t_n;r)) = \max h\,\{y(t_n + \tfrac{h}{2}, u) | u \in (q_{1,1}(t_n; y(t_n,r)), q_{1,2}(t_n; y(t_n,r)))\}$$
$$k_{3,1}(t_n, y(t_n;r)) = \min h\,\{y(t_n + \tfrac{h}{2}, u) | u \in (q_{2,1}(t_n; y(t_n,r)), q_{2,2}(t_n; y(t_n,r)))\}$$
$$k_{3,2}(t_n, y(t_n;r)) = \max h\,\{y(t_n + \tfrac{h}{2}, u) | u \in (q_{2,1}(t_n; y(t_n,r)), q_{2,2}(t_n; y(t_n,r)))\}$$
$$k_{4,1}(t_n, y(t_n;r)) = \min h\,\{y(t_n + \tfrac{h}{2}, u) | u \in (q_{3,1}(t_n; y(t_n,r)), q_{3,2}(t_n; y(t_n,r)))\}$$
$$k_{4,2}(t_n, y(t_n;r)) = \max h\,\{y(t_n + \tfrac{h}{2}, u) | u \in (q_{3,1}(t_n; y(t_n,r)), q_{3,2}(t_n; y(t_n,r)))\}$$

Where:

$$q_{1,1}(t_n; y(t_n,r)) = \underline{y}(t_n,r) + \tfrac{h}{2}k_{1,1}(t_n, y(t_n;r))$$
$$q_{1,2}(t_n; y(t_n,r)) = \overline{y}(t_n,r) + \tfrac{h}{2}k_{1,2}(t_n, y(t_n;r))$$
$$q_{2,1}(t_n; y(t_n,r)) = \underline{y}(t_n,r) + \tfrac{h}{2}k_{2,1}(t_n, y(t_n;r))$$
$$q_{2,2}(t_n; y(t_n,r)) = \overline{y}(t_n,r) + \tfrac{h}{2}k_{2,2}(t_n, y(t_n;r))$$
$$q_{3,1}(t_n; y(t_n,r)) = \underline{y}(t_n,r) + \tfrac{h}{2}k_{3,1}(t_n, y(t_n;r))$$
$$q_{3,2}(t_n; y(t_n,r)) = \overline{y}(t_n,r) + \tfrac{h}{2}k_{3,2}(t_n, y(t_n;r))$$

Now using the initial condition, we compute:

$$\underline{y}(t_{n+1};r) = \underline{y}(t_n;r) + \tfrac{1}{6}(k_{1,1}(t_n, y(t_n;r)) + 2k_{2,1}(t_n, y(t_n;r)) + 2\,k_{3,1}(t_n, y(t_n;r)) + k_{4,2}(t_n, y(t_n;r))$$

$$\overline{y}(t_{n+1};r) = \overline{y}(t_n;r) + \tfrac{1}{6}(k_{1,2}(t_n, y(t_n;r)) + 2k_{2,2}(t_n, y(t_n;r)) + 2\,k_{3,2}(t_n, y(t_n;r)) + k_{4,2}(t_n, y(t_n;r))$$

The solution at t_n

$$0 \le n \le N \quad and\ a = t_o \le t_1 \le t_2 \le \cdots \le t_n = b, \text{ and } h = \tfrac{b-a}{N} = t_{n+1} - t_n,$$

$$\underline{Y}(t_{n+1};r) = \underline{Y}(t_n;r) + \tfrac{1}{6}F[t_n, y(t_n;r)]$$
$$\overline{Y}(t_{n+1};r) = \overline{Y}(t_n;r) + \tfrac{1}{6}G[t_n, y(t_n;r)], \text{ and}$$
$$\underline{y}(t_{n+1};r) = \underline{y}(t_n;r) + \tfrac{1}{6}F[t_n, y(t_n;r)]$$
$$\overline{y}(t_{n+1};r) = \overline{y}(t_n;r) + \frac{1}{6}G[t_n, y(t_n;r)]$$

Example (1):

Let the fuzzy initial value problem

$$\dot{y}(t) = y(t), \quad t \in [0,1]$$
$$y(0) = (0.75 + 0.25r, \qquad 1.125 - 0.125\,r), \qquad 0 < r \le 1$$

Solve:

The exact solution is obtained as:

$$\underline{Y}(t;r) = \underline{y}(t;r)e^t$$
$$\overline{Y}(t;r) = \overline{y}(t;r)e^t$$

