Advection diffusion equation for Nutrient Uptake by aquatic plant root with nonlinear boundary condition *

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Abstract

In this article, we use advection diffusion equation for nutrient uptake by the aquatic root which is considered as cylindrical, i.e, we obtain concentration of nutrient entering into the root surface by advection diffusion equation. The equation is written in the radial form and solved using Michal Menten boundary condition, which is nonlinear boundary condition. It is found that generally advection diffusion is solved taking Peclet number as zero, then equation reduces to the diffusion equation and solved by Laplace method [10]. But we solve the advection diffusion equation without taking Peclet number as zero and solved by re-scaling and using separation of variable which reduces it into Bessel's equation. For particular solution, we use extreme parameters.

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Key words and Phrases: Solution of advection diffusion equation, Re-scaling variable, nutrient uptake.

1 Introduction

The primary physiological function of root is uptaking the water as well as nutrients and transport to leaves for photosynthesis. Investigations and observation of the uptake of water and nutrient in plant root and stem can be traced back to many years ago, it possesses importance in point of view of agricultural production and economical development[4,8-10]. In traditional farming like planting and agricultural the mechanism of water and nutrients is invaluable for utilizing water and fertilizer for increasing production. Now a new trend of planting inedible plant, emerge on industrial basis. also Aquatic plant help to purify water in the Dam or River.

In recent years, a number of researchers from various fields, such as physics, applied mathematics and plant physiology, paid more attention to develop mathematical model for water and nutrient uptake. The outstanding work in this field is done by T.Roose and proposed a mathematical model for uptake of water and nutrient. Roose work is the development of Nye, Tinker and Barber model for water and nutrient uptake assuming that the root is an infinitely long cylinder. To develop Mathematical model, we first derive advection diffusion equation of nutrient transport in the water and then try to solve the advection diffusion equation by transforming it into non-dimensional form and using Michal Menten boundary condition as boundary condition. We re-scale the equation and reduce into the Bessel's equation, so we write solution in terms of Bessel's function.

I already workout this problem for nutrient uptake by plant root from groundwater in my Ph.D. Thesis. In this paper we tried to this problem for aquatic plant.In this paper we derived mathematical models and obtain solutions with different situation.

2 Advection diffusion equation for nutrient in water

The aquatic root are mainly surrounded by liquid and gas. We indicate ϕ_l volume fraction of soil occupied by the liquid, and ϕ_g volume fraction of soil occupied by gas. Other phases like microbes, mucigel etc are neglected. The conservation of soil volume equation is written as:

$$\phi_g + \phi_l = 1. \tag{2.1}$$

Nutrient comes in contact with surface of the root by flow of water in which diffusion of nutrient takes place. Then the equation for ions in the liquid

phase is written as

$$\frac{\partial}{\partial t}(\phi_l c_l) + \nabla.(c_l \mathbf{u}) = \nabla.(\phi_l D \nabla c_l), \tag{2.2}$$

where $\mathbf{u}$ is the Darcy flux of water, c_l is the nutrient concentration and D is the diffusion coefficient in the liquid, Hence, the equation (2.2) in terms of c_l becomes,

$$(\phi_l)\frac{\partial c_l}{\partial t} + \nabla.(c_l \mathbf{u}) = \nabla.(\phi_l D \nabla c_l). \tag{2.3}$$

Noting $c_l = c$ and writing equation (2.3) in radial polar coordinates we get

$$(\phi_l)\frac{\partial c}{\partial t} - \frac{aV}{r}\frac{\partial c}{\partial r} = D\phi_l \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial c}{\partial r}), \tag{2.4}$$

where a is the radius of the root. The water flux is given by $\mathbf{u} = -\frac{aV}{r}$, which derives from the law of mass conservation for water, i.e, $\nabla.\mathbf{u} = 0$. The quantity V is the Darcy flux of water into the root [1].

3   Boundary condition

Root surface accept the nutrient up to a certain level even if the nutrient concentration in liquid increases indefinitely. It is also verified that the root surface accept nutrient up to a critical level(low) of nutrient in liquid phase near the root surface below which first it stop the uptake of nutrient and then start bleeding in the liquid. The experimentally measured, heuristic Michaelis-Menten type nutrient uptake boundary condition is therefore given by, see [6]

$$\phi_l D \frac{\partial c}{\partial r} + Vc = \frac{F_m c}{K_m + c} - E, \tag{3.1}$$

at $r = a$.

