Prey-Predator Model with General Holling Type Response Function and Optimal Harvesting Policy

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Abstract -*In this paper, we have considered a very general Holling type predator-prey system with selective harvesting and where both of the species follow logistic growth. The uniform boundedness of the system has been studied together with the conditions of existence. Also, we have obtained the criteria for local stability of various equilibrium points then considering suitable Lyapunov function, the global stability of the system has been discussed. After that using Pontryagin Maximal Principle, we have studied the optimal harvesting policy for the system. At the end, the problem has been illustrated through some numerical examples.*

Keywords-*Prey-predator system, Stability, Bionomic equilibrium, Optimal harvesting, Holling type response.*

I. INTRODUCTION

The way which was first introduced by Lotka [1] and Volterra [2] in population biology, is in recent time a very important branch in modern science. From the simplest model of Lotka-Volterra, researchers gradually developed different kinds of population models which are more realistic and much needful for demand-supply chain in between human population and natural resources. To serve these purpose Jogensen [3] has done the pioneering work for the dynamical behaviour of population biology. Since then the population biology has started to take a formal shape. Now-a-days in population biology the study of fishery and harvesting plays an important role. For this purpose, bio economic modelling is a dynamic area of study in biomathematics. Initially the techniques of formulating the model and its solution procedure have been discussed by Clerk ([4],[5]).

After that many researchers have worked with several single and multi-species interacting population model with more than one state variable. The problem of combine harvesting of two ecologically independent species first studied by Clerk [4]. Also, some multi species harvesting models have been developed by Mesterton-Gibbons [7]. Then many types of combined harvesting of two and three species have also been discussed by Chowdhuri and Kar ([8]- [10]), Kar *et.al.* [11], Sadhukhan *et.al.*[12] etc.

In this present work, we have considered a prey predator model with selecting harvesting of prey population only and having logistic growth rate for both species with the assumption that the predator has some other source of food except the prey under consideration, which is more realistic for real life phenomenon like shark, salmon, harring etc. type of fishes. Till now almost all researchers have considered simple response functions like Holling type II and ratio dependent function. But in the present model we have considered a general type of Holling function ([13]- [15]) which is more realistic like in real life problems. The model has been discussed about its uniform boundedness, local stability, Global stability, bionomic equilibrium position, optimal harvesting condition and at the end, with some numerical examples based on hypothetical data model has been illustrated.

II. FORMULATION OF THE MODEL

Assuming that the harvesting of prey populations is subjected to as harvesting effort E, we can formulate the model as

$$\frac{dx}{dt} = r_1 x \left(1 - \frac{x}{k_1}\right) - \frac{\alpha x^p y}{a + x^p} - Eqx$$
$$\frac{dy}{dt} = r_2 y \left(1 - \frac{y}{k_2}\right) + \frac{\beta x^p y}{a + x^p} \qquad (1)$$

with, $x(0) \geq 0, y(0) \geq 0$.

Where,
$x = x(t)$ = size of the prey population at time t,
$y = y(t)$ = size of the predator population at time t,
k_1 =environmental carrying capacity of the prey,
k_2 =environmental carrying capacity of the predator,
r_1 =intrinsic growth rate of prey,

r_2 =intrinsic growth rate of predator,
a = Michaelies-Menten constant,
α = predation coefficient,
β = conversion factor, assumed to be less than 1,
q =are catchability coefficients of the prey species respectively,

also, the response function $g(x) = \frac{x^p}{a+x^p}$ ($p > 0$) is built by assuming that the per capita rate of predation depends on the prey numbers only. Here all parameters are positive. Here the functional response is said to belong to Holling type II if $p \leq 1$; to Holling type III if $p > 1$.

III. STEADY STATES

The possible steady states of the system (1) are $E_0(0,0)$, $E_1(\overline{x}, 0)$, where $\overline{x} = \frac{k_1}{r_1}(r_1 - Eq)$, $E_2(0, k_2)$ and non-trivial steady state $E_3(x^*, y^*)$, where

$$r_1\left(1 - \frac{x^*}{k_1}\right) - \frac{\alpha x^{*p-1}y^*}{a+x^{*p}} - Eq = 0$$
$$r_2\left(1 - \frac{y^*}{k_2}\right) - \frac{\beta x^{*p}}{a+x^{*p}} = 0 \quad (2)$$

IV. BOUNDEDNESS OF THE SOLUTIONS

A. Theorem 1. All the solutions of the system (1) in $\Re_+^2$ are uniformly bounded.

Proof: Let us introduce a function $W = \frac{x}{\alpha} + \frac{y}{\beta}$.