Att=1:

$$Y(1;r) = [(0.75 + 0.25r)e, \qquad (1.125 - 0.125\,r)e]$$

The approximation solution is at h=0.01 and a=0, b=1, then we have:

$0 \le n \le N \quad and \quad 0 = t_o \le t_1 \le t_2 \le \cdots \le t_n = 1$, and $0.01 = \frac{1-0}{N} = t_{n+1} - t_n$

$$k_{1,1}(t_n, y(t_n; r)) = \min(0.01)\,\{y(t_n\,, u) | u \in (0.75 + 0.25r, 1.125 - 0.125\,r)\}$$
$$k_{1,2}(t_n, y(t_n; r)) = \max(0.01)\,\{y(t_n\,, u) | u \in (0.75 + 0.25r, 1.125 - 0.125\,r)\}$$
$$q_{1,1}(t_n; y(t_n, r)) = 0.75 + 0.25r + \frac{0.01}{2} k_{1,1}(t_n, y(t_n; r))$$
$$q_{1,2}(t_n; y(t_n, r)) = 1.125 - 0.125\; + \frac{0.01}{2} k_{1,2}(t_n, y(t_n; r))$$
$$k_{2,1}(t_n, y(t_n; r)) = \min h\,\{y(t_n + \tfrac{h}{2}, u) |$$
$$u \in \left(q_{1,1}(t_n; y(t_n, r)), q_{1,2}(t_n; y(t_n, r))\right)\}$$
$$k_{2,1}(t_n, y(t_n; r)) = \min 0.01\,\{y(t_n + \tfrac{0.01}{2}, u) | u \in \left(q_{1,1}(t_n; y(t_n, r)), q_{1,2}(t_n; y(t_n, r))\right)\}$$
$$k_{2,2}(t_n, y(t_n; r)) = \max 0.01\,\{y(t_n + \tfrac{0.01}{2}, u) | u \in \left(q_{1,1}(t_n; y(t_n, r)), q_{1,2}(t_n; y(t_n, r))\right)\}$$
$$q_{2,1}(t_n; y(t_n, r)) = 0.75 + 0.25r + \frac{0.01}{2} k_{2,1}(t_n, y(t_n; r))$$
$$q_{2,2}(t_n; y(t_n, r)) = 1.125 - 0.125\,r + \frac{0.01}{2} k_{2,2}(t_n, y(t_n; r))$$
$$k_{3,1}(t_n, y(t_n; r)) = \min\; 0.01\,\{y(t_n + \tfrac{h}{2}, u) | u \in \left(q_{2,1}(t_n; y(t_n, r)), q_{2,2}(t_n; y(t_n, r))\right)\}$$
$$k_{3,2}(t_n, y(t_n; r)) = max\; 0.01\,\{y(t_n + \tfrac{h}{2}, u) | u \in \left(q_{2,1}(t_n; y(t_n, r)), q_{2,2}(t_n; y(t_n, r))\right)\}$$
$$q_{3,1}(t_n; y(t_n, r)) = 0.75 + 0.25r + \frac{0.01}{2} k_{3,1}(t_n, y(t_n; r))$$
$$q_{3,2}(t_n; y(t_n, r)) = 1.125 - 0.125\; + \frac{0.01}{2} k_{3,2}(t_n, y(t_n; r))$$
$$k_{4,1}(t_n, y(t_n; r)) = \min 0.01\,\{y(t_n + \tfrac{h}{2}, u) | u \in \left(q_{3,1}(t_n; y(t_n, r)), q_{3,2}(t_n; y(t_n, r))\right)\}$$
$$k_{4,2}(t_n, y(t_n; r)) = max\, 0.01\,\{y(t_n + \tfrac{h}{2}, u) | u \in \left(q_{3,1}(t_n; y(t_n, r)), q_{3,2}(t_n; y(t_n, r))\right)\}$$

Then:

$$\underline{y}(t_{n+1}; r) = \underline{y}(t_n; r) + \tfrac{1}{6}(k_{1,1}(t_n, y(t_n; r)) + 2k_{2,1}(t_n, y(t_n; r)) + 2\,k_{3,1}(t_n, y(t_n; r)) + k_{4,2}(t_n, y(t_n; r))$$
$$\overline{y}(t_{n+1}; r) = \overline{y}(t_n; r) + \tfrac{1}{6}(k_{1,2}(t_n, y(t_n; r)) + 2k_{2,2}(t_n, y(t_n; r)) + 2\,k_{3,2}(t_n, y(t_n; r)) + k_{4,2}(t_n, y(t_n; r))$$

Numerical results

We used MATLAB software in all the calculations which done in this section.