    Where c indicate the concentration of nutrient in the liquid, K_m indicate the Michaelis-Menten constant that is equal to the root surface nutrient concentration when the flux of nutrient into the root is half of the maximum possible, F_m indicate the maximum flux of nutrient into the root, $E = \frac{F_m c_{min}}{K_m + c_{min}}$ where c_{min} indicate the minimum concentration when the roots stop the uptake of nutrients, and a is the radius of the root.

4   Initial Condition and boundary condition

Initial condition can be write as for $t = 0$

$$c = c_0 \ at \ t = 0 \ for \ a < r < \infty, \tag{4.1}$$

for later time

$$c \to c_0 \; as \; r \to \infty \; for \; t > 0. \tag{4.2}$$

5 Non-dimensionalisation of Nutrient Transport equation

Choosing time, space, and concentration-scale as follows and substitute in (2.4)

$$t = \frac{a^2}{D}t^*, \; r = ar^*, \; c = K_m c^*. \tag{5.1}$$

Where c^*, $t^* \, and \, r^*$are dimensionless nutrient concentration, time, and radial variables,respectively, we obtain (after dropping $^*s$) the following dimensionless model

$$\frac{\partial c}{\partial t} - P_e \frac{1}{r}\frac{\partial c}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial c}{\partial r}), \tag{5.2}$$

with boundary conditions

$$\frac{\partial c}{\partial r} + P_e c = \lambda \frac{c}{1+c} - \epsilon \; at \; r = 1. \tag{5.3}$$

$$c \to c_\infty \; as \; r \to \infty \; for \; t > 0, \tag{5.4}$$

the dimensionless initial condition is given by

$$c = c_\infty \; at \; t = 0 \; for \; 1 < r < \infty. \tag{5.5}$$

the dimensionless parameters in above equations are defined as

$$P_e = \frac{aV}{D\phi_l}, \; \lambda = \frac{F_m a}{DK_m \phi_l}, \; \epsilon = \frac{Ea}{DK_m \phi_l}, \; c_\infty = \frac{c_0}{K_m}. \tag{5.6}$$

equation (5.2) write as

$$\frac{\partial c}{\partial t} - (\frac{P_e + 1}{r})\frac{\partial c}{\partial r} = \frac{\partial^2 c}{\partial r^2}, \tag{5.7}$$

implies

$$\frac{\partial c}{\partial t} = (\frac{P_e + 1}{r})\frac{\partial c}{\partial r} + \frac{\partial^2 c}{\partial r^2}, \tag{5.8}$$

re-scaling with $r = (1+P_e)R$, then $\partial r = (1+P_e)\partial R$. Then equation (5.8) become

$$(1 + P_e)\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial R^2} + \frac{1}{R}\frac{\partial c}{\partial R}, \tag{5.9}$$

Corresponding boundary condition changes

$$\frac{\partial c}{\partial R} + (1 + P_e)P_e c = \lambda(1 + P_e)[\frac{c}{1+c} - \epsilon], \ at \ R = \frac{1}{1 + P_e}, \qquad (5.10)$$

for $\lambda = \frac{F_m a}{DK_m \phi_l}$ value of λ with large value of ϕ and small radius R we have

$$\lambda \equiv 0. \qquad (5.11)$$

Then the boundary condition becomes

$$\frac{\partial c}{\partial R} + (1 + P_e)P_e c = 0, \qquad (5.12)$$

Consider $c(R,t) = U(R)T(t)$ substituting in (5.9) and (5.12) then it becomes

$$\frac{1}{T}(1 + P_e)\frac{\partial T}{\partial t} = \frac{1}{U}[\frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R}], \qquad (5.13)$$

corresponding boundary condition becomes

$$\frac{\partial U}{\partial R} + (1 + P_e)P_e U = 0. \qquad (5.14)$$

From the equation (5.9) we can write

$$\frac{1}{T}(1 + P_e)\frac{\partial T}{\partial t} = \frac{1}{U}[\frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R}] = -\beta^2. \qquad (5.15)$$

