# Representation Theorem On Γ-Hilbert Space

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**Abstract**

In this paper we discuss about the simple properties of Γ-Hilbert space,introduced by D.K.Bhattacharya and T.E.Aman in their paper Γ-Hilbert Space and linear Quadratic Control problem in 2003. We have defined the orthogonality in Γ-Hilbert space and discuss about the closest point property,Unique Decomposition Theorem following the defined orthogonality on that space . Further we discuss the representation of any bounded linear functionals on Γ-Hilbert space in terms of Γ-inner product in that space.

Key word : Hilbert space, Γ-Hilbert Space , Unique Decomposition Theorem , Riesz Representation Theorem.

## 1 INTRODUCTION:

The definition of Γ-Hilbert space was introduced by Bhattacharya D.K and T.E.Aman in their paper " Γ- Hilbert space and linear quadratic control problem" in 2003[1].But we found no litarature on Γ-Hilbert Space after that. So it is very essential to develop the study on Γ-Hilbert Space, mainly Orthogonality and representation of any bounded linear functional on Γ-Hilbert Space .In his paper we have defined orthogonality of elements of Γ-Hilbert Space and following this definition we have developed the Closest Point Property ([2],[3],[4]) , Unique Decomposition Theorem ([2],[3],[4]) and Representation Theorem ([2],[3],[4]) in Γ-Hilbert Space .

## 2 Γ-Hilbert space[1]:

**Definition** Let E, Γ be two linear space over the field R. A mapping $\langle .,.,. \rangle$ : E x Γ x E $\longrightarrow$ R is called a Γ inner product on E if

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(i) $\langle .,.,. \rangle$ is linear in each variable.
(ii)$\langle u, \gamma, v \rangle = \langle v, \gamma, u \rangle$ $\forall$ u,v belongs to E and $\gamma$ belongs to $\Gamma$.
(iii)$\langle u, \gamma, u \rangle > 0$, $\forall$ $\gamma \neq 0$ and u $\neq 0$ .

[(E,$\Gamma$),$\langle .,.,. \rangle$] is called a $\Gamma$-inner product space over R .
A complete $\Gamma$-inner product space is called $\Gamma$-Hilbert space.

It follows that for each $\gamma \in \Gamma$, $+\sqrt{\langle u, \gamma, u \rangle}$ gives the properties of the norm for u$\in$ E. It is called the $\gamma$ norm of u and is denoted by $\|u\|_\gamma$
We observe that $\inf\{\langle u, \gamma, u \rangle : \gamma \in \Gamma\}$ satisfies all conditions of a norm, we called it the $\Gamma$-norm and is denoted by
$\|u\|_\Gamma = \inf\{\langle u, \gamma, u \rangle : \gamma \in \Gamma\}$ .

# 3 Orthogonality of $\Gamma$-Hilbert Spaces

## 3.1 Definition :($\gamma$-Orthogonal)

Let L be a non-empty subset of a $\Gamma$ - Hilbert space $H_\Gamma$.Two elements x and y of $H_\Gamma$ are said to be $\gamma$-orthogonal if their inner product $\langle x, \gamma, y \rangle = 0$ . In symbol, we write $x \perp_\gamma y$.If x is $\gamma$-orthogonal to every element of L then we say that x is $\gamma$-orthogonal to L and in symbol we write $x \perp_\gamma L$.

## 3.2 Definition:($\Gamma$-Orthogonal)

Let L be a non-empty subset of a $\Gamma$ - Hilbert space $H_\Gamma$.Two elements x and y of $H_\Gamma$ are said to be $\Gamma$-orthogonal or simply orthogonal if $\langle x, \gamma, y \rangle = 0$ for all $\gamma \in \Gamma$. In symbol, we write $x \perp_\Gamma y$.If x is $\Gamma$-orthogonal to every element of L then we say that x is $\Gamma$-orthogonal to L and in symbol we write $x \perp_\Gamma L$.

## 3.3 Theorem:

If x and y are two $\gamma$-orthogonal elements of H$_\Gamma$, *then*
$\| x + y \|_\gamma^2 = \| x \|_\gamma^2 + \| y \|_\gamma^2$
and
$\| x - y \|_\gamma^2 = \| x \|_\gamma^2 + \| y \|_\gamma^2$ .