Now using the system (1) and above relation we have

$$\frac{dW}{dt} = \frac{r_1 x}{\alpha}\left(1 - \frac{x}{k_1}\right) - \frac{Eqx}{\alpha} + \frac{r_2}{\beta}\left(1 - \frac{y}{k_2}\right)$$

Therefore, for $\eta > 0$

$$\frac{dW}{dt} + \eta W = \frac{r_1 x}{\alpha}\left(1 - \frac{x}{k_1}\right) - \frac{Eq}{\alpha} + \frac{r_2}{\beta}\left(1 - \frac{y}{k_2}\right) + \eta\frac{x}{\alpha} + \eta\frac{y}{\beta}$$

$$= \frac{x}{\alpha}\left[r_1\left(1 - \frac{x}{k_1}\right) + \eta - Eq\right] + \frac{y}{\beta}\left[r_2\left(1 - \frac{y}{k_2}\right) + \eta\right].$$

Now using the relations

$$[k_1(r_1 + \eta - Eq) - 2r_1 x]^2 \geq 0 \text{ and } [k_2(r_2 + \eta) - 2r_2 y]^2 \geq 0$$

we can respectively derive

$$\frac{x}{\alpha}\left[r_1\left(1 - \frac{x}{k_1}\right) + \eta - Eq\right] \leq \frac{k_1}{4r_1\alpha}(r_1 + \eta - Eq)^2$$

and

$$\frac{y}{\beta}\left[r_2\left(1 - \frac{y}{k_2}\right) + \eta\right] \leq \frac{k_2}{4r_2\beta}(r_2 + \eta)^2$$

Therefore, $\frac{dW}{dt} + \eta W \leq \gamma$.

Where, $\gamma = \frac{k_1}{4r_1\alpha}(r_1 + \eta - Eq)^2 + \frac{k_2}{4r_2\beta}(r_2 + \eta)^2$.

Now applying the theorem of differential inequalities [17], we get $0 \leq W(x,y) \leq \frac{\gamma}{\eta} + \frac{W(x(0),y(0))}{e^{\eta t}}$.

Therefore, for $t \to \infty$, $0 \leq W(x,y) \leq \frac{\gamma}{\eta}$.

So, $\left\{(x,y) \in \Re_+^2 : 0 \leq W(x,y) \leq \frac{\gamma}{\eta} + \epsilon, \text{for } \epsilon > 0\right\} = B$ is the region where all solutions of the system of equation (1) are bounded in $\Re_+^2$. Hence the theorem.

V. LOCAL STABILITY ANALYSIS

In this section, we investigate the local stability of the system (1) around its steady states.
The variational matrix of the system (1) is given by

$$V(x,y) = \begin{bmatrix} r_1 - \frac{2r_1}{k_1}x - \frac{\alpha a x^{p-1}y}{(a+x^p)^2} - Eq & -\frac{\alpha x^p}{a+x^p} \\ \frac{a\beta p x^{p-1}y}{(a+x^p)^2} & r_2 - \frac{2r_2}{k_2}y + \frac{\beta x^p}{a+x^p} \end{bmatrix} (3)$$

From (3), we get,

$$V(0,0) = \begin{bmatrix} r_1 - Eq & 0 \\ 0 & r_2 \end{bmatrix}, \ p \geq 1 (4)$$

From the variational matrix (3), using the relation $\overline{x} = \frac{k_1}{r_1}(r_1 - Eq)$ we get,

$$V(\overline{x},0) = \begin{bmatrix} -\frac{r_1}{k_1}\overline{x} & -\frac{\alpha\overline{x}^p}{a+\overline{x}^p} \\ 0 & r_2 + \frac{\beta\overline{x}^p}{a+\overline{x}^p} \end{bmatrix} \quad (5)$$

