Deriving Shape Functions and Verified Hermite Functions for a Two Node Element with Three Primary Variables at Each Node

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***Abstract* —** *In this paper, I derived shape functions for a two node element with three primary variables* $W,\theta,k,\theta=\frac{\partial w}{\partial x}$ *and also I verified three verification conditions for shape functions.*

First verification condition at node 1 is

$$(i)\ N_1=1 \text{ and } N_2=0,\ N_3=0, N_4=0, N_5=0, N_6=0$$

$$(ii)\ \frac{\partial N_2}{\partial x}=1 \text{ and } \frac{\partial N_1}{\partial x}=0, \frac{\partial N_3}{\partial x}=0, \frac{\partial N_4}{\partial x}=0, \frac{\partial N_5}{\partial x}=0, \frac{\partial N_6}{\partial x}=0$$

$$(iii)\ \frac{\partial^2 N_3}{\partial x^2}=1 \text{ and } \frac{\partial^2 N_1}{\partial x^2}=0, \frac{\partial^2 N_2}{\partial x^2}=0, \frac{\partial^2 N_4}{\partial x^2}=0, \frac{\partial^2 N_5}{\partial x^2}=0, \frac{\partial^2 N_6}{\partial x^2}=0$$

Second Verification condition at node 2 is

$$(iv)\ N_4=1 \text{ and } N_1=0,\ N_2=0, N_3=0, N_5=0, N_6=0$$

$$(v)\ \frac{\partial N_5}{\partial x}=1 \text{ and } \frac{\partial N_1}{\partial x}=0, \frac{\partial N_2}{\partial x}=0, \frac{\partial N_3}{\partial x}=0, \frac{\partial N_4}{\partial x}=0, \frac{\partial N_6}{\partial x}=0$$

$$(vi)\ \frac{\partial^2 N_6}{\partial x^2}=1 \text{ and } \frac{\partial^2 N_1}{\partial x^2}=0, \frac{\partial^2 N_2}{\partial x^2}=0, \frac{\partial^2 N_3}{\partial x^2}=0, \frac{\partial^2 N_4}{\partial x^2}=0, \frac{\partial^2 N_5}{\partial x^2}=0$$

Third Verification condition is $N_1+N_4=1$. *For computational purpose I used Mathematica 9 Software [2].*

Keywords — *Primary variables, Hermite Functions, Two node element, Shape functions.*

I. Introduction

Two-dimensional problems involve the specification of the secondary variables on discrete portions of the boundary of the domain. In such cases, one can use appropriate one-dimensional interpolation functions to compute the contributions of the specified secondary variables to the nodal force.

II. GEOMETRICAL DESCRIPTION

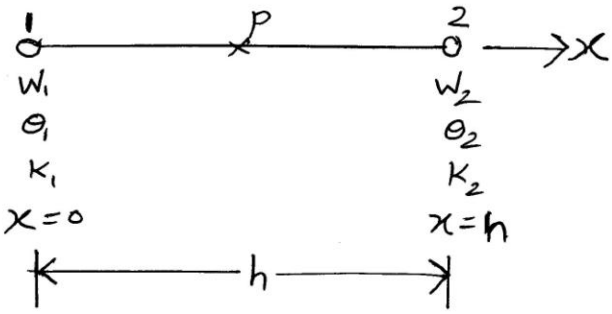

*Figure.*1 Two noded Beam element varying from 0 to h

A two noded beam element shown in Figure.1 in which nodal unknowns are the displacement W and Slope $\theta=\frac{\partial W}{\partial x}$ and primary unknown $k=\frac{\partial^2 W}{\partial x^2}$

III. DERIVING SHAPE FUNCTIONS FOR A TWO NODE ELEMENT WITH THREE PRIMARY VARIABLES AT EACH NODE

Since the element in figure.1 has six degrees of freedom,We have to select the polynomial with only 6 constants. In this polynomial after substituting boundary conditions we get shape functions this we can take as first order Hermitian Polynomials as shape functions.

$$W(x) = A_1 + A_2 x + A_3 x^2 + A_4 x^3 + A_5 x^4 + A_6 x^5 \qquad (1)$$

Where W is the transverse displacement and $A_1, A_2, A_3, A_4, A_5, A_6$ are polynomial Coefficients

If higher order (i.e., higher than cubic) approximation of w is desired, one must identify additional dependent (primary) unknowns at each of the two nodes. For example, if we add $\frac{\partial^2 w}{\partial x^2}$ *as* the primary unknown at each of the two nodes, there will be total of six conditions, and a fifth-order polynomial is required to interpolate the end conditions.

