

Representation Theorem On Γ -Hilbert Space

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Abstract

In this paper we discuss about the simple properties of Γ -Hilbert space, introduced by D.K. Bhattacharya and T.E. Aman in their paper Γ -Hilbert Space and linear Quadratic Control problem in 2003. We have defined the orthogonality in Γ -Hilbert space and discuss about the closest point property, Unique Decomposition Theorem following the defined orthogonality on that space. Further we discuss the representation of any bounded linear functionals on Γ -Hilbert space in terms of Γ -inner product in that space.

Key word : Hilbert space, Γ -Hilbert Space, Unique Decomposition Theorem, Riesz Representation Theorem.

1 INTRODUCTION:

The definition of Γ -Hilbert space was introduced by Bhattacharya D.K and T.E. Aman in their paper " Γ - Hilbert space and linear quadratic control problem" in 2003[1]. But we found no literature on Γ -Hilbert Space after that. So it is very essential to develop the study on Γ -Hilbert Space, mainly Orthogonality and representation of any bounded linear functional on Γ -Hilbert Space. In his paper we have defined orthogonality of elements of Γ -Hilbert Space and following this definition we have developed the Closest Point Property ([2],[3],[4]), Unique Decomposition Theorem ([2],[3],[4]) and Representation Theorem ([2],[3],[4]) in Γ -Hilbert Space.

2 Γ -Hilbert space[1]:

Definition Let E, Γ be two linear space over the field R . A mapping $\langle \cdot, \cdot \rangle : E \times \Gamma \times E \rightarrow R$ is called a Γ inner product on E if

- (i) $\langle \cdot, \cdot, \cdot \rangle$ is linear in each variable.
- (ii) $\langle u, \gamma, v \rangle = \langle v, \gamma, u \rangle \forall u, v$ belongs to E and γ belongs to Γ .
- (iii) $\langle u, \gamma, u \rangle > 0, \forall \gamma \neq 0$ and $u \neq 0$.

$[(E, \Gamma), \langle \cdot, \cdot, \cdot \rangle]$ is called a Γ -inner product space over R .

A complete Γ -inner product space is called Γ -Hilbert space.

It follows that for each $\gamma \in \Gamma, \sqrt{\langle u, \gamma, u \rangle}$ gives the properties of the norm for $u \in E$. It is called the γ norm of u and is denoted by $\|u\|_\gamma$

We observe that $\inf\{\langle u, \gamma, u \rangle : \gamma \in \Gamma\}$ satisfies all conditions of a norm, we called it the Γ -norm and is denoted by

$$\|u\|_\Gamma = \inf\{\langle u, \gamma, u \rangle : \gamma \in \Gamma\}.$$

3 Orthogonality of Γ -Hilbert Spaces

3.1 Definition :(γ -Orthogonal)

Let L be a non-empty subset of a Γ - Hilbert space H_Γ . Two elements x and y of H_Γ are said to be γ -orthogonal if their inner product $\langle x, \gamma, y \rangle = 0$. In symbol, we write $x \perp_\gamma y$. If x is γ -orthogonal to every element of L then we say that x is γ -orthogonal to L and in symbol we write $x \perp_\gamma L$.

3.2 Definition:(Γ -Orthogonal)

Let L be a non-empty subset of a Γ - Hilbert space H_Γ . Two elements x and y of H_Γ are said to be Γ -orthogonal or simply orthogonal if $\langle x, \gamma, y \rangle = 0$ for all $\gamma \in \Gamma$. In symbol, we write $x \perp_\Gamma y$. If x is Γ -orthogonal to every element of L then we say that x is Γ -orthogonal to L and in symbol we write $x \perp_\Gamma L$.