Again, from the variational matrix (3), for the point $E_2(0,k_2)$

$$V(0,k_2) = \begin{bmatrix} r_1 - Eq & 0 \\ 0 & -r_2 \end{bmatrix}, \ p \geq 1 (6)$$

Also, for the non-trivial steady state $E_3(x^*,y^*)$, using (2) in (3) we have,

$$V(x^*,y^*) = \begin{bmatrix} -\frac{r_1}{k_1}x^* + \frac{\alpha x^{*p-1}y^*}{(a+x^{*p})} - \frac{\alpha a p\, x^{*p-1}y^*}{(a+x^{*p})^2} & -\frac{\alpha x^{*p}}{a+x^{*p}} \\ \frac{a\beta p\, x^{*p-1}y^*}{(a+x^{*p})^2} & -\frac{r_2}{k_2}y^* \end{bmatrix} (7)$$

A. Theorem 2. The steady state $E_0(0,0)$ is unstable.

Proof: The eigen values of V $(0, 0)$ are $r_1 - Eq$ and r_2. So, irrespective of the sign of $r_1- Eq$, the intrinsic growth rate of predator r_2 is always positive.

Therefore, the steady state $E_0(0,0)$ is unstable.

B. Theorem 3. The steady state $E_1(\overline{x},0)$ is unstable.

Proof: The eigen values of V $(\overline{x},0)$ are $-(r_1 - Eq)$ and $r_2 + \frac{\beta\overline{x}^p}{a+\overline{x}^p}$. So, irrespective of the sign of $-(r_1- Eq)$, the other eigen value $r_2 + \frac{\beta\overline{x}^p}{a+\overline{x}^p}$ is always positive.

Therefore, the steady state $E_1(\overline{x},0)$ is unstable.

C. Theorem 4. The steady state $E_2(0,k_2)$ will be an asymptotically stable if and only if $p \geq 1$ and $E > BTP_x$ and unstable otherwise. Where, $BTP_x = \frac{r_1}{q}$, is the biotic potential for x.

Proof: The eigen values of V $(0, k_2)$ are $(r_1 - Eq)$ and $-r_2$. So, the steady state will be a stable node if $(r_1 - Eq) < 0$. That is if $E > \frac{r_1}{q}$, which imply $E > BTP_x$.

This completes the proof.

D. Theorem 5. The steady state $E_3(x^*,y^*)$ will be an asymptotically stable if and only if $\frac{r_1}{k_1} > \frac{\alpha x^{*p-2}y^*}{(a+x^{*p})}$ and unstable otherwise.

Proof: The characteristic equation for the variational matrix $V(x^*,y^*)$ can be written as

$$\lambda^2 - \frac{r_1}{k_1}x^* + \left[\frac{\alpha x^{*p-1}y^*}{(a+x^{*p})} - \frac{\alpha a p\, x^{*p-1}y^*}{(a+x^{*p})^2} - \frac{r_2}{k_2}y^*\right]\lambda$$
$$+ \left[\frac{r_1 r_2}{k_1 k_2}x^*y^* - \frac{r_2}{k_2}\frac{\alpha x^{*p-1}y^{*2}}{(a+x^{*p})} + \frac{r_2}{k_2}\frac{\alpha a p\, x^{*p-1}y^{*2}}{(a+x^{*p})^2} + \frac{a\beta a p\, x^{*2p-1}y^*}{(a+x^{*p})^3}\right] = 0 \quad (8)$$

From the equation (8), the sum and product of the roots are respectively

$$\lambda_1 + \lambda_2 = \left[-\frac{r_1}{k_1}x^* + \frac{\alpha x^{*p-1}y^*}{(a+x^{*p})} - \frac{\alpha a p\, x^{*p-1}y^*}{(a+x^{*p})^2} - \frac{r_2}{k_2}y^*\right] (9)$$

and

$$\lambda_1\lambda_2 = \frac{r_1 r_2}{k_1 k_2}x^*y^* - \frac{r_2}{k_2}\frac{\alpha x^{*p-1}y^{*2}}{(a+x^{*p})} + \frac{r_2}{k_2}\frac{\alpha a p\, x^{*p-1}y^{*2}}{(a+x^{*p})^2} + \frac{a\beta a p\, x^{*2p-1}y^*}{(a+x^{*p})^3} (10)$$