Differentiating partially eq(1) w.r.t. 'x'

$$(1) \Rightarrow \frac{\partial W}{\partial x} = 0 + A_2(1) + A_3(2x) + A_4(3x^2) + A_5(4x^3) + A_6(5x^4)$$

$$\frac{\partial W}{\partial x} = A_2 + 2A_3 x + 3A_4 x^2 + 4A_5 x^3 + 5A_6 x^4 \qquad (2)$$

Differentiating partially eqation(2) w.r.t. 'x'

$$\frac{\partial^2 w}{\partial x^2} = 0 + 2A_3(1) + 3A_4(2x) + 4A_5(3x^2) + 5A_6(4x^3)$$

$$\frac{\partial^2 W}{\partial x^2} = 2A_3 + 6A_4 x + 12A_5 x^2 + 20A_6 x^3 \qquad (3)$$

Applying the nodal conditions such that

$W = W_1$, $\frac{\partial W}{\partial x} = \theta_1$ and $\frac{\partial^2 W}{\partial x^2} = k_1$ at x= 0

and $W = W_2$ and $\frac{\partial W}{\partial x} = \theta_2, \frac{\partial^2 W}{\partial x^2} = k_2$ at x=h , where h is step length

in equations (1), (2) and (3) we get

When $W = W_1$ and x = 0

$$(1) \Rightarrow W_1 = A_1 + A_2(0) + A_3(0)^2 + A_4(0)^3$$

$$W_1 = A_1 + 0 + 0 + 0$$

$$W_1 = A_1 \qquad (4)$$

When $\frac{\partial W}{\partial x} = \theta_1$ and x = 0

$$(2) \Rightarrow \theta_1 = A_2 + 2A_3(0) + 3A_4(0)^2$$

$$\theta_1 = A_2 + 0 + 0$$

$$\theta_1 = A_2 \qquad (5)$$

When $\frac{\partial^2 W}{\partial x^2} = k_1$ at x=0

$$(3) \Rightarrow k_1 = 2A_3 \qquad (6)$$

When $W = W_2$ and x = h

$$(1) \Rightarrow W_2 = A_1 + A_2(h) + A_3 h^2 + A_4 h^3 + A_5 h^4 + A_6 h^5 \qquad (7)$$

When $\frac{\partial W}{\partial x} = \theta_2$ and x = h

$$(2) \Rightarrow \theta_2 = A_2 + 2A_3 h + 3A_4 h^2 + 4A_5 h^3 + 5A_6 h^4 \quad (8)$$

When $\frac{\partial^2 W}{\partial x^2} = k_2$ and x=h

$$(3) \Rightarrow 2A_3 + 6A_4 h + 12A_5 h^2 + 20A_6 h^3 \quad (9)$$

Using Mathematica 9 Software Solving (4),(5) ,(6),(7),(8) and (9) we get $A_1, A_2, A_3, A_4, A_5, A_6$

Input

$Solve[A_1 - W_1 == 0 \&\& A_2 - \theta_1 == 0 \&\& 2*A_3 - k_1 == 0 \&\& A_1 + A_2 * h + A_3 * h^2 + A_4 * h^3 + A_5 * h^4 + A_6 * h^5 - W_2 == 0 \&\& A_2 + 2*A_3*h + 3*A_4 * h^2 + 4*A_5*h^3 + 5*A_6*h^4 - \theta_2 == 0 \&\& A_2 + 2*A_3 + 6*A_4*h + 12*A_5*h^2 + 20*A_6*h^3 - k_2 == 0, \{A_1, A_2, A_3, A_4, A_5, A_6\}]$

Output

$\{\{A_1 -> W_1, A_2 -> \theta_1,$

$A_3 -> \frac{k_1}{2}, A_4 -> -\frac{3h^2 k_1 - h^2 k_2 + 20W_1 - 20W_2 + 12h\theta_1 + 8h\theta_2}{2h^3},$

$A_5 -> -\frac{-3h^2 k_1 + 2h^2 k_2 - 30W_1 + 30W_2 - 16h\theta_1 - 14h\theta_2}{2h^4},$

$A_6 -> -\frac{h^2 k_1 - h^2 k_2 + 12W_1 - 12W_2 + 6h\theta_1 + 6h\theta_2}{2h^5}\}\}$