**Proof:** we have
$\| x + y \|_\gamma^2$
$= \langle x + y, \gamma, x + y \rangle$
$= \langle x, \gamma, x \rangle + \langle x, \gamma, y \rangle + \langle y, \gamma, x \rangle + \langle y, \gamma, y \rangle$ ; by (i) of 2.
$= \langle x, \gamma, x \rangle + \langle y, \gamma, y \rangle$ ; as $x \perp_\gamma y$ , $\langle x, \gamma, y \rangle = 0$
$= \| x \|_\gamma^2 + \| y \|_\gamma^2$
and
$\| x - y \|_\gamma^2$

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$= \langle x-y, \gamma, x-y \rangle$
$= \langle x, \gamma, x \rangle$ - $\langle x, \gamma, y \rangle$ - $\langle y, \gamma, x \rangle + \langle y, \gamma, y \rangle$
$= \langle x, \gamma, x \rangle + \langle y, \gamma, y \rangle$
$= \| x \|_\gamma^2 + \| y \|_\gamma^2$

## 3.4 Corollary

$\| -x \|_\gamma = \| x \|_\gamma$
*Proof* :
$As\ 0, x \in H_\Gamma, \| 0 - x \|_\gamma^2 = \| 0 \|_\gamma^2 + \| x \|_\gamma^2 = \| x \|_\gamma^2$ .

## 3.5 Theorem:

If x and y are two $\Gamma$-orthogonal elements of $\mathrm{H}_\Gamma, then$
$\| x+y \|_\Gamma^2 = \| x \|_\Gamma^2 + \| y \|_\Gamma^2$
and
$\| x-y \|_\Gamma^2 = \| x \|_\Gamma^2 + \| y \|_\Gamma^2$ .

**Proof:** we have
$\| x+y \|_\Gamma^2$
$= \inf\{ \langle x+y, \gamma, x+y \rangle : \gamma \in \Gamma \}$
$= \inf\{ \langle x, \gamma, x \rangle + \langle x, \gamma, y \rangle + \langle y, \gamma, x \rangle + \langle y, \gamma, y \rangle : \gamma \in \Gamma\}$
$= \inf\{ \langle x, \gamma, x \rangle + \langle y, \gamma, y \rangle : \gamma \in \Gamma \}$
$= \inf\{ \langle x, \gamma, x : \rangle : \gamma \in \Gamma \} + \inf\{ \langle y, \gamma, y \rangle : \gamma \in \Gamma \}$
$= \| x \|_\Gamma^2 + \| y \|_\Gamma^2$
and
$\| x-y \|_\Gamma^2$
$= \inf\{ \langle x-y, \gamma, x-y \rangle : \gamma \in \Gamma \}$
$= \inf\{ \langle x, \gamma, x \rangle$ - $\langle x, \gamma, y \rangle$ - $\langle y, \gamma, x \rangle + \langle y, \gamma, y \rangle : \gamma \in \Gamma \}$
$= \inf\{ \langle x, \gamma, x \rangle + \langle y, \gamma, y \rangle : \gamma \in \Gamma \}$
$= \inf\{ \langle x, \gamma, x \rangle : \gamma \in \Gamma\} + \inf\{ \langle y, \gamma, y \rangle : \gamma \in \Gamma \}$
$= \| x \|_\Gamma^2 + \| y \|_\Gamma^2$ .

## 3.6 Defination:

Let $S \subset H_\Gamma$. Then the set of all elements of $H_\Gamma$, $\gamma$-orthogonal to S is called the $\gamma$-orthogonal complement of S and is denoted by $S^{\perp_\gamma}$ .
The $\gamma$- orthogonal compliment of $S^{\perp_\gamma}$ is denoted by $S^{\perp_\gamma \perp_\gamma}$.

## 3.7 Theorem:

a)$\{\theta\}^{\perp_\gamma} = H_\Gamma$
and $H_\Gamma^{\perp_\gamma} = \{\theta\}$

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b) if $S \subset H_\Gamma$, then $S^{\perp_\gamma}$ is a closed subspace of of $H_\Gamma$ .