Therefore, $E_3(x^*,y^*)$ will be asymptotically stable node if

$$\lambda_1 + \lambda_2 < 0 \ and \ \lambda_1\lambda_2 > 0 \ (11)$$

which implies the conditions

$$\frac{r_1}{k_1}x^* - \frac{\alpha x^{*p-1}y^*}{(a+x^{*p})} + \frac{\alpha a p\, x^{*p-1}y^*}{(a+x^{*p})^2} + \frac{r_2}{k_2}y^* > 0 \ (12)$$

and

$$\frac{r_1 r_2}{k_1 k_2}x^*y^* - \frac{r_2}{k_2}\frac{\alpha x^{*p-1}y^{*2}}{(a+x^{*p})} + \frac{r_2}{k_2}\frac{\alpha a p\, x^{*p-1}y^{*2}}{(a+x^{*p})^2} + \frac{a\beta a p\, x^{*2p-1}y^*}{(a+x^{*p})^3} > 0 \ (13)$$

These two conditions hold together if and only if

$$\frac{r_1}{k_1} > \frac{\alpha x^{*p-2}y^*}{(a+x^{*p})} (14)$$

which completes the theorem.

VI. GLOBAL STABILITY ANALYSIS

In this section, to check the global stability of the system (1) we assume that the equilibrium point $E_3(x^*,y^*)$ exists and it is locally asymptotically stable.

Now let us consider the functions

$G(x,y) = \frac{1}{xy}$ (15)

$h_1(x,y) = r_1 x \left(1 - \frac{x}{k_1}\right) - \frac{\alpha x^p y}{a+x^p} - Eqx$ (16)

and

$h_2(x,y) = r_2 y \left(1 - \frac{y}{k_2}\right) + \frac{\beta x^p y}{a+x^p}$ (17)

Clearly, $G(x,y) > 0$ in the interior of the positive quadrant of $x - y$ plane.
Therefore, we have

$$\Delta(x,y) = \frac{\partial(h_1 G)}{\partial x} + \frac{\partial(h_2 G)}{\partial y} = -\frac{r_1}{k_1 y} - \frac{\alpha}{(a+x^p)^2}[a(p-1)x^{p-2} - x^{2p-2}] - \frac{r_2}{k_2 x}$$

Clearly for $p > 1, \Delta(x,y) < 0$ if $a(p-1)x^{p-2} - x^{2p-2} > 0$.

That is if $x < [a(p-1)]^{\frac{1}{p}}$.

Therefore, in this circumstance, by Bendixson-Dulac criterion there exists no limit cycle in the region

$$B \equiv \left\{(x,y): x < [a(p-1)]^{\frac{1}{p}}, y > 0\right\}$$

So, with the assumption of the existence and local asymptotic stability of the interior equilibrium point $E_3(x^*, y^*)$, it is also globally asymptotically stable in the above region B of $x - y$plane [18].

Again, for $p > 1, \Delta(x,y) = -\frac{r_1}{k_1 y} + \frac{\alpha}{(a+x^p)^2} - \frac{r_2}{k_2 x}$.

So, in this case the region for globally asymptotically stable will be

$$B_1 \equiv \left\{(x,y): \frac{r_1}{k_1 y} + \frac{r_2}{k_2 x} > \frac{\alpha}{(a+x^p)^2}, y > 0\right\}.$$

If $0 < p < 1$ then the condition for global stability is $x^p < a(p-1)$, which is not possible as, $x$is non-negative.

VII. BIONOMIC EQUILIBRIUM

The concept of bionomic equilibrium is the combination of biological and economic equilibriums. Biological equilibrium is the solution of the system $\dot{x} = 0, \dot{y} = 0$and the economic equilibrium occurs when the total revenue (TR) obtained by selling the harvested biomass is equal to the total cost (TC) for the effort of harvesting. Now considering C as constant fishing cost per unit effort and c as constant price per unit biomass of the first and second species respectively, the net economic rent is given by

$\pi(x,y,E) = cqxE - CE$ (18)

As the harvesting cost per unit effort (C) is constant, now we get from (1)

