Comment on the paper "Preserves of eigenvalue inclusion sets of matrix products"

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Abstract

In [3], Theorem 2.1 deals with characterization of mappings $\phi: M_n \to M_n$ which satisfies $O_\varepsilon(\phi(A)\phi(B)) = O_\varepsilon(AB)$, where $O_\varepsilon(A)$, $\varepsilon \in [0,1]$, denotes Ostrowski set of A. In the proof of this theorem an assertion was made (assertion 2.6) whose proof contains an error. In this paper an example is provided to substantiate our claim and the error also has been rectified.

Keywords: *Eigen value, Inclusion sets, Preservers, Gershgorin Set, Ostrowski set.*

Introduction

First we introduce the notations used in the paper.
Let M_n be the set of n × n complex matrices and $E_{11}, E_{12}, \dots\dots E_{nn}$ be the standard basis of M_n.
For any matrix $A \in M_n$, Eigen values inclusion set is a set which includes all its eigenvalues.

There are three main Eigen values inclusion sets of any matrix $A \in M_n$, namely Gershgorin set, Ostrowski set and Brauer's set which are denoted by $G(A), O_\varepsilon(A)$ and $C(A)$ respectively (For definitions of $G(A)$, and $O_\varepsilon(A)$ see section 1). It is known that $O_1(A) = G(A)$.

In [3, Theorem2.1], it has been proved that, a mapping $\phi: M_n \to M_n$ satisfies $O_\varepsilon(\phi(A)\phi(B)) = O_\varepsilon(AB)$ for all $A, B \in M_n, \varepsilon \in [0,1]$ if and only if there exist $c = \pm 1$, a permutation matrix P and an invertible diagonal matrix D, where D is unitary matrix unless $(n, \varepsilon) = (2, \frac{1}{2})$, such that $\phi(A) = c(DP)A(DP)^{-1}$.
While Assertions 2.1 to 2.3 prove the theorem for $\varepsilon = 1$ i.e, when ϕ satisfies $G(\phi(A)\phi(B)) = G(AB)$ for all $A, B \in M_n$, Assertions 2.4 to 2.6 prove the theorem for $\varepsilon \in (0,1)$. Essentially proof of assertion (2.6) is the proof of the theorem for $\varepsilon \in (0,1)$.

The matrices X and Y considered in 2.6 do not satisfy $O_{\frac{1}{2}}(A) = O_{\frac{1}{2}}(B)$ as claimed which is crucial for the proof of the assertion. A counter example has been provided in this note to this effect. Further, X and Y have been defined so that
$O_{\frac{1}{2}}(A) = O_{\frac{1}{2}}(B)$ and the rest of the proof goes through.

This paper has been divided into 2 sections,
Section 1 deals with basic definitions, statements of key results from [3] and Section 2 with counter example to show that the claim made in Assertion 2.6 is false and rectification of proof of the assertion.

Section 1: Basic Definitions and Statements

Given matrix $A = [a_{ij}] \in M_n$, we define

$$R_k(A) = \text{Row deleted sum of } A = \sum_{j\neq k, j=1}^{n} |a_{kj}|$$
$$C_k(A) = \text{Column deleted sum of } A = \sum_{j\neq k, j=1}^{n} |a_{jk}|$$

The Gershgorin set of A (see [1], [2]) is defined as

$$G(A) = \cup_{k=1}^{n} G_k(A) \text{ , where } G_k(A) = \{\mu \in \mathbb{C}\, ; |\mu - a_{kk}| \le R_k(A)\}$$

It is well known that $G(A)$ contains all the Eigen values of A.
Let $\varepsilon \in [0, 1]$ the Ostrowski set of A (see [1]) is defined by

$$O_\varepsilon(A) = \cup_{k=1}^{n} O_{\varepsilon,k}(A) = O(A), \text{ where}$$
$$O_{\varepsilon,k}(A) = \{\mu \in \mathbb{C}\, ;|\mu - a_{kk}| \le R_k^{\varepsilon}(A)C_k^{1-\varepsilon}(A)\}$$

It is also known that the Ostrowski set contains all the eigenvalues of A. It is clear that $O_1(A) = G(A)$.

If a mapping $\phi: M_n \to M_n$ satisfies $O_\varepsilon(\phi(A)\phi(B)) = O_\varepsilon(AB)$ for all $A, B \in M_n$, $\varepsilon \in (0,1)$ the following assertions have been proved in [3].

Assertion 2.4: Let $D = ?diag\,(1,2,\dots,n)$ with $? > 1$. Then there exist a permutation matrix P and a diagonal matrix $R = diag\,(r_1,\dots,r_n)$ with $r_k \in \{1,-1\}$ such that $\phi(D) = PRDP^T$ and

$\phi(D + E_{ij}) = P(RD + v_{ij}E_{ij})P^t$ for all $i \ne j$, where v_{ij}'s are non zero numbers such that $v_{ij}.v_{ji} = 1$.