Substituting $A_1, A_2, A_3, A_4, A_5, A_6$ in eq(1)

$A_1 := W_1$

$A_2 := \theta_1$

$A_3 := \frac{k_1}{2}$

$A_4 := -\frac{3h^2 k_1 - h^2 k_2 + 20W_1 - 20W_2 + 12h\theta_1 + 8h\theta_2}{2h^3}$

$A_5 := -\frac{-3h^2 k_1 + 2h^2 k_2 - 30W_1 + 30W_2 - 16h\theta_1 - 14h\theta_2}{2h^4}$

$A_6 := -\frac{h^2 k_1 - h^2 k_2 + 12W_1 - 12W_2 + 6h\theta_1 + 6h\theta_2}{2h^5}$

$W(x) := A_1 + A_2 * x + A_3 * x^2 + A_4 * x^3 + A_5 * x^4 + A_6 * x^5$

$Expand[W(x)]$

Output

$$\frac{x^2 k_1}{2} - \frac{3x^3 k_1}{2h} + \frac{3x^4 k_1}{2h^2} - \frac{x^5 k_1}{2h^3} + \frac{x^3 k_2}{2h} - \frac{x^4 k_2}{h^2} + \frac{x^5 k_2}{2h^3} + W_1 - \frac{10x^3 w_1}{h^3} + \frac{15x^4 W_1}{h^4} - \frac{6x^5 W_1}{h^5}$$

$$+\frac{10x^3 W_2}{h^3} - \frac{15x^4 W_2}{h^4} + \frac{6x^5 W_2}{h^5} + x\theta_1 - \frac{6x^3\theta_1}{h^2} + \frac{8x^4\theta_1}{h^3} - \frac{3x^5\theta_1}{h^4} - \frac{4x^3\theta_2}{h^2} + \frac{7x^4\theta_2}{h^3} - \frac{3x^5\theta_2}{h^4}$$

$$W(x) = \frac{x^2 k_1}{2} - \frac{3x^3 k_1}{2h} + \frac{3x^4 k_1}{2h^2} - \frac{x^5 k_1}{2h^3} + \frac{x^3 k_2}{2h} - \frac{x^4 k_2}{h^2} + \frac{x^5 k_2}{2h^3} + W_1 - \frac{10x^3 w_1}{h^3} + \frac{15x^4 W_1}{h^4} - \frac{6x^5 W_1}{h^5} + \frac{10x^3 W_2}{h^3} - \frac{15x^4 W_2}{h^4} + \frac{6x^5 W_2}{h^5} + x\theta_1 - \frac{6x^3\theta_1}{h^2} + \frac{8x^4\theta_1}{h^3} - \frac{3x^5\theta_1}{h^4} - \frac{4x^3\theta_2}{h^2} + \frac{7x^4\theta_2}{h^3} - \frac{3x^5\theta_2}{h^4}$$

$$W(x)=W_1\left(1-\frac{10x^3}{h^3}+\frac{15x^4}{h^4}-\frac{6x^5}{h^5}\right)$$
$$+\theta_1\left(x-\frac{6x^3}{h^2}+\frac{8x^4}{h^3}-\frac{3x^5}{h^4}\right)$$
$$+k_1\left(\frac{x^2}{2}-\frac{3x^3}{2h}+\frac{3x^4}{2h^2}-\frac{x^5}{2h^3}\right)$$
$$+W_2\left(\frac{10x^3}{h^3}-\frac{15x^4}{h^4}+\frac{6x^5}{h^5}\right)$$
$$+\theta_2\left(-\frac{4x^3\theta_2}{h^2}+\frac{7x^4}{h^3}-\frac{3x^5}{h^4}\right)$$
$$+k_2\left(\frac{x^3}{2h}-\frac{x^4}{h^2}+\frac{x^5}{2h^3}\right) \qquad (10)$$

i.e.,

$$\left.\begin{aligned} W&=N_1W_1+N_2\theta_1+N_3k_1+N_4W_2+N_5\theta_2+N_6k_2\\ &=N_1\delta_1+N_2\delta_2+N_3\delta_3+N_4\delta_4+N_5\delta_5+N_6\delta_6\end{aligned}\right\} (11)$$