$\dot{x} = 0 \Rightarrow x = 0 \; or, E = \frac{1}{q}\left[r_1\left(1 - \frac{x}{k_1}\right) - \frac{\alpha x^{p-1} y}{a+x^p}\right]$ (19)

and

$\dot{y} = 0 \Rightarrow y = 0 \; or \; y = \frac{k_2}{r_2}\left[r_2 + \frac{\beta x^p}{a+x^p}\right]$(20)

So, the bionomic equilibrium $(x_b, y_b)$is determined by solving the (19), (20) along with the equation

$\pi(x,y,E) = cqxE - CE = 0$ (21)

Therefore, we have

$$\left. \begin{array}{l} x_b = \frac{c}{Cq} \\ y_b = \frac{k_2}{r_2}\left[r_2 + \frac{\beta\left(\frac{c}{Cq}\right)^p}{a+\left(\frac{c}{Cq}\right)^p}\right] \end{array} \right\}(22)$$

and the corresponding value of $E$is

$E_b = \frac{1}{q}\left[r_1\left(1 - \frac{x_b}{k_1}\right) - \frac{\alpha x_b^{p-1} y_b}{a+x_b{}^p}\right]$(23)

VIII. OPTIMAL HARVESTING POLICY

In this section with the assumption of existence of$E_3(x^*, y^*)$,the present value of$J$, continuoustime-stream of revenues is given by

$J = \int_0^\infty e^{-\delta t} \pi(x,y,E,t)dt$(24)

where $\pi(x,y,E,t) = (cqx - C)E$ and δ denotes the annual discount rate. Now we have to maximize J subject

to the system of equation (1) using Pontryagin's Maximal Principle [19]. The control variable $E(t)$ is subjected to the constraints $0 \leq E(t) \leq E_{max}$, so that $V_t = [0, E_{max}]$ is the control set and E_{max} is a feasible upper limit for the harvesting effort.

The Hamiltonian for the problem is given by

$$H = e^{-\delta t}(cqx - C)E + \lambda_1 \left\{ r_1 x \left(1 - \frac{x}{k_1}\right) - \frac{\alpha x^p y}{a+x^p} - Eqx \right\} + \lambda_2 \left\{ r_2 x \left(1 - \frac{y}{k_2}\right) + \frac{\beta x^p y}{a+x^p} \right\} (25)$$

where, $\lambda_1(t)$ and $\lambda_2(t)$ are adjoint variables.

Hamiltonian H depends linearly on E with coefficient $\sigma = e^{-\delta t}(cqx - C) - \lambda_1 qx$

Consequently, its maximum value is reached for extremes of E, i.e. the harvest rate must be either 0 or E_{max}. This observation leads to the rule that one must harvest as much as possible when the switching function $\sigma > 0$, and will not harvest at all when $\sigma < 0$. Furthermore, when $\sigma = 0$, the harvest rate is undetermined. In this case three solutions for E are possible, namely $0, E_{max}$ or E^* which is the singular control that maintains the condition $\sigma = 0$. Therefore, the optimal control path will be either "bang-bang" control or singular.

Our objective is to reach optimal solution optimally from the initial state $(x(0), y(0))$. This can be achieved by applying a "bang-bang" control (Pontryagin *et. al.* [19]) to the system as presented below.

Define,

$$\tilde{E}(t) = \begin{cases} E_{max} & for \ \sigma(t) > 0 \\ 0 & for \ \sigma(t) < 0 \end{cases} (26)$$

Moreover, let T be the time at which the path $(x(t), y(t))$, which, generated via the "bang-bang" control $E(t) = \tilde{E}(t)$, reaches the steady state (x_δ, y_δ). Then, the optimal control policy is

$$E(t) = \begin{cases} \tilde{E}(t) & for \ 0 \leq t < T \\ E^* & for \quad t > T \end{cases} (27)$$

and the optimal path is given by the trajectory generated by the above optimal control.

In view of the stability property of the interior equilibrium of the system (1), we can also reach the singular optimal solution through a suboptimal by choosing the control policy $E(t)$ to be equal to E^* for all t. The advantage of choosing the optimal path is that it leads to the optimal singular solution more rapidly than the suboptimal path does.