Assertion 2.5: Following the notation in Assertion 2.4 and letting $v_{kk} = r_k$ for $1 \le k \le n$, we have $\phi(E_{ij}) = v_{ij}PE_{ij}P^t$ for all $1 \le i,j \le n$.

Section 2

In [3] the assertion 2.6 states: A mapping $\phi: M_n \to M_n$ satisfies $O_\varepsilon(\phi(A)\phi(B)) = O_\varepsilon(AB)$ for all $A, B \in M_n$, $\varepsilon \in (0,1)$ if and only if there exist $c = \pm 1$, a permutation matrix P and an invertible diagonal matrix D, where D is unitary matrix unless $(n,\varepsilon) = (2,\frac{1}{2})$, such that $\phi(A) = c(DP)A(DP)^{-1}$.

To show $\phi(A) = c(DP)A(DP)^{-1}$ authors take $\phi(A) = [v_{ij}a_{ij}]$ and observe, in view of the earlier discussion in the paper, that $v_{kk} = 1$ for all k.

The proof of the assertion being direct for $(n,\varepsilon) = \left(2,\frac{1}{2}\right)$, authors prove the assertion for $(n,\varepsilon) \ne \left(2,\frac{1}{2}\right)$. In the proof authors claim $|v_{ij}| = 1$ for $i \ne j$ and observe that once the claim holds, the assertion holds for $n = 2$.

The claim follows easily for $\varepsilon \ne \frac{1}{2}$, so it remains to prove the claim for $\varepsilon = \frac{1}{2}$, $n \ge 3$. To carry out the proof authors take

$X = E_{ii} + 2E_{ij} + E_{ik} + E_{ji}$ and $Y = E_{ii} + E_{ij} + 2E_{ik} + E_{ji}$, where i, j and k are distinct, and claim

$O_{\frac{1}{2}}(X) = O_{\frac{1}{2}}(Y)$, which is not true.

When, $n = 3, i = 1, j = 2, k = 3$

$$O_{\frac{1}{2}}(X) \ne O_{\frac{1}{2}}(Y)$$

Because,

$$X = E_{11}+2E_{12}+E_{13}+E_{21} = \begin{bmatrix}1 & 2 & 1\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \text{ and } Y = E_{11}+E_{12}+2E_{13}+E_{21} = \begin{bmatrix}1 & 1 & 2\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$O_{\frac{1}{2}}(X) = \{z: |z-1| \le \sqrt{3}\} \cup \{z: |z| \le \sqrt{2}\} \cup \{z: |z| \le 0\},$$

$$O_{\frac{1}{2}}(Y) = \{z: |z-1| \le \sqrt{3}\} \cup \{z: |z| \le 1\} \cup \{z: |z| \le 0\}$$

Clearly

$$O_{\frac{1}{2}}(X) \ne O_{\frac{1}{2}}(Y)$$

Instead we take

$$X = E_{ii} + 2E_{ij} + E_{ik} + E_{ji} + E_{ki},$$
$$Y = E_{ii} + E_{ij} + 2E_{ik} + E_{ji} + E_{ki}, \text{ where } k \ne \{i,j\} \text{ and } i \ne j$$

We observe $O_{\frac{1}{2}}(X) = O_{\frac{1}{2}}(Y)$ and the proof of the assertion goes through.

We have

$$X = \begin{bmatrix} 0\cdots 0\cdots 0\cdots 0\cdots 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0\cdots 1\cdots 2\cdots 1\cdots 0 \\ 0\cdots 0\cdots 0\cdots 0\cdots 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0\cdots 1\cdots 0\cdots 0\cdots 0 \\ 0\cdots 0\cdots 0\cdots 0\cdots 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0\cdots 1\cdots 0\cdots 0\cdots 0 \\ 0\cdots 0\cdots 0\cdots 0\cdots 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0\cdots 0\cdots 0\cdots 0\cdots 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0\cdots 0\cdots 0\cdots 0\cdots 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0\cdots 1\cdots 1\cdots 2\cdots 0 \\ 0\cdots 0\cdots 0\cdots 0\cdots 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0\cdots 1\cdots 0\cdots 0\cdots 0 \\ 0\cdots 0\cdots 0\cdots 0\cdots 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0\cdots 1\cdots 0\cdots 0\cdots 0 \\ 0\cdots 0\cdots 0\cdots 0\cdots 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0\cdots 0\cdots 0\cdots 0\cdots 0 \end{bmatrix}$$

Therefore

$$O_{\frac{1}{2}}(X) = \{\mu \in \mathbb{C}: |\mu - 1| \leq \sqrt{6}\} \cup \{\mu \in \mathbb{C}: |\mu| \leq \sqrt{2}\} \cup \{\mu \in \mathbb{C}: |\mu| \leq 1\} \cup \{0\}$$
$$= \{\mu \in \mathbb{C}: |\mu - 1| \leq \sqrt{6}\} \cup \{\mu \in \mathbb{C}: |\mu| \leq \sqrt{2}\},$$

$$O_{\frac{1}{2}}(Y) = \{\mu \in \mathbb{C}: |\mu - 1| \leq \sqrt{6}\} \cup \{\mu \in \mathbb{C}: |\mu| \leq 1\} \cup \{\mu \in \mathbb{C}: |\mu| \leq \sqrt{2}\} \cup \{0\}$$
$$= \{\mu \in \mathbb{C}: |\mu - 1| \leq \sqrt{6}\} \cup \{\mu \in \mathbb{C}: |\mu| \leq \sqrt{2}\}$$

