Composition Operators on Weighted Orlicz Spaces

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Abstract. In this paper, we study the boundedness of composition operators between any two weighted Orlicz spaces.

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1 Introduction

Let $X = (X, \Sigma, \mu)$ be a σ- finite complete measure space. Any measurable nonsingular transformation τ induces a composition operator C_τ from $L^0(X)$ to itself defined by

$$C_\tau f(x) = f(\tau(x)), \qquad x \in X,\ f \in L^0(X),$$

where $L^0(X)$ denotes the linear space of all equivalence classes of all real valued Σ-measurable function on X, where we identify any two functions that are equal μ-almost everywhere on X.

A nondecreasing continuous convex function $\phi : [0,\ \infty) \to [0,\ \infty)$ for which $\phi(0) = 0$ and $\lim_{x\to\infty} \phi(x) = \infty$ is called a $Young\ function$. A function $\Phi : X \times [0, \infty) \longrightarrow [0, \infty)$ is said to be a $generalized\ Young\ function$ or $Musielak - Orlicz\ function$ if

(i) $\Phi(x, \cdot)$ is a Young function for almost every $x \in X$ and

(ii) $\Phi(\cdot, u)$ is Σ- measurable for every $u \geq 0$.

For any generalized Young function Φ, the $Musielak - Orlicz\ space$ associated with Φ, denoted by $L^\Phi(X)$, is defined as the set of all $f \in L^0(X)$ such that

$$I_\Phi(\lambda f) = \int_X \Phi(x, \lambda|f(x)|)d\mu(x) < \infty \qquad \text{for some} \ \ \lambda = \lambda(f) > 0.$$

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The function I_Φ is called the *modular*. The space $L^\Phi(X)$ is a Banach space with the *Luxemburg − Nakano norm*

$$\|f\|_\Phi = \inf\{\lambda > 0 \mid I_\Phi(f/\lambda) \leq 1\}.$$

Let ϕ be an Orlicz function and w be a weight in X i.e. an a.e. positive and integrable real valued function in X. Then $\Phi : X \times [0, \infty) \longrightarrow [0, \infty)$ defined by

$$\Phi(x, u) = \phi(u)w(x), \quad x \in X, u \geq 0,$$

is a generalized Young function. The resulting Musielak-Orlicz space is called *weighted Orlicz space* and is denoted by $L^\phi_w(X)$. In this case, the modular I_Φ is given by

$$I_\Phi(f) = \int_X \phi(|f(x)|)w(x)d\mu(x).$$

If Φ is independent of x, then the resulting Musielak-Orlicz space is simply called *Orlicz space* and is denoted by $L^\phi(X)$.
Composition operators on Orlicz spaces have also been studied in [3], [4], [5],[8] and [14]. The techniques used in this paper essentially depend on the conditions of embedding of one Orlicz space into another (see, [11, Page 45] for details).

2 Boundedness of Composition Operators

In this section, we study the boundedness of composition operators on weighted Orlicz spaces.

Lemma 2.1. (*[11, Lemma 8.3]*) *Let (X, Σ, μ) be a σ-finite nonatomic measure space, $\{\alpha_n\}$ a sequence of positive numbers and $\{s_n\}$ a sequence of measurable, finite, non-negative functions on X such that for $n = 1, 2, \ldots$*

$$\int_X s_n(x)d\mu(x) \geq 2^n\alpha_n.$$

Then there exist an increasing sequence $\{n_k\}$ of integers and a sequence $\{A_k\}$ of pairwise disjoint measurable sets such that for $k = 1, 2, \ldots$

$$\int_{A_k} s_{n_k}(x)d\mu(x) = \alpha_{n_k}.$$

Theorem 2.2. *Let (X, Σ, μ) be a σ-finite nonatomic measure space, w_1 and w_2 be weights in X and $\tau : X \to X$ be a measurable non-singular transformation such that $\tau(X) = X$. Denote by g_τ the Raydon-Nikodym derivative $d\mu \circ \tau^{-1}/d\mu$. Then the composition operator $C_\tau : L^{\phi_1}_{w_1}(X) \longrightarrow L^{\phi_2}_{w_2}(X)$ is bounded if and only if there exist $a, b > 0$ and $0 \leq h \in L^1(X)$ such that*

$$\phi_2(au)(w_2 \circ \tau^{-1})(x)g_\tau(x) \leq b\phi_1(u)w_1(x) + h(x)$$

for almost all $x \in X$ and for all $u \geq 0$.