3.3 Theorem:

If x and y are two γ -orthogonal elements of H_Γ , then

$$\|x + y\|_\gamma^2 = \|x\|_\gamma^2 + \|y\|_\gamma^2$$

and

$$\|x - y\|_\gamma^2 = \|x\|_\gamma^2 + \|y\|_\gamma^2.$$

Proof: we have

$$\|x + y\|_\gamma^2$$

$$= \langle x + y, \gamma, x + y \rangle$$

$$= \langle x, \gamma, x \rangle + \langle x, \gamma, y \rangle + \langle y, \gamma, x \rangle + \langle y, \gamma, y \rangle ; \text{ by (i) of 2.}$$

$$= \langle x, \gamma, x \rangle + \langle y, \gamma, y \rangle ; \text{ as } x \perp_\gamma y, \langle x, \gamma, y \rangle = 0$$

$$= \|x\|_\gamma^2 + \|y\|_\gamma^2$$

and

$$\|x - y\|_\gamma^2$$

$$\begin{aligned}
 &= \langle x - y, \gamma, x - y \rangle \\
 &= \langle x, \gamma, x \rangle - \langle x, \gamma, y \rangle - \langle y, \gamma, x \rangle + \langle y, \gamma, y \rangle \\
 &= \langle x, \gamma, x \rangle + \langle y, \gamma, y \rangle \\
 &= \| x \|_{\gamma}^2 + \| y \|_{\gamma}^2
 \end{aligned}$$

3.4 Corollary

$$\| -x \|_{\gamma} = \| x \|_{\gamma}$$

Proof :

$$\text{As } 0, x \in H_{\Gamma}, \| 0 - x \|_{\gamma}^2 = \| 0 \|_{\gamma}^2 + \| x \|_{\gamma}^2 = \| x \|_{\gamma}^2 .$$

3.5 Theorem:

If x and y are two Γ -orthogonal elements of H_{Γ} , then

$$\| x + y \|_{\Gamma}^2 = \| x \|_{\Gamma}^2 + \| y \|_{\Gamma}^2$$

and

$$\| x - y \|_{\Gamma}^2 = \| x \|_{\Gamma}^2 + \| y \|_{\Gamma}^2 .$$

Proof: we have

$$\begin{aligned}
 &\| x + y \|_{\Gamma}^2 \\
 &= \inf\{ \langle x + y, \gamma, x + y \rangle : \gamma \in \Gamma \} \\
 &= \inf\{ \langle x, \gamma, x \rangle + \langle x, \gamma, y \rangle + \langle y, \gamma, x \rangle + \langle y, \gamma, y \rangle : \gamma \in \Gamma \} \\
 &= \inf\{ \langle x, \gamma, x \rangle + \langle y, \gamma, y \rangle : \gamma \in \Gamma \} \\
 &= \inf\{ \langle x, \gamma, x \rangle : \gamma \in \Gamma \} + \inf\{ \langle y, \gamma, y \rangle : \gamma \in \Gamma \} \\
 &= \| x \|_{\Gamma}^2 + \| y \|_{\Gamma}^2
 \end{aligned}$$

and

$$\begin{aligned}
 &\| x - y \|_{\Gamma}^2 \\
 &= \inf\{ \langle x - y, \gamma, x - y \rangle : \gamma \in \Gamma \} \\
 &= \inf\{ \langle x, \gamma, x \rangle - \langle x, \gamma, y \rangle - \langle y, \gamma, x \rangle + \langle y, \gamma, y \rangle : \gamma \in \Gamma \} \\
 &= \inf\{ \langle x, \gamma, x \rangle + \langle y, \gamma, y \rangle : \gamma \in \Gamma \} \\
 &= \inf\{ \langle x, \gamma, x \rangle : \gamma \in \Gamma \} + \inf\{ \langle y, \gamma, y \rangle : \gamma \in \Gamma \} \\
 &= \| x \|_{\Gamma}^2 + \| y \|_{\Gamma}^2 .
 \end{aligned}$$

3.6 Defination:

Let $S \subset H_{\Gamma}$. Then the set of all elements of H_{Γ} , γ -orthogonal to S is called the γ -orthogonal complement of S and is denoted by $S^{\perp_{\gamma}}$.

The γ -orthogonal compliment of $S^{\perp_{\gamma}}$ is denoted by $S^{\perp_{\gamma} \perp_{\gamma}}$.

3.7 Theorem:

$$a) \{ \theta \}^{\perp_{\gamma}} = H_{\Gamma}$$

$$\text{and } H_{\Gamma}^{\perp_{\gamma}} = \{ \theta \}$$

b) if $S \subset H_\Gamma$, then $S^{\perp\gamma}$ is a closed subspace of H_Γ .

Proof:

a)

As $\langle \theta, \gamma, x \rangle = 0$ for all $x \in H_\Gamma$ and $\gamma \in \Gamma$.