By the maximal principal, there exists adjoint variables $\lambda_1(t)$ and $\lambda_2(t)$ for all $t \geq 0$, such that

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x} = \lambda_1 \left(\frac{r_1 x}{k_1} - \frac{\alpha x^{p-1} y}{a+x^p} + \frac{a\alpha p \, x^{p-1} y}{(a+x^p)^2} \right) - \lambda_2 \frac{a\beta p \, x^{p-1} y}{(a+x^p)^2} - cqe^{-\delta t}E (28)$$

and

$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial y} = \lambda_1 \frac{\alpha x^p}{a+x^p} + \lambda_2 \frac{r_2 y}{k_2} (29)$$

Now eliminating λ_2 from (28) and (29)

$$\frac{d^2\lambda_1}{dt^2} - \left(\frac{r_1 x}{k_1} - \frac{\alpha x^{p-1} y}{a+x^p} + \frac{a\alpha p \, x^{p-1} y}{(a+x^p)^2} + \frac{r_2 y}{k_2} \right) \frac{d\lambda_1}{dt} + \left[\frac{r_2 y}{k_2} \left(\frac{r_1 x}{k_1} - \frac{\alpha x^{p-1} y}{a+x^p} + \frac{a\alpha p \, x^{p-1} y}{(a+x^p)^2} \right) + \frac{\alpha^2 \beta a p \, x^{2p-1} y}{(a+x^p)^3} \right] \lambda_1 = M_1 e^{-\delta t} (30)$$

where, with the help of (23)

$$M_1 = \delta cqE + \frac{r_2 y}{k_2} cqE = \left(\delta cq + \frac{r_2 y}{k_2} cq \right) \left\{ \frac{1}{q} \left(r_1 - \frac{r_1 x_b}{k_1} - \frac{\alpha x_b^{p-1} y_b}{a+x_b^p} \right) \right\} (31)$$

The auxiliary equation corresponding to the differential equation (28) can be written as

$$\mu^2 - \left(\frac{r_1 x}{k_1} - \frac{\alpha x^{p-1} y}{a+x^p} + \frac{a\alpha p \, x^{p-1} y}{(a+x^p)^2} + \frac{r_2 y}{k_2} \right) \mu + \left[\frac{r_2 y}{k_2} \left(\frac{r_1 x}{k_1} - \frac{\alpha x^{p-1} y}{a+x^p} + \frac{a\alpha p \, x^{p-1} y}{(a+x^p)^2} \right) + \frac{\alpha^2 \beta a p \, x^{2p-1} y}{(a+x^p)^3} \right] = 0 (32)$$

From the equation (32) with the help of (12) and (13) under the assumption that the condition of Theorem-5 holds, we have

$$The \ sum \ of \ the \ roots = \left(\frac{r_1 x}{k_1} - \frac{\alpha x^{p-1} y}{a+x^p} + \frac{a\alpha p \, x^{p-1} y}{(a+x^p)^2} + \frac{r_2 y}{k_2} \right) > 0 (33)$$

and

$$The \ product \ of \ the \ roots = \frac{r_2 y}{k_2} \left(\frac{r_1 x}{k_1} - \frac{\alpha x^{p-1} y}{a+x^p} + \frac{a\alpha p \, x^{p-1} y}{(a+x^p)^2} \right) + \frac{\alpha^2 \beta a p \, x^{2p-1} y}{(a+x^p)^3} > 0 (34)$$

So, we conclude that the roots of the equation (32) are either both real and positive or complex conjugates with positive real parts. Therefore, the general solution of (30) is of the form

$$\lambda_1 = A_1 e^{\mu_1 t} + B_1 e^{\mu_2 t} + \frac{M_1}{N} e^{-\delta t} (35)$$

where, A_1 and B_1 are arbitrary constants and $\mu_i's (i = 1,2)$ are the roots of the equation (32)

$$N = \delta^2 + \left(\frac{r_1 x}{k_1} - \frac{\alpha x^{p-1} y}{a+x^p} + \frac{a\alpha p \, x^{p-1} y}{(a+x^p)^2} + \frac{r_2 y}{k_2} \right) \delta + \left[\frac{r_2 y}{k_2} \left(\frac{r_1 x}{k_1} - \frac{\alpha x^{p-1} y}{a+x^p} + \frac{a\alpha p \, x^{p-1} y}{(a+x^p)^2} \right) + \frac{\alpha^2 \beta a p \, x^{2p-1} y}{(a+x^p)^3} \right] \neq 0 (36)$$