Where $N_1, N_2, N_3, N_4, N_5, N_6$ are shape functions for the beam elements and $\delta_1,\delta_2,\delta_3,\delta_4,\delta_5,\delta_6$ are the nodal displacements

$$\text{i.e., } \{\delta\}=\begin{Bmatrix}\delta_1\\ \delta_2\\ \delta_3\\ \delta_4\\ \delta_5\\ \delta_6\end{Bmatrix}=\begin{Bmatrix}W_1\\ \theta_1\\ k_1\\ W_2\\ \theta_2\\ k_2\end{Bmatrix}$$

Comparing (10) and (11) we get

$$N_1=H_{01}^2(x)=1-\frac{10x^3}{h^3}+\frac{15x^4}{h^4}-\frac{6x^5}{h^5} \qquad (12)$$

$$N_2=H_{11}^2(x)=x-\frac{6x^3}{h^2}+\frac{8x^4}{h^3}-\frac{3x^5}{h^4} \qquad (13)$$

$$N_3=H_{21}^2(x)=\frac{x^2}{2}-\frac{3x^3}{2h}+\frac{3x^4}{2h^2}-\frac{x^5}{2h^3} \qquad (14)$$

$$N_4=H_{02}^2(x)=\frac{10x^3}{h^3}-\frac{15x^4}{h^4}+\frac{6x^5}{h^5} \qquad (15)$$

$$N_5=H_{12}^2(x)=-\frac{4x^3\theta_2}{h^2}+\frac{7x^4}{h^3}-\frac{3x^5}{h^4} \qquad (16)$$

$$N_6=H_{22}^2(x)=\frac{x^3}{2h}-\frac{x^4}{h^2}+\frac{x^5}{2h^3} \qquad (17)$$

Here

In $H_{01}^2(x)$, 0 represents Zeroth order derivative, 1 represents node number one and power 2 represents second order Hermitian function.

In $H_{11}^2(x)$, 1 represents first order derivative, 1 represents node number one and power 2 represents second order Hermitian function.

In $H_{21}^2(x)$, 2 represents second order derivative, 1 represents node number one and power 2 represents second order Hermitian function.

In $H_{02}^2(x)$, 0 represents Zeroth order derivative, 2 represents node number two and power 2 represents second order Hermitian function.

In $H_{12}^2(x)$, 1 represents first order derivative, 2 represents node number two and power 2 represents second order Hermitian function.

In $H^2_{22}(x)$, 2 represents second order derivative, 2 represents node number two and power 2 represents second order Hermitian function.

IV. VERIFICATION

(I). 1st VERIFICATION CONDITION

First verification condition at node 1 is

$(i)\ N_1 = 1$ and $N_2 = 0,\ N_3 = 0, N_4 = 0,$
$N_5 = 0, N_6 = 0$

$$(ii)\ \frac{\partial N_2}{\partial x} = 1 \text{ and } \frac{\partial N_1}{\partial x} = 0, \frac{\partial N_3}{\partial x} = 0, \frac{\partial N_4}{\partial x} = 0, \frac{\partial N_5}{\partial x} = 0, \frac{\partial N_6}{\partial x} = 0$$

$$(iii)\ \frac{\partial^2 N_3}{\partial x^2} = 1 \text{ and } \frac{\partial^2 N_1}{\partial x^2} = 0, \frac{\partial^2 N_2}{\partial x^2} = 0, \frac{\partial^2 N_4}{\partial x^2} = 0, \frac{\partial^2 N_5}{\partial x^2} = 0, \frac{\partial^2 N_6}{\partial x^2} = 0$$

At Node 1, x = 0

Input

$$N_1 = H^2_{01}(x) := 1 - \frac{10x^3}{h^3} + \frac{15x^4}{h^4} - \frac{6x^5}{h^5}$$

$$N_2 = H^2_{11}(x) := x - \frac{6x^3}{h^2} + \frac{8x^4}{h^3} - \frac{3x^5}{h^4}$$

$$N_3 = H^2_{21}(x) := \frac{x^2}{2} - \frac{3x^3}{2h} + \frac{3x^4}{2h^2} - \frac{x^5}{2h^3}$$

$$N_4 = H^2_{02}(x) := \frac{10x^3}{h^3} - \frac{15x^4}{h^4} + \frac{6x^5}{h^5}$$

$$N_5 = H^2_{12}(x) := -\frac{4x^3\theta_2}{h^2} + \frac{7x^4}{h^3} - \frac{3x^5}{h^4}$$

$$N_6 = H^2_{22}(x) := \frac{x^3}{2h} - \frac{x^4}{h^2} + \frac{x^5}{2h^3}$$

$x := 0$

N_1

N_2

N_3

N_4

N_5

N_6

Output

1

0

0

0

0

0

Finding first derivatives for (12),(13),(14) , (15),(16), and (17)