Therefore $\theta \perp_\gamma x$; for all $x \in H_\Gamma$ and $\gamma \in \Gamma$. Hence $\{\theta\}^{\perp\gamma} = H_\Gamma$.

Again if $x \perp_\gamma y$ for every $y \in H_\Gamma$ and $\gamma \in \Gamma$, then $x = \theta$ thus $H_\Gamma^{\perp\gamma} = \{\theta\}$.

(b)

Let $x, y \in S^{\perp\gamma}$, then for every $z \in S$ and $\gamma \in \Gamma$, $\langle x, \gamma, z \rangle = 0$; and $\langle y, \gamma, z \rangle = 0$. Now for $\alpha, \beta \in F$

$$\begin{aligned} & \langle \alpha x + \beta y, \gamma, z \rangle \\ &= \langle \alpha x, \gamma, z \rangle + \langle \beta y, \gamma, z \rangle \\ &= \alpha \langle x, \gamma, z \rangle + \beta \langle y, \gamma, z \rangle \\ &= 0. \end{aligned}$$

Thus $\alpha x + \beta y \in S^{\perp\gamma}$. Hence $S^{\perp\gamma}$ is a subspace of H_Γ . Next we prove that $S^{\perp\gamma}$ is closed.

Let $\{x_n\} \in S^{\perp\gamma}$ and $x_n \rightarrow x$ for some $x \in H_\Gamma$. For continuity of the γ -inner product, we have

$$\begin{aligned} & \langle x, \gamma, y \rangle \\ &= \left\langle \lim_{n \rightarrow \infty} x_n, \gamma, y \right\rangle \\ &= \lim_{n \rightarrow \infty} \langle x_n, \gamma, y \rangle \\ &= 0. \end{aligned}$$

For every $y \in S$. This shows that $x \in S^{\perp\gamma}$, and thus $S^{\perp\gamma}$ is closed.

3.8 Defination:

Let $S \subset H_\Gamma$. Then the set of all elements of H_Γ , Γ -orthogonal to S is called the Γ -orthogonal complement of S and is denoted by $S^{\perp\Gamma}$.

The Γ -orthogonal compliment of $S^{\perp\Gamma}$ is denoted by $S^{\perp\Gamma\perp}$.

3.9 Theorem:

a) $\{\theta\}^{\perp\Gamma} = H_\Gamma$

and $H_\Gamma^{\perp\Gamma} = \{\theta\}$

b) if $S \subset H_\Gamma$, then $S^{\perp\Gamma}$ is a closed subspace of H_Γ

Proof: (a) As $\langle \theta, \gamma, x \rangle = 0$ for all $x \in H_\Gamma$ and $\gamma \in \Gamma$.

Therefore $\theta \perp_\Gamma x$; for all $x \in H_\Gamma$. Hence $\{\theta\}^{\perp\Gamma} = H_\Gamma$.

Again if $x \perp_\Gamma y$ for every $y \in H_\Gamma$, then $x = \theta$ thus $H_\Gamma^{\perp\Gamma} = \{\theta\}$.

(b) Let $x, y \in S^{\perp\Gamma}$, then for every $z \in S$ $\inf\{\langle x, \gamma, z \rangle : \gamma \in \Gamma\} = 0$; and $\inf\{\langle y, \gamma, z \rangle : \gamma \in \Gamma\} = 0$.. Now for $\alpha, \beta \in F$

$$\begin{aligned} & \inf\{\langle \alpha x + \beta y, \gamma, z \rangle : \gamma \in \Gamma\} \\ &= \inf\{\langle \alpha x, \gamma, z \rangle : \gamma \in \Gamma\} + \inf\{\langle \beta y, \gamma, z \rangle : \gamma \in \Gamma\} \end{aligned}$$

$$= \alpha \inf\{\langle x, \gamma, z \rangle : \gamma \in \Gamma\} + \beta \inf\{\langle y, \gamma, z \rangle : \gamma \in \Gamma\} \\ = 0 .$$

Thus $\alpha x + \beta y \in S^{\perp\Gamma}$. Hence $S^{\perp\Gamma}$ is a subspace of H_Γ . Next we prove that $S^{\perp\Gamma}$ is closed. Let $\{x_n\} \in S^{\perp\Gamma}$ and $x_n \rightarrow x$ for some $x \in H_\Gamma$. For continuity of the Γ -inner product, we have

$$\inf\{\langle x, \gamma, y \rangle : \gamma \in \Gamma\} \\ = \inf\{\langle \lim_{n \rightarrow \infty} x_n, \gamma, y \rangle : \gamma \in \Gamma\} \\ = \lim_{n \rightarrow \infty} \inf\{\langle x_n, \gamma, y \rangle : \gamma \in \Gamma\} \\ = 0 .$$

For every $y \in S$. This shows that $x \in S^{\perp\Gamma}$, and thus $S^{\perp\Gamma}$ is closed.