Proof:
a)
As $\langle \theta, \gamma, x \rangle = 0$ for all x$\in$ $H_\Gamma$ and $\gamma \in \Gamma$ .
Therefore $\theta \perp_\gamma x$ ; for all x$\in$ $H_\Gamma$ and $\gamma \in \Gamma$ . Hence $\{\theta\}^{\perp_\gamma} = H_\Gamma$.
Again if $x \perp_\gamma y$ for every y $\in H_\Gamma$ and $\gamma \in \Gamma$, then x= $\theta$ thus $H_\Gamma^{\perp_\gamma} = \{\theta\}$.
(b)
Let x,y $\in S^{\perp_\gamma}$ , then for every z $\in$ S and $\gamma \in \Gamma$ ,$\langle x, \gamma, z \rangle = 0$; and $\langle y, \gamma, z \rangle$ =0. . Now for $\alpha$ ,$\beta \in$ F
$\langle \alpha x + \beta y, \gamma, z \rangle$
$= \langle \alpha x, \gamma, z \rangle + \langle \beta y, \gamma, z \rangle$
$= \alpha \ \langle x, \gamma, z \rangle + \beta \ \langle y, \gamma, z \rangle$
$= 0$ .
Thus $\alpha$x + $\beta$y $\in S^{\perp_\gamma}$. Hence $S^{\perp_\gamma}$ is a subspace of $H_\Gamma$. Next we prove that $S^{\perp_\gamma}$ is closed.
Let $\{x_n\} \in S^{\perp_\gamma}$ and x$_n \longrightarrow$ x for some x $\in H_\Gamma$. For continuity of the $\gamma$-inner product, we have
$\langle x, \gamma, y \rangle$
$= \left\langle \lim_{n\to\infty} x_n, \gamma, y \right\rangle$
$= \lim_{n\to\infty} \langle x_n, \gamma, y \rangle$
$= 0$ .
For every y $\in$ S. This shows that x $\in S^{\perp_\gamma}$ , and thus $S^{\perp_\gamma}$ is closed.

### 3.8 Defination:

Let $S \subset H_\Gamma$. Then the set of all elements of $H_\Gamma$, $\Gamma$-orthogonal to S is called the $\Gamma$-orthogonal complement of S and is denoted by $S^{\perp_\Gamma}$ .
The $\Gamma$- orthogonal compliment of $S^{\perp_\Gamma}$ is denoted by $S^{\perp_\Gamma \perp_\Gamma}$.

### 3.9 Theorem:

a)$\{\theta\}^{\perp_\Gamma} = H_\Gamma$
and $H_\Gamma^{\perp_\Gamma} = \{\theta\}$
b) if $S \subset H_\Gamma$, then $S^{\perp_\Gamma}$ is a closed subspace of of $H_\Gamma$

**Proof:** (a) As $\langle \theta, \gamma, x \rangle = 0$ for all x$\in$ $H_\Gamma$ and $\gamma \in \Gamma$ .
Therefore $\theta \perp_\Gamma x$ ; for all x$\in$ $H_\Gamma$ . Hence $\{\theta\}^{\perp_\Gamma} = H_\Gamma$.
Again if $x \perp_\Gamma y$ for every y $\in H_\Gamma$ , then x= $\theta$ thus $H_\Gamma^{\perp_\Gamma} = \{\theta\}$.

(b) Let x,y $\in S^{\perp_\Gamma}$ , then for every z $\in$ S inf$\{\langle x, \gamma, z \rangle : \gamma \in \Gamma \} = 0$; and inf$\{\langle y, \gamma, z \rangle : \gamma \in \Gamma \}$ =0.. Now for $\alpha$ ,$\beta \in$ F
inf$\{\langle \alpha x + \beta y, \gamma, z \rangle$: $\gamma \in \Gamma$ }
$=$ inf$\{\langle \alpha x, \gamma, z \rangle$ :$\gamma \in \Gamma$ } + inf$\{\langle \beta y, \gamma, z \rangle : \gamma \in \Gamma \}$

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$= \alpha \inf\{\langle x, \gamma, z\rangle : \gamma \in \Gamma \} + \beta \inf\{\langle y, \gamma, z\rangle : \gamma \in \Gamma \}$
$= 0$ .
Thus $\alpha$x + $\beta$y $\in S^{\perp_\Gamma}$. Hence $S^{\perp_\Gamma}$ is a subspace of $H_\Gamma$. Next we prove that $S^{\perp_\Gamma}$ is closed. Let $\{x_n\} \in S^{\perp_\Gamma}$ and $x_n \longrightarrow$ x for some x $\in H_\Gamma$. For continuity of the $\Gamma$-inner product, we have
$\inf\{\langle x, \gamma, y\rangle : \gamma \in \Gamma \}$
$= \inf\{\left\langle \lim\limits_{n\to\infty} x_n, \gamma, y \right\rangle : \gamma \in \Gamma \}$
$= \lim\limits_{n\to\infty} \inf\{\langle x_n, \gamma, y\rangle : \gamma \in \Gamma \}$
$= 0$ .
For every y $\in$ S. This shows that x $\in S^{\perp_\Gamma}$ , and thus $S^{\perp_\Gamma}$ is closed.