Therefore, the shadow price [4], $\lambda_1 e^{\delta t}$ remains bounded as $t \to \infty$ if and only if $A_1 = B_1 = 0$ and therefore

$$\lambda_1 e^{\delta t} = \frac{M_1}{N} = constant (37)$$

Similarly, we get

$$\lambda_2 e^{\delta t} = \frac{M_2}{N} = constant \quad (38)$$

again here, with the help of (23)

$$M_2 = -\frac{\alpha x^p}{a+x^p} \left\{ \frac{1}{q} \left(r_1 - \frac{r_1 x_b}{k_1} - \frac{\alpha x_b^{p-1} y_b}{a+x_b^p} \right) \right\} \quad (39)$$

Thus, the shadow price, $\lambda_i e^{\delta t}$ ($i = 1, 2.$), remain constant over time in optimal equilibrium when they satisfy the transversality condition at infinity [5], when they remain bounded as $t \to \infty$. The Hamiltonian (25) must maximized for E belongs to V_t. Assuming that the control constraints are not binding (that is optimal solution does not occur at $E = 0$ or $E = E_{max}$) we have singular control [6] given by

$$\frac{\partial H}{\partial E} = e^{-\delta t} (cqx - C) - \lambda_1 qx = 0 \quad (40)$$

i.e.

$$\lambda_1 qx = e^{-\delta t} \frac{\partial \pi}{\partial E} \quad (41)$$

which indicates that the total user cost of harvest per unit effort must be equal to the discounted value of the future profit at the steady state effort level. Using the values of $\lambda_1(t)$ and $\lambda_2(t)$ in equation (40), we get

$$\left(c - \frac{M_1}{N} \right) qx = C \quad (42)$$

The above equation together with the equations (22) and (23) gives the optimal equilibrium population densities as $x = x_\delta$, $y = y_\delta$. Now when $\delta \to \infty$, the above equation leads to the result

$$cqx_\infty = C \quad (43)$$

which gives that $\pi(x_\infty, y_\infty, E) = 0$.

This shows that an infinite discount rate leads to complete dissipation of economic revenue. Which is a similar conclusion drawn by Clark [4] for combined harvesting of two ecologically independent populations and by Chaudhuri [16] in the combined harvesting of two competing species. Using (42), we get

$$\pi = (cqx - C)E = \frac{M_1 qxE}{N} \quad (44)$$

As each of M_1 is of $o(\delta)$ and N is of $o(\delta^2)$, therefore π is of $o(\delta^{-1})$. Thus, π is a decreasing function of δ (≥ 0). We therefore conclude that $\delta = 0$ leads to maximization of π.

We have established here the existence of an optimal equilibrium solution and discussed about the optimal approach path (*cf.* Clark [6], Kar *et. al.* [22], Srinivasu ([20],[21]), Kar and Chattopadhyay [23]) using "bang-bang" control.

IX. NUMERICAL EXPERIMENTS

We illustrate the model by the examples for the following set of parametric values with suit- able unit: $r_1 = 7.5, r_2 = 6, k_1 = 500, k_2 = 400, a = 30, \alpha = 0.01, \beta = 0.01, q = 0.3, E = 10, c = 0.7, \delta = 0.5$ and $C = 20$

A. Example-1: For the above values of parameters and for $p = 0.5$, we get
(1) from theorem-2, (0,0) is an unstable node.
(2) from theorem-3, (300, 0) is unstable.
(3) from theorem-4, (0, 400) is unstable.
(4) from theorem-5, the nontrivial steady state (296.7, 400) is stable.
(5) the bionomic equilibriums are (0.12, 400) and corresponding harvesting effort $E_b = 3.64$.
(6) the optimal equilibriums are (363.64, 400.25) and corresponding optimal harvesting effort $E^* = 5.57$.

In next two examples, we are going to discuss about the behaviour of the system for non-trivial steady states only.

B. Example-2: For the above values of parameters and for $p = 1$, we get
(1) from theorem-5, the nontrivial steady state (291.7, 400) is stable.
(2) the bionomic equilibriums are (0.12, 400.60) and corresponding harvesting effort is not feasible.
(3) the optimal equilibrium is also not feasible in this case.