3.10 Theorem:(The closest point property)

Let S be a closed convex subset of a Γ -Hilbert space H_Γ . For every point $x \in H_\Gamma$ there exist a unique point $y \in S$ such that

$$\|x - y\|_\Gamma = \inf_{z \in S} \|x - z\|_\Gamma .$$

Proof: As $\|x\|_\Gamma \geq 0, \forall x \in H_\Gamma, \{\|x - z\|_\Gamma : z \in S\}$ is bounded below by 0; and hence $\inf_{z \in S} \|x - z\|_\Gamma$ is exist.

Let $\{y_n\}$ be a sequence in S such that

$$\lim_{n \rightarrow \infty} \|x - y_n\|_\Gamma = \inf_{z \in S} \|x - z\|_\Gamma .$$

Let $d = \inf_{z \in S} \|x - z\|_\Gamma$. Since S is convex $\frac{1}{2}(y_m + y_n) \in S$, we have

$$\|x - \frac{1}{2}(y_m + y_n)\|_\Gamma \geq d \text{ for all } m, n \in \mathbb{N} .$$

Moreover, by the parallelogram law

$$\|y_m - y_n\|_\Gamma^2 \\ = 4\|x - \frac{1}{2}(y_m + y_n)\|_\Gamma^2 + \|y_m - y_n\|_\Gamma^2 - 4\|x - \frac{1}{2}(y_m + y_n)\|_\Gamma^2 \\ = \|(x - y_m) + (x - y_n)\|_\Gamma^2 + \|(x - y_m) - (x - y_n)\|_\Gamma^2 - 4\|x - \frac{1}{2}(y_m + y_n)\|_\Gamma^2 \\ = 2(\|x - y_m\|_\Gamma^2 + \|x - y_n\|_\Gamma^2) - 4\|x - \frac{1}{2}(y_m + y_n)\|_\Gamma^2$$

Since

$$2(\|x - y_m\|_\Gamma^2 + \|x - y_n\|_\Gamma^2) \rightarrow 4d^2, \text{ as } m, n \rightarrow \infty,$$

and

$$\|x - \frac{1}{2}(y_m + y_n)\|_\Gamma^2 \geq d^2,$$

we have $\|y_m - y_n\|_\Gamma^2 \rightarrow 0$, as $m, n \rightarrow \infty$. Thus $\{y_n\}$ is a Cauchy sequence. Since H_Γ is complete and S is closed, the limit $\lim_{n \rightarrow \infty} y_n = y$ exist and $y \in S$. From the continuity of the Γ -norm, we obtain

$$\|x - y\|_\Gamma = \|x - \lim_{n \rightarrow \infty} y_n\|_\Gamma = \|x - y_n\|_\Gamma = d = \inf_{z \in S} \|x - z\|_\Gamma .$$

We have proved that there exist point in S satisfying the required condition. It remains to

prove the uniqueness. suppose that there is another point y_1 in S satisfying the required condition. Then since $\frac{1}{2} (y + y_1) \in S$, we have

$$\|y - y_1\|_{\Gamma}^2 = 4d^2 - 4\|x - \frac{1}{2}(y+y_1)\|_{\Gamma}^2 \leq 0.$$

This can only happen if $y = y_1$.