### 3.10 Theorem:(The closest point property)

Let S be a closed convex subset of a $\Gamma$-Hilbert space $H_\Gamma$ . For every point x $\in H_\Gamma$ there exist a unique point y $\in$ S such that

$\|x - y\|_\Gamma = \inf_{z\in S}\|x - z\|_\Gamma$ .

**Proof:** As $\|x\|_\Gamma \geq 0$, $\forall$ x $\in$ $H_\Gamma$, $\{\|x - z\|_\Gamma$ : z $\in$ S $\}$ is bounded below by 0; and hence $\inf_{z\in S}\|x - z\|_\Gamma$ is exist.
Let $\{y_n\}$ be a sequence in S such that
$\lim\limits_{n\to\infty} \|x - y_n\|_\Gamma = \inf_{z\in S}\|x - z\|_\Gamma$ .
Let $d$ =$\inf_{z\in S}\|x - z\|_\Gamma$. Since S is convex $\frac{1}{2}$ $(y_m + y_n) \in$ S , we have

$\|$ x - $\frac{1}{2}(y_m + y_n)\|_\Gamma \geq$ d for all m, n $\in$ N .

Moreover, by the parallelogram law

$\| (y_m - y_n)\|_\Gamma^2$
$= 4\|$ x - $\frac{1}{2}(y_m + y_n)\|_\Gamma^2 + \| (y_m - y_n)\|_\Gamma^2 - 4\|$ x - $\frac{1}{2}(y_m + y_n)\|_\Gamma^2$
$=\|$ (x - $y_m) + (x - y_n)\|_\Gamma^2 + \|$ (x - $y_m) - (x - y_n)\|_\Gamma^2 - 4\|$ x - $\frac{1}{2}(y_m + y_n)\|_\Gamma^2$
$= 2(\|$x - $y_m\|_\Gamma^2 + \|$x - $y_n\|_\Gamma^2) - 4\|$ x - $\frac{1}{2}(y_m + y_n)\|_\Gamma^2$
*Since*
$2(\|$x - $y_m\|_\Gamma^2 + \|$x - $y_n\|_\Gamma^2) \longrightarrow 4d^2$, as m,n $\longrightarrow \infty$,
and
$\|$ x - $\frac{1}{2}(y_m + y_n)\|_\Gamma^2 \geq d^2$,

we have $\| (y_m - y_n)\|_\Gamma^2 \longrightarrow 0$, as m, n $\longrightarrow \infty$ . Thus $\{y_n\}$ is a cauchy sequence . Since $H_\Gamma$ is complete and S is closed, the limit $\lim_{n\longrightarrow\infty} y_n$ = y exist and y $\in$ S . From the continuity of the $\Gamma$-norm , we obtain
$\|x - y\|_\Gamma = \|x - \lim\limits_{n\to\infty} y_n\|_\Gamma = \|x - y_n\|_\Gamma$ = d = $\inf_{z\in S}\|x - z\|_\Gamma$.

We have proved that there exist point in S satisfying the required condition.It remains to prove the uniqueness.suppose that there is another point $y_1$ in S satisfying the required condition. Then since $\frac{1}{2}$ ( y + $y_1$) $\in$ S , we have

$\|$ y- $y_1\|_\Gamma^2$ =4d$^2$ - 4$\|$ x - $\frac{1}{2}$(y+y$_1$)$\|_\Gamma^2 \leq 0$.
This can only happen if y = $y_1$.