We have the Bessel equation with boundary condition

$$\frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R} + \beta^2 U = 0. \qquad (5.16)$$

$$\frac{\partial U}{\partial R} + (1 + P_e)P_e U = 0, \ at \ R = \frac{1}{1 + P_e} \qquad (5.17)$$

and

$$\frac{\partial T}{\partial t} = -\frac{\beta^2}{1 + P_e}T. \qquad (5.18)$$

$$c = c_\infty \ at \ t = 0 \ as \ 1 < R < \frac{1}{1 + P_e} \qquad (5.19)$$

Solution of Bessels equation is given by,

$$U(\beta, R) = J_0(\beta R)[\beta Y_1(\beta \frac{1}{(1 + P_e)}) + P_e(-1 - P_e)Y_0(\beta \frac{1}{(1 + P_e)})]$$

$$-Y_0(\beta R)[\beta J_1(\beta \frac{1}{(1 + P_e)}) + P_e(-1 - P_e)J_0(\beta \frac{1}{1 + P_e})], \quad (5.20)$$

also

$$N(\beta) = [\beta J_1(\beta \frac{1}{(1+P_e)}) + (-1-P_e)J_0(\beta \frac{1}{(1+P_e)})]^2$$
$$+[\beta Y_1(\beta \frac{1}{(1+P_e)}) + (-1-P_e)Y_0(\beta \frac{1}{(1+P_e)})]^2. \qquad (5.21)$$

Replacing R by $R = \frac{r}{(1+P_e)}$ in equation (5.20)
Above solution of Bessels equation become

$$U(\beta,r) = J_0(\beta \frac{r}{(1+P_e)})[\beta Y_1(\beta \frac{1}{(1+P_e)}) + P_e(-1-P_e)Y_0(\beta \frac{1}{(1+P_e)})]$$
$$-Y_0(\beta \frac{r}{(1+P_e)})[\beta J_1(\beta \frac{1}{(1+P_e)}) + P_e(-1-P_e)J_0(\beta \frac{1}{1+P_e})] \quad (5.22)$$

Then the complete solution is given by, see [5]

$$c(r,t) = \int_{\beta=0}^{\infty} \frac{\beta}{N(\beta)} e^{-\frac{1}{(1+P_e)}\beta^2 t} U(\beta,r)d\beta \int_{r'=1}^{\infty} r'U(\beta,r')c_{\infty}dr'. \qquad (5.23)$$

Amount of nutrient absorb by root is given as, [2,3]

$$M = 2\pi rt \frac{\partial c}{\partial t}. \qquad (5.24)$$

6 Steady state uptake of nutrient

Consider equation (5.7) with boundary condition (5.3) and (5.5) in steady state it takes the form

$$\frac{\partial^2 c}{\partial r^2} + \frac{(1+P_e)}{r}\frac{\partial c}{\partial r} = 0, \qquad (6.1)$$

with the substitution $r = (1+P_e)R$ equation (6.1) changes to the form

$$\frac{\partial^2 c}{\partial R^2} + \frac{1}{R}(\frac{\partial c}{\partial R}) = 0. \qquad (6.2)$$

With the assumption of section (5.1), λ approaches to zero and ϵ is of order zero then boundary condition for (5.12) is the equation changes to the form,

$$\frac{\partial c}{\partial R} + (1+P_e)P_e c = 0. \qquad (6.3)$$

And initial condition changes to $c \to c_\infty$ as $R \to \infty$ for $t > 0$

$$c = c_\infty, \ at \ t = o \ for \ \frac{1}{1 + P_e} < R < \infty, \qquad (6.4)$$

we may take for large R as L Solution of equation (6.2) is given by, see [2,3],

$$c = A + B log R. \qquad (6.5)$$

We can find the arbitrary constant A and B by applying initial and boundary condition as follwes
$\frac{B}{R} + (1 + P_e)P_e(A + B log R) = 0$ at $R = \frac{1}{1+P_e}$,

$$B = -\frac{(1 + P_e)P_e c_\infty}{[(1 + P_e) + (1 + P_e)P_e \ log\frac{1}{L(1+P_e)}]}. \qquad (6.6)$$

$$A = c_\infty + \frac{(1 + P_e)P_e c_\infty}{[(1 + P_e) + (1 + P_e)P_e \ log\frac{1}{L(1+P_e)}]} log L. \qquad (6.7)$$