3.11 Unique Decomposition Theorem :

If H_1 is a closed subspace of a Γ -Hilbert space H_{Γ} , then every element $x \in H_{\Gamma}$ has a unique decomposition in the form $x = y + z$ where $y \in H_1$ and $z \in H_1^{\perp\Gamma}$

Proof :

If $x \in H_1$, then the obvious decomposition is $x = x + 0$. suppose now that $x \notin H_1$. Let y be the unique point of H_1 satisfying $\|x - y\|_{\Gamma} = \inf_{w \in H_1} \|x - w\|_{\Gamma}$, as in closest point Theorem of Γ -Hilbert space. We will show that $x = y + (x - y)$ is the desired decomposition. If $w \in H_1$ and $\lambda > 0$, then $y + \lambda w \in H_1$ and thus

$$\begin{aligned} & \|x - y\|_{\Gamma}^2 \\ & \leq \|x - y - \lambda w\|_{\Gamma}^2 \\ & = \inf\{\langle x - y - \lambda w, \gamma, x - y - \lambda w \rangle : \gamma \in \Gamma\} \\ & = \inf\{\langle x - y, \gamma, x - y \rangle + 2\lambda \langle x - y, \gamma, -w \rangle + \lambda^2 \langle w, \gamma, w \rangle : \gamma \in \Gamma\} \\ & = \|x - y\|_{\Gamma}^2 + \lambda^2 \|w\|_{\Gamma}^2 + 2\lambda \inf\{\langle x - y, \gamma, -w \rangle : \gamma \in \Gamma\}. \end{aligned}$$

Hence

$$\lambda^2 \|w\|_{\Gamma}^2 + 2\lambda \inf\{\langle x - y, \gamma, -w \rangle : \gamma \in \Gamma\} \geq 0.$$

Now dividing λ and letting $\lambda \rightarrow 0$ we get

$$\inf\{\langle x - y, \gamma, w \rangle : \gamma \in \Gamma\} \leq 0$$

Since $w \in H_1$ implies $-w \in H_1$, thus the above inequality is also hold with $-w$ instead of w .

$$\text{i.e } \inf\{\langle x - y, \gamma, w \rangle : \gamma \in \Gamma\} \geq 0.$$

Therefore $\inf\{\langle x - y, \gamma, w \rangle : \gamma \in \Gamma\} = 0$;

Which means $x - y \in H_1^{\perp\Gamma}$.

To prove the uniqueness note that if $x = y_1 + z_1, y_1 \in H_1$ and $z_1 \in H_1^{\perp\Gamma}$, then $y - y_1 \in H_1$ and $z - z_1 \in H_1^{\perp\Gamma}$. Since $y - y_1 = z_1 - z$, we must have $y - y_1 = z_1 - z = 0$

3.12 Theorem:

If S is a closed subspace of a Γ -Hilbert space H_{Γ} , then $S^{\perp\Gamma\perp\Gamma} = S$.

Proof:

If $x \in S$, then for every $z \in S^{\perp\Gamma}$, we have $\inf\{\langle x, \gamma, z \rangle : \gamma \in \Gamma\} = 0$, which means $x \in S^{\perp\Gamma\perp\Gamma}$. Thus $S \subset S^{\perp\Gamma\perp\Gamma}$. To prove $S^{\perp\Gamma\perp\Gamma} \subset S$ consider an $x \in S^{\perp\Gamma\perp\Gamma}$. Since S is closed, by unique decomposition theorem, $x = y + z$ for some $y \in S$ and $z \in S^{\perp\Gamma}$. In view of the inclusion $S \subset S^{\perp\Gamma\perp\Gamma}$, we have $y \in S^{\perp\Gamma\perp\Gamma}$ and thus $z = x - y \in S^{\perp\Gamma\perp\Gamma}$, because

$S^{\perp\Gamma\perp\Gamma}$ is a vector space . but $z \in S^{\perp\Gamma}$, so we must have $z = 0$, which means $x = y \in S$. this shows that $S^{\perp\Gamma\perp\Gamma} \subset S$. which complites the proof.

3.13 Theorem

:

If f is a non trivial bounded linear functional on a Γ -Hilbert space H_Γ then $dim\mathcal{N}(f)^{\perp\Gamma} = 1$ where $\mathcal{N}(f)$ is the null space of f .