### 3.11 Unique Decomposition Theorem :

If $H_1$ is a closed subspace of a $\Gamma$-Hilbert space $H_\Gamma$, then every element x $\in H_\Gamma$ has a unique decomposition in the form x= y + z where y $\in H_1$ and z $\in H_1^{\perp_\Gamma}$

Proof :
If x $\in H_1$ , then the obvious decomposition is x= x+0 .suppose now that x $\notin H_1$ . Let y be the unique point of $H_1$ satisfying $\|x-y\|_\Gamma = \inf_{w\in H_1}\|x-w\|_\Gamma$ , as in closest point Theorem of $\Gamma$-Hilbert space . We will show that x= y + (x-y) is the desired decomposition. If w $\in H_1$ and $\lambda > 0$ , then y+ $\lambda$w $\in H_1$ and thus
$\|x-y\|_\Gamma^2$
$\leq \|$ x-y - $\lambda$w $\|_\Gamma^2$
$= \inf\{\langle x-y-\lambda w, \gamma, x-y-\lambda w\rangle : \gamma \in \Gamma\}$
$= \inf\{\langle x-y, \gamma, x-y\rangle + 2\lambda \langle x-y, \gamma, -w\rangle + \lambda^2\langle w, \gamma, w\rangle : \gamma \in \Gamma\}$
$=\|x-y\|_\Gamma^2 + \lambda^2\|$ w $\|_\Gamma^2 + 2\lambda \inf\{\langle x-y, \gamma, -w\rangle : \gamma \in \Gamma\}$.
Hence
$\lambda^2\|$ w $\|_\Gamma^2 + 2\lambda \inf\{\langle x-y, \gamma, -w\rangle : \gamma \in \Gamma\} \geq 0$ .
Now dividing $\lambda$ and leeting $\lambda \longrightarrow 0$ we get
$\inf\{\langle x-y, \gamma, w\rangle : \gamma \in \Gamma\} \leq 0$
Since w $\in$H$_1$ implies -w $\in H_1$ , thus the above inequality is also hold with -w instead of w .
i.e $\inf\{\langle x-y, \gamma, w\rangle : \gamma \in \Gamma\} \geq 0$ .
Therefore $\inf\{\langle x-y, \gamma, w\rangle : \gamma \in \Gamma\} = 0$ ;
Which means x-y $\in H_1^{\perp}{}_\Gamma$.
To prove the uniqueness note that if x= y$_1$ + $z_1$, y$_1$$\in H_1$ and z$_1$$\in H_1^{\perp_\Gamma}$, then y-y$_1$$\in H_1$ and z-z$_1$$\in H_1^{\perp_\Gamma}$. Since y-y$_1$ =z$_1$-z , we must have y-y$_1$ = $z_1 - z = 0$

### 3.12 Theorem:

If S is a closed subspace of a $\Gamma$-Hilbert space $H_\Gamma$ , then $S^{\perp_\Gamma\perp_\Gamma}$ = S.

Proof:
If x $\in$ S , then for every z $\in S^{\perp_\Gamma}$, we have $\inf\{\langle x, \gamma, z\rangle : \gamma \in \Gamma\} = 0$, which means x$\in$ $S^{\perp_\Gamma\perp_\Gamma}$ . Thus S $\subset S^{\perp_\Gamma\perp_\Gamma}$. To prove $S^{\perp_\Gamma\perp_\Gamma} \subset$ S consider an x $\in S^{\perp_\Gamma\perp_\Gamma}$. Since S is closed , by unique decomposition theorem , x = y+ z for some y $\in$ S and z $\in S^{\perp_\Gamma}$. In view of the inclusion S $\subset S^{\perp_\Gamma\perp_\Gamma}$ , we have y $\in S^{\perp_\Gamma\perp_\Gamma}$ and thus z= x-y $\in S^{\perp_\Gamma\perp_\Gamma}$, because

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$S^{\perp_\Gamma \perp_\Gamma}$ is a vector space . but z $\in S^{\perp_\Gamma}$, so we must have z = 0 , which means x = y $\in$ S. this shows that $S^{\perp_\Gamma \perp_\Gamma} \subset$ S. which complites the proof.

### 3.13 Theorem

:
If f is a non trivial bounded linear functional on a Γ-Hilbert space $H_\Gamma$ then $dim\mathcal{N}(f)^{\perp_\Gamma}$ = 1 where $\mathcal{N}$(f) is the null space of f.