Then the general solution for equation is given by
$c = c_\infty + \frac{(1+P_e)P_e c_\infty}{[(1+P_e)+(1+P_e)P_e \ log\frac{1}{L(1+P_e)}]} log L - \frac{(1+P_e)P_e c_\infty}{[(1+P_e)+(1+P_e)P_e \ log\frac{1}{L(1+P_e)}]} log R.$

$$c = c_\infty + \frac{P_e c_\infty}{[1 + P_e \ log\frac{1}{L(1+P_e)}]} log L - \frac{P_e c_\infty}{[1 + P_e \ log\frac{1}{L(1+P_e)}]} log R, \qquad (6.8)$$

solution modified as

$$c(R) = c_\infty[1 + \frac{P_e log\frac{L}{R}}{[1 + P_e \ log\frac{1}{L(1+P_e)}]}], \qquad (6.9)$$

replacing value of R is

$$c(r) = c_\infty[1 + \frac{P_e log\frac{L(1+P_e)}{r}}{[1 + P_e \ log\frac{1}{L(1+P_e)}]}]. \qquad (6.10)$$

Solution of steady state advection diffusion equation is written as

$$c(r) = c_\infty[\frac{1 - P_e log r}{1 - P_e log L(1 + P_e)}], \qquad (6.11)$$

total nutrient uptake per unit length is given by

$$Q = -2\pi D c_\infty \frac{r - P_e}{1 - P_e log L(1 + P_e)}. \qquad (6.12)$$

7   Nutrient transport equation with $c_\infty << 1$ and $\epsilon < P_e << 1$

In this section we consider P_e, ϵ and c_∞ are negligible. If Michaelis-Menten coefficient K_∞ much larger than the far field concentration c_0 ,i.e., $c_\infty << 1$, the equation (5.2) reduces to the form

$$\frac{\partial c}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial c}{\partial r}). \tag{7.1}$$

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial r^2} + \frac{1}{r}\frac{\partial c}{\partial r}. \tag{7.2}$$

Corresponding boundary condition reduces to the form

$$\frac{\partial c}{\partial r} = \lambda\frac{c}{1+c}, \tag{7.3}$$

re-scaling $c = c_\infty C$ then the model in scaled concentration is written as

$$\frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial r^2} + \frac{1}{r}\frac{\partial C}{\partial r}, \tag{7.4}$$

scaled boundary condition are as follows

$$\frac{\partial C}{\partial r} = \lambda\frac{C}{1+c_\infty C}, \; r = 1 \; and \; C \to 1 \; as \; r \to \infty. \tag{7.5}$$

for $c_\infty << 1$ we can approximate the root surface boundary condition,using the binomial expansion,at the leading order given by

$$\frac{\partial C}{\partial r} \approx \lambda C \; at \; r = 1. \tag{7.6}$$

Initial condition scaled in following manner

$$C = 1 \; at \; t = 0 \; for \; 1 < r < \infty. \tag{7.7}$$

We solve the above boundary value problem by separation of the variables .substituting the substitution $C(r,t) = T(t)U(r)$ the value in equation (7.4) we have

$$\frac{1}{U}[\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}] = \frac{1}{T}[\frac{\partial T}{\partial t}] = -\beta^2. \tag{7.8}$$

Now consider the boundary value problem

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \beta^2 U = 0. \tag{7.9}$$

With the boundary condition

$$\frac{dU}{dr} - \lambda U = 0. \tag{7.10}$$

The complete solution is given by, see [5],

$$C(r,t) = \int_{\beta=0}^{\infty} \frac{\beta}{N(\beta)} e^{-\beta^2 t} U(\beta,r) d\beta \int_{r=1}^{\infty} r' U(\beta,r') dr', \tag{7.11}$$

where $U(\beta_m, r)$ is eigenvalue function.

