Projectively semisymmetric and ϕ-projectively semisymmetric 3-dimensional LP-Sasakian Manifolds

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Abstract— *In this paper, we study projective curvature tensor on 3-dimensional LP-Sasakian manifolds. Mainly, we consider projectively semisymmetric and ϕ-projectively semisymmetric 3-dimensional LP-Sasakian manifolds and it is proved that in both the situations the manifold is infinitesimally spatially isotropic relative to the unit timelike vector field ξ.*

MSC (2000): 53C15, 53C25.

Keywords — *LP-Sasakian manifold, projective curvature tensor, Einstein manifold.*

I. INTRODUCTION

The Sasakian geometry is a special kind of contact metric geometry such that the structure vector field transverse to the Reeb vector filed ξ is a Kahler structure invariant under the flow of ξ. By analogy with Sasakian manifolds, in 1989, Matsumoto [4] introduced the notion of LP-Sasakian manifolds. Then the same notion has been introduced by Mihai and Rosca [5] independently and obtained interesting results. These manifolds have also been studied by Aqeel, et al.[1], De, et al. [2], Mihai, et al. [6], Murathan, et al. [8], Shaikh, et al. [10, 11, 12], Taleshian, et al. [15, 16, 17, 18] and others.

The projective curvature tensor is an important tensor from the differential geometric point of view. Let M be a n-dimensional Riemannian manifold. If there exist an one-to-one correspondence between each coordinate neighborhood of M and a domain in Euclidian space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 3$, M is locally projectively flat if and only if the well known projective curvature tensor P vanishes. Here P is defined by [7]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y], \quad (1)$$

for all $X,Y,Z \in T(M)$, where R is the curvature tensor and S is the Ricci tensor. In fact M is projectively flat if and only if it is of constant curvature [20]. Thus the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature.

The object of the present paper is to study 3-dimensional LP-Sasakian manifolds satisfyinag certain conditions on the projective curvature tensor. Section 2 is devoted to preliminaries. In section 3 we study projectively semisymmetric 3-dimensional LP-Sasakian manifolds and obtained interesting results.

II. PRELIMINARIES

An n-dimensional differentiable manifold M is said to be an LP-Sasakian manifold [4] if it admits a $(1,1)$ tensor field ϕ, a unit timelike contravariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy

$$\eta(\xi) = -1, \quad g(X,\xi) = \eta(X), \quad \phi^2 X = X + \eta(X)\xi, \quad (2)$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \ \nabla_X \xi = \phi X, \quad (3)$$

$$(\nabla_X \phi)(Y) = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X, \quad (4)$$

for all vector fields X and Y, where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g. It can be easily seen that in an LP-Sasakian manifold, the following relations hold:

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad rank\phi = n-1. \quad (5)$$

Again, if we put $\Omega(X,Y)=g(X,\phi Y)$ for any vector fields X,Y, then the tensor field $\Omega(X,Y)$ is a symmetric $(0,2)$ tensor field [4]. Also, since the vector field η is closed in an LP-Sasakian manifold, we have ([4, 5])

$$(\nabla_X\eta)(Y)=\Omega(X,Y), \Omega(X,\xi)=0 \tag{6}$$

for any vector fields X and Y.

In a 3-dimensional LP-Sasakian manifold, it follows that [13]

$$R(X,Y)Z=\frac{r-4}{2}[g(Y,Z)X-g(X,Z)Y]$$

$$+\frac{r-6}{2}[g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi$$

$$+\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y]. \tag{7}$$

where r is the scalar curvature of the manifold. Clearly, (7) shows that the manifold is of quasi-constant curvature. In addition, if $r=6$ then the manifold is of constant curvature.

Contracting the equation (7) with respect to Z, we have

$$S(X,Y)=\frac{1}{2}\{(r-2)g(X,Y)+(r-6)\eta(X)\eta(Y)\}.$$

Let M be a 3-dimensional LP-Sasakian manifold with structure (ϕ,ξ,η,g). Then the following relations hold ([13]) :

$$R(\xi,X)Y=g(X,Y)\xi-\eta(Y)X, \tag{9}$$

$$R(X,Y)\xi=\eta(Y)X-\eta(X)Y, \tag{10}$$

$$S(X,\xi)=2\eta(X), \tag{11}$$

$$Q\xi=2\xi \tag{12}$$

for any vector fields X and Y on M

An LP-Sasakian manifold M is said to be an Einstein if its Ricci tensor S of type $(0,2)$ is of the form $S(X,Y)=\alpha g(X,Y)$ for any vector fields X,Y, where α is a constant.