Proof. Suppose that the given condition holds. Let $0 \neq f \in L^{\phi_1}_{w_1}(X)$. Let $M \geq 1$ be a real number satisfying $M(b + \|h\|_1) \geq 1$. Then

$$
\begin{aligned}
&I_{\Phi_2}\left(\frac{C_\tau f}{(M(b+\|h\|_1)\|f\|_{\Phi_1})/a}\right)\\
&= \int_X \phi_2\left(\frac{a|C_\tau f(x)|}{M(b+\|h\|_1)\|f\|_{\Phi_1}}\right) w_2(x)d\mu(x)\\
&\leq \frac{1}{M(b+\|h\|_1)} \int_X \phi_2\left(\frac{a|f(\tau(x))|}{\|f\|_{\Phi_1}}\right) w_2(x)d\mu(x)\\
&= \frac{1}{M(b+\|h\|_1)} \int_{\tau(X)} \phi_2\left(\frac{a|f(y)|}{\|f\|_{\Phi_1}}\right) (w_2\circ\tau^{-1})(y)d(\mu\circ\tau^{-1})(y)\\
&= \frac{1}{M(b+\|h\|_1)} \int_X \phi_2\left(\frac{a|f(y)|}{\|f\|_{\Phi_1}}\right) (w_2\circ\tau^{-1})(y)d(\mu\circ\tau^{-1})(y)\\
&= \frac{1}{M(b+\|h\|_1)} \int_X \phi_2\left(\frac{a|f(y)|}{\|f\|_{\Phi_1}}\right) (w_2\circ\tau^{-1})(y)g_\tau(y)d\mu(y)\\
&\leq \frac{1}{M(b+\|h\|_1)} \int_X \left(b\ \phi_1\left(\frac{|f(y)|}{\|f\|_{\Phi_1}}\right) w_1(x) + h(y)\right) d\mu(y)\\
&\leq 1.
\end{aligned}
$$

Thus $\|C_\tau f\|_{\Phi_2} \leq \frac{M}{a}(b + \|h\|_1)\|f\|_{\Phi_1}$. This shows that C_τ is bounded.
Consider the function

$$h_n(x) = \sup_{u\geq 0}\left(\phi_2(2^{-n}u)(w_2\circ\tau^{-1})(x)g_\tau(x) - 2^n\phi_1(u)w_1(x)\right).$$

Write $X = \bigcup_{i=1}^{\infty} X_i$, where $\{X_i\}_{i=1}^{\infty}$ is a pairwise disjoint sequence of measurable subsets of X with $\mu(X_i) < \infty$ for every $i = 1, 2, \ \ldots$
For every $q \in \mathbb{Q}^+$,we put $f_{q,i}(x) = q\chi_{X_i}(x)$, where χ_{X_i} is the characteristic function of X_i. Then it can be shown that

$$h_n(x) = \sup_{\substack{q\in\mathbb{Q}^+\\ i\in\mathbb{N}}}\left(\phi_2(2^{-n}f_{q,i}(x))(w_2\circ\tau^{-1})(x)g_\tau(x) - 2^n\phi_1(f_{q,i}(x))w_1(x)\right).$$

Taking (f_k) to be a rearrangement of $(f_{q,i})$ with $f_1 = f_{0,i}$, the above equation can be rewritten as

$$h_n(x) = \sup_{k\in\mathbb{N}}\left(\phi_2(2^{-n}f_k(x))(w_2\circ\tau^{-1})(x)g_\tau(x) - 2^n\phi_1(f_k(x))w_1(x)\right).$$

Then h_n are measurable and $h_n(x) \geq 0$ for each $x \in X$. To complete the proof, we have to show that $\int_X h_n(x)d\mu(x) < \infty$ for some n. Suppose this is not true.
Denote

$$r_{m,n}(x) = \max_{1\leq k\leq m}\left(\phi_2(2^{-n}f_k(x))(w_2\circ\tau^{-1})(x)g_\tau(x) - 2^n\phi_1(f_k(x))w_1(x)\right).$$