$$U(\beta,r) = J_0(\beta r)[\beta Y_1(\beta) + \lambda Y_0(\beta)] - Y_0(\beta r)[\beta J_1(\beta) + \lambda J_0(\beta)]. \tag{7.12}$$

$$N(\beta) = [\beta J_1(\beta) + \lambda J_0(\beta)]^2 + [\beta Y_1(\beta) + \lambda Y_0(\beta)]^2. \tag{7.13}$$

So the general solution of equation is given by

$$c(r,t) = \int_{\beta=0}^{\infty} \frac{\beta}{N(\beta)} e^{-\beta^2 t} R(\beta,r) d\beta \int_{r=1}^{\infty} r' R(\beta,r') dr'. \tag{7.14}$$

8 Advection diffusion equation with Case $c_\infty <<$ 1and $\epsilon << 1$

With the very very small space concentration ϵ value is negligible for the advection diffusion equation (5.2) with boundary condition (5.3) can be reduced in the diffusion equation by re-scaling $r = (1 + P_e)R$ and $c = c_\infty C$

$$`(1 + P_e)\frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial R} + \frac{1}{R}\frac{\partial C}{\partial R}. \tag{8.1}$$

$$\frac{\partial C}{\partial R} + (1 + P_e)P_e C = \lambda(1 + P_e)[\frac{C}{1 + c_\infty C}] \; at \; R = \frac{1}{1 + P_e}, \tag{8.2}$$

for $c_\infty << 1$ we can approximate the root surface boundary condition,using the binomial expansion,at the leading order given by

$$\frac{\partial C}{\partial R} + (1 + P_e)P_e C = \lambda(1 + P_e)C \; at \; R = \frac{1}{1 + P_e}. \tag{8.3}$$

$$\frac{\partial C}{\partial R} + [(1 + P_e)P_e - \lambda(1 + P_e)]C = 0 \; at \; R = \frac{1}{1 + P_e}. \tag{8.4}$$

$$\frac{\partial C}{\partial R} + (1 + P_e)(P_e - \lambda)C = 0 \; at \; R = \frac{1}{1 + P_e}. \tag{8.5}$$

The complete solution is given by separation of variable as similar to equation (7.8) with the substitution $C(R,t) = U(R)T(t)$

$$C(R,t) = \int_{\beta=0}^{\infty} \frac{\beta}{N(\beta)} e^{-\beta^2 t} U(\beta, R) d\beta \int_{R=\frac{1}{1+P_e}}^{\infty} R'U(\beta, R') dR' \qquad (8.6)$$

where $U(\beta_m, R)$ is solution of Bessel equation.

$$U(\beta, R) = J_0(\beta R)[\beta Y_1(\beta \frac{1}{1+P_e}) + (1+P_e)(P_e - \lambda)Y_0(\beta R)]$$
$$- Y_0(\beta r)[\beta J_1(\beta \frac{1}{1+P_e}) + (1+P_e)(P_e - \lambda)J_0(\beta \frac{1}{1+P_e})]. \qquad (8.7)$$

$$N(\beta) = [\beta J_1(\beta \frac{1}{1+P_e}) + (1+P_e)(P_e - \lambda)J_0(\beta \frac{1}{1+P_e})]^2$$
$$+ [\beta \frac{1}{1+P_e} Y_1(\beta) + (1+P_e)(P_e - \lambda)Y_0(\beta \frac{1}{1+P_e})]^2. \qquad (8.8)$$

Re-substituting value of $R = \frac{r}{1+P_e}$

$$U(\beta, r) = J_0(\beta \frac{r}{1+P_e})[\beta Y_1(\beta \frac{1}{1+P_e}) + (1+P_e)(P_e - \lambda)Y_0(\beta R)]$$
$$- Y_0(\beta r)[\beta J_1(\beta \frac{1}{1+P_e}) + (1+P_e)(P_e - \lambda)J_0(\beta \frac{1}{1+P_e})], \qquad (8.9)$$

so the general solution of equation is given by

$$c(r,t) = c_\infty \int_{\beta=0}^{\infty} \frac{\beta}{N(\beta)} e^{-\beta^2 t} U(\beta, r) d\beta \int_{r=1}^{\infty} r'U(\beta, r') dr'. \qquad (8.10)$$