Exact solutions

Table1: Table of results

R	$\underline{Y}$	$\overline{Y}$
0	2.054868862	3.09206463
0.1	2.123854919	3.057238443
0.2	2.192979466	3.022644241
0.3	2.261817072	3.015118406
0.4	2.345129849	2.878698699
0.5	2.399837083	2.844467729
0.6	2.469154584	2.82803846
0.7	2.538293745	2.794102002
0.8	2.607270746	2.745640185
0.9	2.676324662	2.707516136
1	2.744280115	2.677562061

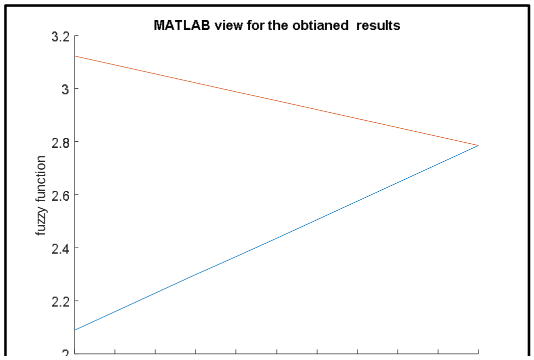

Figure 1

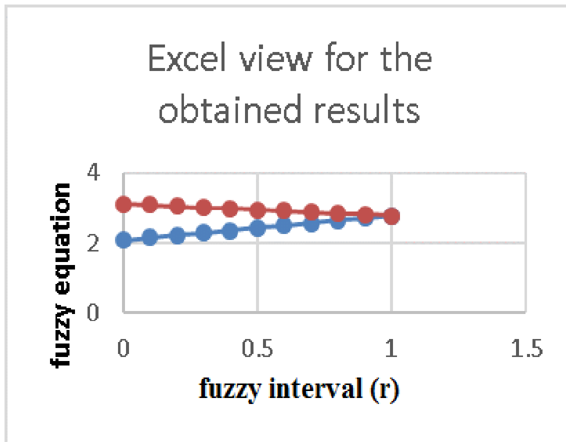

Figure 2

Approximated solution at h =0.01 (proposed method)

Table2: Table of result

r	$\underline{Y}$	$\overline{Y}$
0	2.0885223	3.1235008
0.10	2.158139	3.089620
0.2	2.227757	3.0557395
0.30	2.2973745	3.0218595
0.4	2.3669919	2.98797905
0.500	2.436609	2.954098
0.6	2.506226	2.920218
0.7	2.57584418	2.88633774
0.8	2.6454615	2.8524573
0.9	2.7150790	2.8185768
1	2.78469641	2.78469641

The flowchart of the proposed Runge-Kutta method is shown below:

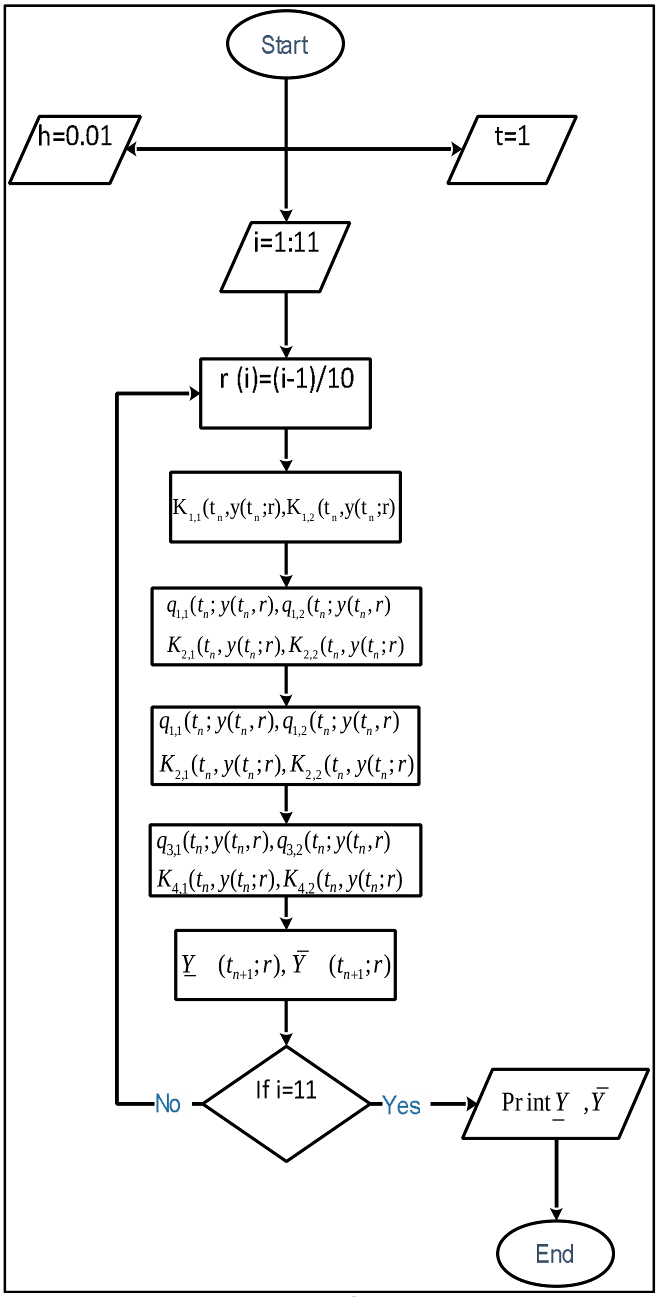

Figure 3 Proposed Runge-Kutta

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VI. Appendix

The program for Solving first order fuzzy differential equation for example (1) asfollows:

$i = 1{:}11;$

$Y(i) = (i-1)|10;$

$\underline{y}(t;r) = \underline{y}(0) = \big(0.75 + 0.25 * (r)\big) * \exp(t)\,;$

$\overline{y}(t;r) = \overline{y}(0) = \big(1.125 - 0.125 * (r)\big) * \exp(t)\,;$

$K_{11} = \min\Big(h * \underline{y}(0), h * \overline{y}(0)\Big);$

$K_{12} = \max\Big(h * \underline{y}(0), h * \overline{y}(0)\Big);$

$q_{11} = \underline{y}(0) + (\frac{h}{2} * K_{11});$

$q_{12} = \overline{y}(0) + (\frac{h}{2} * K_{12});$

$K_{21} = min\left[\left(h * \underline{y}(0) * \exp\left(t + \frac{h}{2}\right)\right) * q_{11}\,, \quad \left(h * \overline{y}(0) * \exp\left(t + \frac{h}{2}\right)\right) * q_{12}\right];$

$K_{22} = max\left[\left(h * \underline{y}(0) * \exp\left(t + \frac{h}{2}\right)\right) * q_{11}\,, \quad \left(h * \overline{y}(0) * \exp\left(t + \frac{h}{2}\right)\right) * q_{12}\right];$

$q_{21} = \underline{y}(0) + (\frac{h}{2} * K_{21});$

$q_{22} = \overline{y}(0) + (\frac{h}{2} * K_{22});$

$K_{31} = min\left[\left(h * \underline{y}(0) * \exp\left(t + \frac{h}{2}\right)\right) * q_{21}\,, \quad \left(h * \overline{y}(0) * \exp\left(t + \frac{h}{2}\right)\right) * q_{22}\right];$

$K_{32} = max\left[\left(h * \underline{y}(0) * \exp\left(t + \frac{h}{2}\right)\right) * q_{21}, \quad \left(h * \bar{y}(0) * \exp\left(t + \frac{h}{2}\right)\right) * q_{22}\right]$

$q_{31} = \underline{y}(0) + (\frac{h}{2} * K_{31});$

$q_{32} = \bar{y}(0) + (\frac{h}{2} * K_{32});$

$K_{41} = min\left[\left(h * \underline{y}(0) * \exp\left(t + \frac{h}{2}\right)\right) * q_{31}, \quad \left(h * \bar{y}(0) * \exp\left(t + \frac{h}{2}\right)\right) * q_{32}\right];$

$K_{42} = max\left[\left(h * \underline{y}(0) * \exp\left(t + \frac{h}{2}\right)\right) * q_{31}, \quad \left(h * \bar{y}(0) * \exp\left(t + \frac{h}{2}\right)\right) * q_{32}\right]$

$\underline{y}(t_{n+1}; r) = \underline{y}(t_n; r) + \frac{1}{6}[K_{11} + 2K_{21} + 2K_{31} + K_{41}]$

$\bar{y}(t_{n+1}; r) = \bar{y}(t_n; r) + \frac{1}{6}[K_{12} + 2K_{22} + 2K_{32} + K_{42}]$

$Plot(r, \underline{y}; r, \bar{y})$