Proof:
Since f is continuous , $\mathcal{N}$(f) is a closed proper subspace of $H_\Gamma$ and thus $\mathcal{N}(f)^{\perp_\Gamma}$ is not empty. let $x_1$,$x_2 \in \mathcal{N}(f)^{\perp_\Gamma}$ be nonzero vectors. since $f(x_1) \neq 0$ and $f(x_2) \neq 0$ there exist a scalar a $\neq$ 0 suh that $f(x_1)$ + a$f(x_1) = f(x_1 + ax_2) = 0$. thus, $x_1 + ax_2 \in \mathcal{N}$(f). On the other hand since $\mathcal{N}(f)^{\perp_\Gamma}$ is a vector space and $x_1, x_2 \in \mathcal{N}(f)^{\perp_\Gamma}$, we must have $x_1 + ax_2 \in \mathcal{N}(f)^{\perp_\Gamma}$.This is only possible if $x_1 + ax_2 = 0$ , which shows that $x_1$ and $x_2$ are linearly dependent , because a $\neq$ 0. Hence $dim\mathcal{N}(f)^{\perp_\Gamma} = 1$, .

### 3.14 Representation theorem:

Let f be a bounded linear functional on a Γ-Hilbert space $H_\Gamma$. Then there exist exactly one $x_0 \in H_\Gamma$ such that f(x)= $\inf\{\langle x, \gamma, x_0\rangle$ : $\gamma \in \Gamma$ for all x $\in H_\Gamma$. moreover, we have $\| f \|_\Gamma = \| x_0 \|_\Gamma$ .

Proof:
If f(x) = 0 for all x $\in H_\Gamma$ , then $x_0 = 0$ has the desired properties. Assume now that f is a nontrivial functional. Then $dim\mathcal{N}(f)^{\perp_\Gamma} = 1$ , by Theorem 3.13 . Let $z_0$ be a unit vector in $\mathcal{N}(f)^{\perp_\Gamma}$ . Then , for every x $\in H_\Gamma$ , we have
x = x - $\inf\{\langle x, \gamma, z_0\rangle$: $\gamma \in \Gamma$ $\}z_0$ + $\inf\{\langle x, \gamma, z_0\rangle$: $\gamma \in \Gamma$ $\}z_0$.
Since $\inf\{\langle x, \gamma, z_0\rangle$: $\gamma \in \Gamma$ $\}z_0 \in \mathcal{N}(f)^{\perp_\Gamma}$, we must have x - $\inf\{\langle x, \gamma, z_0\rangle$: $\gamma \in \Gamma$ $\}z_0 \in$ $\mathcal{N}$(f), which means that
f( x -$\inf\{\langle x, \gamma, z_0\rangle$ :$\gamma \in \Gamma$ $\}z_0$) = 0.
Consequently,

f(x) = f( $\inf\{\langle x, \gamma, z_0\rangle$: $\gamma \in \Gamma$ $\}z_0$) = $\inf\{\langle x, \gamma, z_0\rangle$ : $\gamma \in \Gamma$ $\}$ $f(z_0)$ = $\inf\{\langle x, \gamma, f(z_0)z_0\rangle$ :$\gamma \in \Gamma$ $\}$ .
Therefore if we put $x_0 = f(z_0)z_0$ ,
then f(x)= $\inf\{\langle x, \gamma, x_0\rangle$ : $\gamma \in \Gamma$ $\}$ for all x $\in H_\Gamma$.
Suppose now there is another point $x_1$ such that f(x)= $\inf\{\langle x, \gamma, x_1\rangle$:$\gamma \in \Gamma$ $\}$ for all x $\in$ $H_\Gamma$. Then $\inf\{\langle x, \gamma, x_0 - x_1\rangle$ :$\gamma \in \Gamma$ $\}$ = 0 for all x $\in H_\Gamma$ , and thus $\langle x_0 - x_1, \gamma, x_0 - x_1\rangle$ = 0. This is only posssible if $x_0 = x_1$.
Finally, we have

$\| f \|_\Gamma$
= $\sup_{\|x\|_\Gamma = 1} \mid f(x) \mid$

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$= \sup_{\|x\|_\Gamma=1} | \inf\{\langle x, \gamma.x_0\rangle : \gamma \in \Gamma \} |$
$\leq \sup_{\|x\|_\Gamma=1}(\| x \|_\Gamma \| x_0 \|_\Gamma)$
$= \| x_0 \|_\Gamma$ .

*and*
$\| x_0 \|_\Gamma^2 = \inf\{\langle x_0, \gamma, x_0\rangle : \gamma \in \Gamma \} = | f(x_0) | \leq \| f \|_\Gamma \| x_0 \|_\Gamma$.

Therefore $\| f \|_\Gamma = \| x_0 \|_\Gamma$.

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