9 High Nutrient uptake for $\lambda \gg 1$

If the gradient of nutrient concentration near root surface is high, i.e., $\frac{\partial c}{\partial r}|_{r=1} = \lambda \gg 1$ for $c \sim O(1)$. Then re-scaling the independent variables r and t to stretched variables R and T i.e. $r = 1 + \frac{R}{\lambda}$ and $t = \frac{T}{\lambda^2}$, the problem reduces to

$$\frac{\partial c}{\partial T} = \frac{\partial^2 c}{\partial R^2} + \frac{1}{R+\lambda}\frac{\partial c}{\partial R}. \qquad (9.1)$$

Which at the leading order simplifies to

$$\frac{\partial c}{\partial T} = \frac{\partial^2 c}{\partial R^2}, \qquad (9.2)$$

since $\frac{1}{\lambda+R} << 1$ for $\lambda >> 1$. The re-scaled boundary conditions are

$$\frac{\partial c}{\partial R} = c \ at \ R = 0 \ and \ c \to 1 \ as \ R \to \infty, \tag{9.3}$$

and the initial condition is $c = 1$ at $T = 0$ for $0 < R < \infty$. Then the general solution to this leading order problem is given by

$$c(R,T) = erf(\frac{R}{2\sqrt{T}}) + e^{R+T} erfc(\frac{R}{2\sqrt{T}} + \sqrt{T}), \tag{9.4}$$

with the flux $F(T) = \frac{\partial c}{\partial R}\frac{\partial R}{\partial r}|_{R=0}$, of nutrient into the root given by

$$F(T) = \lambda e^T erfc(\sqrt{T}). \tag{9.5}$$

As $T \to \infty$, the concentration of nutrient at the surface $c \to 0$ and $F \to 0$, since $e^T erfc(\sqrt{T}) \to 0$ as $T \to \infty$.

10   Zero-sink Model

For $t > t_c \sim \frac{1}{\lambda^2}$ the root surface nutrient concentration has dropped to a very low level then we take the boundary condition at the root surface at the leading order to be $c = 0$ at $r = 1$, i.e, the problem to be solved is, see [7],

$$\frac{\partial c}{\partial t} + \frac{(-P_e)}{r}\frac{\partial c}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial c}{\partial r}) \tag{10.1}$$

$$c = 0 \ at \ r = 1 \ and \ c \to 1 \ as \ r \to \infty, \tag{10.2}$$

Let $q = P_e + 1$ the equation (10.1) becomes

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial r^2} + \frac{q}{r}\frac{\partial c}{\partial r}. \tag{10.3}$$

Using variable separation technique where λ is the separation constant yield

$$\frac{1}{T}\frac{\partial T}{\partial t} = \frac{1}{U}[\frac{\partial^2 U}{\partial r^2} + \frac{q}{r}\frac{\partial U}{\partial r}] = -\lambda. \tag{10.4}$$

Then above equation reduces to the equations

$$\frac{\partial T}{\partial t} + \lambda T = 0. \tag{10.5}$$

$$r\frac{\partial^2 U}{\partial r^2} + q\frac{\partial U}{\partial r} + r\lambda U = 0, \tag{10.6}$$

$$R(1) = 0 \qquad (10.7)$$

the time function $T(t)$ is the exponential solution of equation (10.5) is

$$T_i(t) = e^{-\lambda_i t}. \qquad (10.8)$$

The solution of spatial function $R(r)$ is obtained by power series method used for bessel equation

$$R_i(r) = \sum_{n=0}^{\infty} \frac{(-1)^n (r\sqrt{\lambda_i})^{2n}}{2^{2n-\gamma} n! . \Gamma(\nu - \gamma + 1) . \lambda_i^{\frac{\gamma}{2}}} \; with \; \gamma = \frac{1-q}{2} = -\frac{P_e}{2}, \qquad (10.9)$$

given solution can be represented using a negative γ -order Bessel function $J_{-\gamma}$ of the first kind. The separation constant λ_i of a specific problem is a scaled version of the general Bessel function roots to accommodate the boundary condition at r=1

$$R_i(r) = r^\gamma . J_{-\gamma}(r\sqrt{\lambda_i})_{r=1} = 0, \; \sqrt{\lambda_i} = s_i, \qquad (10.10)$$

combining the spatial and time function solution we get desired solution as an infinite sum of eigenfunctions as

$$C(r,t) = \Sigma_{t=0}^{\infty} [A_i r^\gamma . J_{-\gamma}(r\sqrt{\lambda_i}) e^{-\lambda_i t}]. \qquad (10.11)$$