C. Example-3: For the above values of parameters and for $p = 1.5$, we get
(1) from theorem-5, the nontrivial steady state (290.9, 400) is stable.
(2) the bionomic equilibriums are (0.12, 401) and corresponding harvesting effort is $E_b = 9.97$.
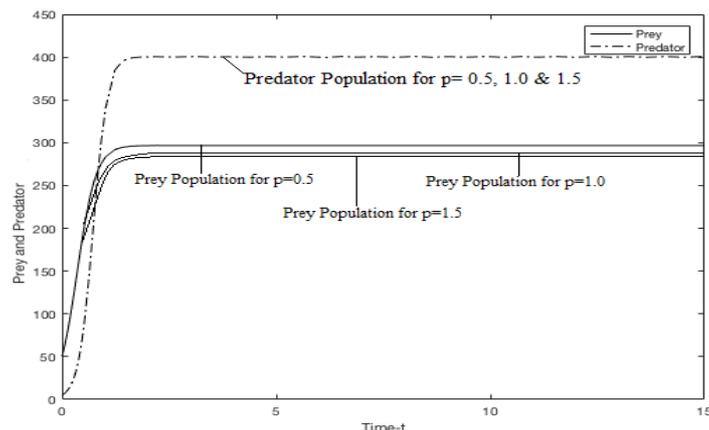(3) the optimal equilibrium is (256.41, 400.67) and corresponding optimal harvesting effort $E^* = 11.67$.

Fig. 1: Stability diagram of the system for p = 0.5, 1.0 & 1.5 with x(0) = 50 and y(0) = 5.

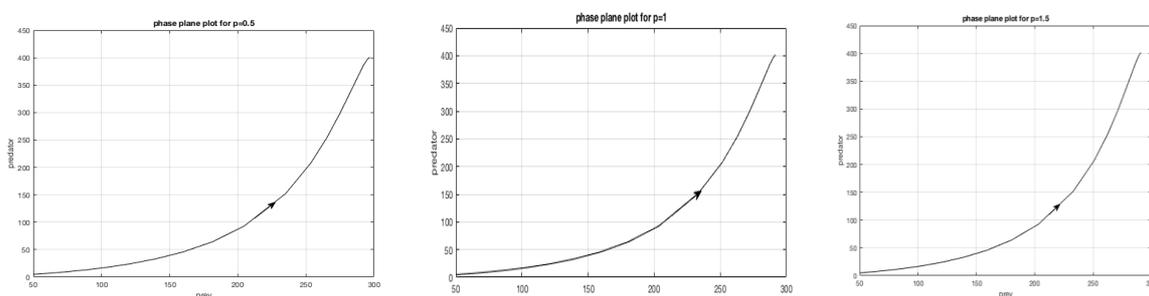Fig.2: Phase portrait for p = 0.5, 1.0 & 1.5 with x(0) = 50 and y(0) = 5.

X. CONCLUSIONS

The present paper mainly deals with the problem of selective harvesting with same catching effort(E)in a two-species prey-predator system in which the growth of the both species is governed by the logistic law of growth. The distinguishing feature of this prey-predator model is the response function, which is a most general Holling type form because for different values of p we can get different Holling type response functions. Firstly, we have studied the boundedness of the solutions for the system in $\mathfrak{R}_+^2$, then we have examined the conditions of existence and stability of the several steady states. Also selecting a suitable Lyapunov function, we analysed the global stability for the system. After that, we have examined the existence of the Bionomic (i.e. biological as well as economic) equilibrium.

Through the optimal harvesting policy, we have tried to find out the condition to maximize the monetary benefit together with the condition to save each of the species from extinction to keep ecological balance right. The optimal tax policy within the range of variation of tax for the interior equilibrium has been studied by Pontryagin's maximal principal.

Finally, we have discussed the problem with the help of a numerical example by using arbitrary feasible parametric values and using MATLAB, we observed from stability diagram (Fig.-1) and phase portrait (Fig.-2)that as the values of $p$decreases, the steady state value of Prey population increases. This is quite good result for ecological sustainability of species. Our model may be extended by incorporating time delay and stochasticity in the system.

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