Proof:

Since f is continuous , $\mathcal{N}(f)$ is a closed proper subspace of H_Γ and thus $\mathcal{N}(f)^{\perp\Gamma}$ is not empty. let $x_1, x_2 \in \mathcal{N}(f)^{\perp\Gamma}$ be nonzero vectors. since $f(x_1) \neq 0$ and $f(x_2) \neq 0$ there exist a scalar $a \neq 0$ suh that $f(x_1) + af(x_2) = f(x_1 + ax_2) = 0$. thus, $x_1 + ax_2 \in \mathcal{N}(f)$. On the other hand since $\mathcal{N}(f)^{\perp\Gamma}$ is a vector space and $x_1, x_2 \in \mathcal{N}(f)^{\perp\Gamma}$, we must have $x_1 + ax_2 \in \mathcal{N}(f)^{\perp\Gamma}$. This is only possible if $x_1 + ax_2 = 0$, which shows that x_1 and x_2 are linearly dependent , because $a \neq 0$. Hence $dim\mathcal{N}(f)^{\perp\Gamma} = 1$, .

3.14 Representation theorem:

Let f be a bounded linear functional on a Γ -Hilbert space H_Γ . Then there exist exactly one $x_0 \in H_\Gamma$ such that $f(x) = \inf\{\langle x, \gamma, x_0 \rangle : \gamma \in \Gamma\}$ for all $x \in H_\Gamma$. moreover, we have $\|f\|_\Gamma = \|x_0\|_\Gamma$.

Proof:

If $f(x) = 0$ for all $x \in H_\Gamma$, then $x_0 = 0$ has the desired properties. Assume now that f is a nontrivial functional. Then $dim\mathcal{N}(f)^{\perp\Gamma} = 1$, by Theorem 3.13 . Let z_0 be a unit vector in $\mathcal{N}(f)^{\perp\Gamma}$. Then , for every $x \in H_\Gamma$, we have

$$x = x - \inf\{\langle x, \gamma, z_0 \rangle : \gamma \in \Gamma\}z_0 + \inf\{\langle x, \gamma, z_0 \rangle : \gamma \in \Gamma\}z_0.$$

Since $\inf\{\langle x, \gamma, z_0 \rangle : \gamma \in \Gamma\}z_0 \in \mathcal{N}(f)^{\perp\Gamma}$, we must have $x - \inf\{\langle x, \gamma, z_0 \rangle : \gamma \in \Gamma\}z_0 \in \mathcal{N}(f)$, which means that

$$f(x - \inf\{\langle x, \gamma, z_0 \rangle : \gamma \in \Gamma\}z_0) = 0.$$

Consequently,

$$f(x) = f(\inf\{\langle x, \gamma, z_0 \rangle : \gamma \in \Gamma\}z_0) = \inf\{\langle x, \gamma, z_0 \rangle : \gamma \in \Gamma\} f(z_0) = \inf\{\langle x, \gamma, f(z_0)z_0 \rangle : \gamma \in \Gamma\}.$$

Therefore if we put $x_0 = f(z_0)z_0$,

then $f(x) = \inf\{\langle x, \gamma, x_0 \rangle : \gamma \in \Gamma\}$ for all $x \in H_\Gamma$.

Suppose now there is another point x_1 such that $f(x) = \inf\{\langle x, \gamma, x_1 \rangle : \gamma \in \Gamma\}$ for all $x \in H_\Gamma$. Then $\inf\{\langle x, \gamma, x_0 - x_1 \rangle : \gamma \in \Gamma\} = 0$ for all $x \in H_\Gamma$, and thus $\langle x_0 - x_1, \gamma, x_0 - x_1 \rangle = 0$. This is only possible if $x_0 = x_1$.

Finally, we have

$$\begin{aligned} \|f\|_\Gamma &= \sup_{\|x\|_\Gamma=1} |f(x)| \end{aligned}$$

$$\begin{aligned} &= \sup_{\|x\|_{\Gamma}=1} | \inf\{ \langle x, \gamma \cdot x_0 \rangle : \gamma \in \Gamma \} | \\ &\leq \sup_{\|x\|_{\Gamma}=1} (\|x\|_{\Gamma} \|x_0\|_{\Gamma}) \\ &= \|x_0\|_{\Gamma} . \end{aligned}$$

and

$$\|x_0\|_{\Gamma}^2 = \inf\{ \langle x_0, \gamma \cdot x_0 \rangle : \gamma \in \Gamma \} = |f(x_0)| \leq \|f\|_{\Gamma} \|x_0\|_{\Gamma} .$$

Therefore $\|f\|_{\Gamma} = \|x_0\|_{\Gamma}$.

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