According to the Sturm-Liouville theory orthogonal base functions correspond to the weights r^q. The coefficient A_i can be adjusted using a Fourier-Bessel decomposition

$$A_i = \frac{\int_0^1 J(s_i r) . r^{\gamma+q} dr}{\int_0^1 [J(s_i r)]^2 . r^{2\gamma+q} dr}. \qquad (10.12)$$

11 Zero-sink Model with $P_e << 1$

The equation (10.1) is reduced to the form as,

$$\frac{\partial c}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial c}{\partial r}). \qquad (11.1)$$

$$c = 0 \; at \; r = 1 \; and \; c \to 1 \; as \; r \to \infty, \qquad (11.2)$$

$c = 1 \; at \; t = 0 \; as \; 1 < r < \infty.$
Separating the variables solution for time-variable function is given by $e^{-\beta^2 t}$ and space variable function $U(\beta, r)$ is the solution of the following problem

$$\frac{d^2 U}{dr^2} + \frac{1}{r} \frac{dU}{dr} + \beta^2 U = 0 \; for \; 1 < r < \infty, \qquad (11.3)$$

$$c = 0 \ at \ r = 1. \tag{11.4}$$

Then the complete solution for c(r, t) is constructed as

$$c(r,t) = \int_{\beta=0}^{\infty} C(\beta) e^{-\beta^2 t} R(\beta, r) d\beta, \tag{11.5}$$

with the application of initial condition we get

$$1 = \int_{\beta=0}^{\infty} c(\beta) U(\beta, r) d\beta \ in \ 1 < r < \infty, \tag{11.6}$$

using the orthogonality of eigenvalue functions we have

$$C(\beta) \equiv \frac{1}{N(\beta)} \beta \int_{r'=1}^{\infty} r' R(\beta, r') dr'. \tag{11.7}$$

Substituting equation (10.7) into equation (10.5) gives

$$c(r,t) = \int_{\beta=0}^{\infty} \frac{\beta}{N(\beta)} e^{-\beta^2 t} U(\beta, r) d\beta \int_{r'=1}^{\infty} r' R(\beta, r') dr'. \tag{11.8}$$

Where
$$U(\beta, r) = J_0(\beta r) Y_0(\beta) - Y_0(\beta r) J_0(\beta), \tag{11.9}$$

and
$$N(\beta) = [J_0^2(\beta) + Y_0^2(\beta)]. \tag{11.10}$$

Then complete integral is given by

$$c(r,t) = \int_{\beta=0}^{\infty} \frac{\beta}{J_0^2(\beta) + Y_0^2(\beta)} e^{-\beta^2 t} [Y_0(\beta r) J_0(\beta)] d\beta - J_0(\beta r) Y_0(\beta)$$
$$. \int_{r'=1}^{\infty} r' [J_0(\beta r') Y_o(\beta) - Y_0(\beta r') J_0(\beta)] dr'. \tag{11.11}$$

12   Conclusion

Similarly to nutrient uptake mathematical model for plant root from ground-water, In this paper, we developed mathematical model for nutrient uptake by aquatic plant root.

We solved radial advection diffusion by re-scaling and reduced it by separation of variables into Bessel's equation rather than Laplace method used in [10], in which whenever Laplace method is used for solving advection diffusion, we have to choose always $P_e << 1$. The method used in this article is one of the best alternative to Laplace method used in [10] and not always

necessary to choose $P_e << 1$ due to which it reducing the advection diffusion equation into diffusion form.

we also developed the model for high sink of nutrient by root and model for zero sink of nutrient by root of aquatic plant

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