Then $r_{m,n}$ are measurable, $r_{m,n}(x) \geq 0$ and $r_{m,n}(x)$ is a nondecreasing sequence tending to $h_n(x)$ as $m \to \infty$ for every $x \in X$. Thus for any n, there exists m_n such that $\int_X r_{m_n,n}(x)d\mu(x) \geq 2^n$. Taking $r_n = r_{m_n,n}$, we have $\int_X r_n(x)d\mu(x) \geq 2^n$ for $n = 1, 2, \ldots$
Let

$$E_{n,k} = \left\{x \in X \mid \phi_2(2^{-n}f_k(x))(w_2 \circ \tau^{-1})(x)g_\tau(x) - 2^n\phi_1(f_k(x))w_1(x) = r_n(x)\right\}$$

and

$$E_n = X \setminus (E_{n,1} \cup E_{n,2} \cup \ldots \cup E_{n,m_n}).$$

Then $\mu(E_n) = 0$.
Let

$$\tilde{f}_n(x) = \begin{cases} 0 & \text{if } x \in E_{n,1} \cup E_n \\ f_k(x) & \text{if } x \in E_{n,k} \setminus \bigcup_{j=1}^{k-1} E_{n,j},\ k = 2, 3, \ldots, m_n. \end{cases}$$

Then

$$\begin{aligned} r_n(x) &= \phi_2(2^{-n}\tilde{f}_n(x))(w_2 \circ \tau^{-1})(x)g_\tau(x) - 2^n\phi_1(\tilde{f}_n(x))w_1(x) \\ &\geq 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_X \phi_2(2^{-n}\tilde{f}_n(x))(w_2 \circ \tau^{-1})(x)g_\tau(x)d\mu(x) &= 2^n \int_X \phi_1(\tilde{f}_n(x))w_1(x)d\mu(x) \\ &\quad + \int_X r_n(x)d\mu(x) \\ &\geq \int_X r_n(x)d\mu(x) \\ &\geq 2^n. \end{aligned}$$

By Lemma 2.1, we obtain an increasing sequence $\{n_k\}$ and a sequence $\{A_k\}$ of pairwise disjoint measurable sets such that

$$\int_{A_k} \phi_2(2^{-n_k}\tilde{f}_{n_k}(x))(w_2 \circ \tau^{-1})(x)g_\tau(x)d\mu(x) = 1, \quad k = 1, 2, \ldots$$

Put

$$f(x) = \begin{cases} \tilde{f}_{n_k}(x) & \text{if } x \in A_k \\ 0 & \text{otherwise.} \end{cases}$$

Let $\lambda > 0$. Choose p large enough that $2^{-n_p} \leq \lambda$. Then

$$\begin{aligned}\int_X \phi_2\left(\lambda C_\tau f(x)\right) w_2(x) d\mu(x) &= \int_X \phi_2\left(\lambda f(\tau(x))\right) w_2(x) d\mu(x)\\ &= \int_{\tau(X)} \phi_2\left(\lambda f(y)\right)(w_2 \circ \tau^{-1})(y) d(\mu \circ \tau^{-1})(y)\\ &= \int_X \phi_2(\lambda f(y))(w_2 \circ \tau^{-1})(y) g_\tau(y) d\mu(y)\\ &= \sum_{k=1}^{\infty} \int_{A_k} \phi_2(\lambda \tilde{f}_{n_k}(y))(w_2 \circ \tau^{-1})(y) g_\tau(y) d\mu(y)\\ &\geq \sum_{k=p}^{\infty} \int_{A_k} \phi_2(2^{-n_k} \tilde{f}_{n_k}(y))(w_2 \circ \tau^{-1})(y) g_\tau(y) d\mu(y)\\ &= \infty.\end{aligned}$$

And

$$\begin{aligned}\int_X \phi_1(f(x)) w_1(x) d\mu(x) &= \sum_{k=1}^{\infty} 2^{-n_k} \int_{A_k} \phi_2(2^{-n_k} \tilde{f}_{n_k}(x))(w_2 \circ \tau^{-1})(x) g_\tau(x) d\mu(x)\\ &\quad - \sum_{k=1}^{\infty} 2^{-n_k} \int_{A_k} r_{n_k}(x) d\mu(x)\\ &\leq \sum_{k=1}^{\infty} 2^{-n_k} \int_{A_k} \phi_2(2^{-n_k} \tilde{f}_{n_k}(x))(w_2 \circ \tau^{-1})(x) g_\tau(x) d\mu(x)\\ &= \sum_{k=1}^{\infty} 2^{-n_k}\\ &\leq 1.\end{aligned}$$

Thus, $f \in L^{\phi_1}_{w_1}(X)$ but $C_\tau(f) \notin L^{\phi_2}_{w_2}(X)$, which is a contradiction. Hence,

$$\int_X h_n(x) d\mu(x) < \infty \text{ for some } n.$$

This completes the proof. □

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