We note that for the above X and Y

$$O_{\frac{1}{2}}(X) = O_{\frac{1}{2}}(Y).$$

Hence

$$O_{\frac{1}{2}}(I_n X) = O_{\frac{1}{2}}(I_n Y)$$
$$\Rightarrow O_{\frac{1}{2}}(\phi(I_n)\phi(X)) = O_{\frac{1}{2}}(\phi(I_n)\phi(Y))$$
$$\Rightarrow O_{\frac{1}{2}}(I_n\phi(X)) = O_{\frac{1}{2}}(I_n\phi(Y)) \quad \text{as } \phi(I_n) = I_n$$
$$\Rightarrow O_{\frac{1}{2}}(\phi(X)) = O_{\frac{1}{2}}(\phi(Y))$$

As $\phi(A) = [v_{ij}\, a_{ij}]$,

$$\phi(X) = \begin{bmatrix} 0\cdots\cdots 0\cdots\cdots 0\cdots\cdots 0\cdots\cdots 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0\cdots\cdots v_{ii}\cdots\cdots 2v_{ij}\cdots v_{ik}\cdots 0 \\ 0\cdots\cdots 0\cdots\cdots 0\cdots\cdots 0\cdots\cdots 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0\cdots\cdots v_{ji}\cdots\cdots 0\cdots\cdots 0\cdots\cdots 0 \\ 0\cdots\cdots 0\cdots\cdots 0\cdots\cdots 0\cdots\cdots 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0\cdots\cdots v_{ki}\cdots\cdots 0\cdots\cdots 0\cdots\cdots 0 \\ 0\cdots\cdots 0\cdots\cdots 0\cdots\cdots 0\cdots\cdots 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0\cdots\cdots 0\cdots\cdots 0\cdots\cdots 0\cdots\cdots 0 \end{bmatrix}, \phi(Y) = \begin{bmatrix} 0\cdots\cdots 0\cdots\cdots 0\cdots\cdots 0\cdots\cdots 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0\cdots\cdots v_{ii}\cdots\cdots v_{ij}\cdots 2v_{ik}\cdots 0 \\ 0\cdots\cdots 0\cdots\cdots 0\cdots\cdots 0\cdots\cdots 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0\cdots\cdots v_{ji}\cdots\cdots 0\cdots\cdots 0\cdots\cdots 0 \\ 0\cdots\cdots 0\cdots\cdots 0\cdots\cdots 0\cdots\cdots 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0\cdots\cdots v_{ki}\cdots\cdots 0\cdots\cdots 0\cdots\cdots 0 \\ 0\cdots\cdots 0\cdots\cdots 0\cdots\cdots 0\cdots\cdots 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0\cdots\cdots 0\cdots\cdots 0\cdots\cdots 0\cdots\cdots 0 \end{bmatrix}$$

As $O_{\frac{1}{2}}(\phi(X)) = O_{\frac{1}{2}}(\phi(Y))$, equating the radii of the disks, we obtain an equation

$$(|v_{ij}| + |2v_{ik}|)^{\frac{1}{2}} (|v_{ji}| + |v_{ki}|)^{\frac{1}{2}} = (|2v_{ij}| + |v_{ik}|)^{\frac{1}{2}} (|v_{ji}| + |v_{ki}|)^{\frac{1}{2}}$$
$$\Rightarrow (|v_{ij}| + |2v_{ik}|)^{\frac{1}{2}} = (|2v_{ij}| + |v_{ik}|)^{\frac{1}{2}}$$
$$\Rightarrow |v_{ij}| = |v_{ik}| \ \ldots\ldots\ldots (1)$$

Taking inverse both sides

$$|v_{ji}| = |v_{ki}| \ \ldots\ldots\ldots\ldots (2)$$

Interchanging the role of i and j in (1), we obtain

$$|v_{ji}| = |v_{jk}| \ \ldots\ldots\ldots\ldots (3)$$

Interchanging the role of j and k, we obtain

$$|v_{ki}| = |v_{kj}| \ldots\ldots\ldots (4)$$

From equations (1), (2), (3) and (4), we get

$$|v_{ij}| = |v_{ik}| = |v_{ki}|^{-1} = |v_{kj}|^{-1} = |v_{jk}| = |v_{ji}|$$ [as $v_{ij} \cdot v_{ji} = 1$ from Assertion 2.4]

Hence,

$$|v_{ij}| = |v_{ji}| = |v_{ij}|^{-1}$$

$\Rightarrow |v_{ij}| = 1$ and the rest of the proof follows.

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