Now, we state two results which will be used in the next section.

Theorem 1. [20] A 3-dimensional Riemannian manifold is Einstein if and only if it is manifold of constant curvature.

Theorem 2. [20] A Riemannian manifold is projectively flat if and only if the manifold is of constant curvature.

III. Projectively Semisymmetric 3-Dimensional LP-Sasakian Manifolds

A Riemannain manifold M is called locally symmetric if its curvature tensor R is parallel, that is, $\nabla R=0$, where ∇ denotes the Levi-Civita connection. As a proper generalization of locally symmetric manifolds the notion of semisymmetric manifolds was defined by $R\cdot R=0$ and studied by many authors [9, 19]. A complete intrinsic classification of these spaces was given by Z. I. Szabo [14].

A 3-dimensional LP-Sasakian manifold M is said to be projectively semisymmetric if it satisfies the condition

$$(R(X,Y).P)(U,V)=0. \tag{13}$$

for all vector fileds X,Y,U,V on M.

Let a 3-dimensional LP-Sasakian manifold M be projectively semisymmetric. Then it follows from the equation (13) that

(8) $$R(X,Y)P(U,V)W-P(R(X,Y)U,V)W$$

$$-P(U,R(X,Y)V)W-P(U,V)R(X,Y)W=0. \tag{14}$$

Using $Y=U=W=\xi$ in (14) we have

$$R(X,\xi)P(\xi,V)\xi-P(R(X,\xi)\xi,V)\xi$$

$$-P(\xi,R(X,\xi)V)\xi-P(\xi,V)R(X,\xi)\xi=0. \tag{15}$$

From (9) and (15) we get

$$P(\xi,V)R(X,\xi)\xi=0. \tag{16}$$

In view of (1), (9) and (11) we obtain from (16) that

$$P(\xi,V)X=0. \tag{17}$$

Therefore

$$R(\xi,V)X-\frac{1}{2}[S(V,X)\xi-S(\xi,X)V]=0. \tag{18}$$

Applying (8), (9) and (11) and in (18) we get

$$\left(\frac{r-6}{4}\right)g(\phi V,\phi X)\xi = 0. \quad (19)$$

Since $g(\phi V,\phi X)\neq 0$, in (19) taking inner product with ξ we have

$$r = 6. \quad (20)$$

Substituting (20) in (8) we obtain

$$S(X,Y) = 2g(X,Y). \quad (21)$$

Thus, the manifold is an Einstein manifold with constant curvature. Consequently, by virtue of theorem 2, the manifold is projectively flat. Hence we state the following:

Corollary 3.1: If a 3-dimensional LP-Sasakian manifold is projectively semisymmetric, then the manifold is projectively flat.

Next, in a projectively flat 3-dimensional LP-Sasakian manifold, we obtain from (1) that

$$R(X,Y)Z = \frac{1}{2}[S(Y,Z)X - S(X,Z)Y]. \quad (22)$$

In this case, by using (21) in (22) we obtain

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y. \quad (23)$$

According to Karcher [3], a Lorentzian manifold is called infinitesimally spartial isotropic relative to a unit timelike vector field U if its Riemann curvature tensor R satisfies the relations

$$R(X,Y)Z = \delta g(Y,Z)X - g(X,Z)Y.$$

for all $X,Y,Z\in U^{\perp}$ and $R(X,U)U = \gamma X$ for $X\in U^{\perp}$, where δ and γ are real-valued functions on the manifold. Hence, we arrive at the following theorem:

Theorem 3. A 3-dimensional projectively semisymmetric LP-Sasakian manifold is infinitesimally spatially isotropic relative to the unit timelike vector field ξ.

Let M be a projectively semisymmetric 3-dimensional LP-Sasakian manifold. Thus we can write

$$R(X,Y)\cdot R = R(X,Y)R(Z,U)V - R(R(X,Y)Z,U)V$$
$$-R(Z,R(X,Y)U)V - R(Z,U)R(X,Y)V \quad (24)$$

for all vector fields X,Y,Z,U,V on M. Hence, it follows from (23) that Hence, it follows from (24) that

$$R(X,Y)R(Z,U)V$$
$$=[g(U,V)g(Y,Z)X - g(Z,V)g(Y,U)X$$
$$-g(U,V)g(X,Z)Y - g(Z,V)g(X,U)Y], \quad (25)$$

$$R(R(X,Y)Z,U)V$$
$$=[g(U,V)g(Y,Z)X - g(Y,Z)g(X,V)U$$
$$-g(X,Z)g(U,V)Y + g(Y,V)g(X,Z)U], \quad (26)$$

$$R(Z,R(X,Y)U)V$$
$$=[g(U,Y)g(X,V)Z - g(U,Y)g(Z,V)X$$
$$-g(U,X)g(Y,V)Z + g(X,U)g(Z,V)Y] \quad (27)$$

and

$$R(Z,U)R(X,Y)V$$
$$=[g(U,X)g(Y,V)Z - g(X,Z)g(Y,V)U$$
$$-g(X,V)g(U,Y)Z + g(X,V)g(Z,Y)U]. \quad (28)$$

Thus, in view of (25)-(28), we obtain from (24) that

$$R(X,Y)\cdot R = 0. \quad (29)$$

That is, the manifold is semisymmetric, This leads to the following statement:

Theorem 4. A projectively semisymmetric 3-dimensional LP-Sasakian manifold is semisymmetric i.e, the condition $R(X,Y)\cdot R = 0$ is satisfied.

It is also known that every semisymmetric LP-Sasakian manifold is Ricci semisymmetric. Hence we can state the following:

Corollary 3.2. A projectively semisymmetric 3-dimensional LP-Sasakian manifold is Ricci semisymmetric.

IV. ϕ-PROJECTIVELY SEMISYMMETRIC 3-DIMENSIONAL LP-SASAKIAN MANIFOLDS

A 3-dimensional LP-Sasakian manifold M is said to be ϕ-projectively semisymmetri if it satisfies the condition

$$(P(X,Y).\phi)Z = 0, \quad (30)$$

for all vector fields X and Y on M.

Let M be a ϕ-projectively semisymmetric 3-dimensional LP-Sasakian manifold. Then the condition (30) turns into

$$(P(X,Y).\phi)Z = P(X,Y)\phi Z - \phi P(X,Y)Z = 0, \quad (31)$$

for any vector fields X, Y and Z on M.

Now, in view of (1), (7) and (8) we have

$$P(X,Y)\phi Z = \left(\frac{r-4}{2}\right)[g(Y,\phi Z)X - g(X,\phi Z)Y]$$

$$+\left(\frac{r-6}{2}\right)[g(Y,\phi Z)\eta(X) - g(X,\phi Z)\eta(Y)]\xi$$

$$-\left(\frac{r-2}{4}\right)[g(Y,\phi Z)X - g(X,\phi Z)Y]. \quad (32)$$

Similarly, we obtain

$$\phi(P(X,Y)Z) = \phi[R(X,Y)Z - \frac{1}{2}\{S(Y,Z)X - S(X,Z)Y\}]. \quad (33)$$

By virue of (32) and (33), we get from (31)

$$\phi[R(X,Y)Z - \frac{1}{2}\{S(Y,Z)X - S(X,Z)Y\}]$$

$$= \left(\frac{r-4}{2}\right)[g(Y,\phi Z)X - g(X,\phi Z)Y]$$

$$+\left(\frac{r-6}{2}\right)[g(Y,\phi Z)\eta(X) - g(X,\phi Z)\eta(Y)]\xi$$

$$-\left(\frac{r-2}{4}\right)[g(Y,\phi Z)X - g(X,\phi Z)Y]. \quad (34)$$

Putting $X = \xi$ in (34) we have

$$\phi[R(\xi,Y)Z - \frac{1}{2}\{S(Y,Z)\xi - S(\xi,Z)Y\}]$$

$$= \left(\frac{6-r}{4}\right)g(Y,\phi Z)\xi. \quad (35)$$

Using (8), (9) and (11) in (35) yields

$$\left(\frac{6-r}{4}\right)g(Y,\phi Z)\xi = 0. \quad (36)$$

Since $g(Y,\phi Z)\xi \neq 0$, in (36) taking inner product with ξ we have $r = 6$. Thus, the manifold is of constant curvature and consequently Einstein. Hence, from theorems 1 and 2, we arrive at the following:

Corollary 4.3. If a 3-dimensional LP-Sasakian manifold is ϕ-projectively semisymmetric, then the manifold is projectively flat.

By taking account of discussion held in previous section, we conclude that

Theorem 5. A 3-dimensiona ϕ-projectively semisymmetric LP-Sasakian manifold is infinitesimally spatially isotropic relative to the unit timelike vector field ξ.

Acknowledgement: The second author (DGP) is thankful to UGC, New Delhi for financial support to the Department of Mathematics, K. U. Dharwad in the form of UGC-SAP-DRS-III Programme.

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