Input

$$N_1 = H^2_{01}(x) := 1 - \frac{10x^3}{h^3} + \frac{15x^4}{h^4} - \frac{6x^5}{h^5}$$

$$N_2 = H^2_{11}(x) := x - \frac{6x^3}{h^2} + \frac{8x^4}{h^3} - \frac{3x^5}{h^4}$$

$$N_3 = H^2_{21}(x) := \frac{x^2}{2} - \frac{3x^3}{2h} + \frac{3x^4}{2h^2} - \frac{x^5}{2h^3}$$

$$N_4 = H^2_{02}(x) := \frac{10x^3}{h^3} - \frac{15x^4}{h^4} + \frac{6x^5}{h^5}$$

$$N_5 = H_{12}^2(x) := -\frac{4x^3\theta_2}{h^2} + \frac{7x^4}{h^3} - \frac{3x^5}{h^4}$$

$$N_6 = H_{22}^2(x) := \frac{x^3}{2h} - \frac{x^4}{h^2} + \frac{x^5}{2h^3}$$

$\partial_x(N_1)$

$\partial_x(N_2)$

$\partial_x(N_3)$

$\partial_x(N_4)$

$\partial_x(N_5)$

$\partial_x(N_6)$

Output

$$-\frac{30x^2}{h^3} + \frac{60x^3}{h^4} - \frac{30x^4}{h^5}$$

$$1 - \frac{18x^2}{h^2} + \frac{32x^3}{h^3} - \frac{15x^4}{h^4}$$

$$x - \frac{9x^2}{2h} + \frac{6x^3}{h^2} - \frac{5x^4}{2h^3}$$

$$\frac{30x^2}{h^3} - \frac{60x^3}{h^4} + \frac{30x^4}{h^5}$$

$$-\frac{12x^2}{h^2} + \frac{28x^3}{h^3} - \frac{15x^4}{h^4}$$

$$\frac{3x^2}{2h} - \frac{4x^3}{h^2} + \frac{5x^4}{2h^3}$$

$$\therefore \partial_x(N_1) = -\frac{30x^2}{h^3} + \frac{60x^3}{h^4} - \frac{30x^4}{h^5} \qquad (18)$$

$$\therefore \partial_x(N_2) = 1 - \frac{18x^2}{h^2} + \frac{32x^3}{h^3} - \frac{15x^4}{h^4} \qquad (19)$$

$$\therefore \partial_x(N_3) = x - \frac{9x^2}{2h} + \frac{6x^3}{h^2} - \frac{5x^4}{2h^3} \qquad (20)$$

$$\therefore \partial_x(N_4) = \frac{30x^2}{h^3} - \frac{60x^3}{h^4} + \frac{30x^4}{h^5} \qquad (21)$$

$$\therefore \partial_x(N_5) = -\frac{12x^2}{h^2} + \frac{28x^3}{h^3} - \frac{15x^4}{h^4} \qquad (22)$$

$$\therefore \partial_x(N_6) = \frac{3x^2}{2h} - \frac{4x^3}{h^2} + \frac{5x^4}{2h^3} \qquad (23)$$

Partial first derivative condition at node 1, x = 0

Input

$$\frac{\partial N_1}{\partial x} := -\frac{30x^2}{h^3} + \frac{60x^3}{h^4} - \frac{30x^4}{h^5}$$

$$\frac{\partial N_2}{\partial x} := 1 - \frac{18x^2}{h^2} + \frac{32x^3}{h^3} - \frac{15x^4}{h^4}$$

$$\frac{\partial N_3}{\partial x} := x - \frac{9x^2}{2h} + \frac{6x^3}{h^2} - \frac{5x^4}{2h^3}$$

$$\frac{\partial N_4}{\partial x} := \frac{30x^2}{h^3} - \frac{60x^3}{h^4} + \frac{30x^4}{h^5}$$

$$\frac{\partial N_5}{\partial x} := -\frac{12x^2}{h^2} + \frac{28x^3}{h^3} - \frac{15x^4}{h^4}$$

$$\frac{\partial N_6}{\partial x} := \frac{3x^2}{2h} - \frac{4x^3}{h^2} + \frac{5x^4}{2h^3}$$

x = 0

$$\frac{\partial N_1}{\partial x}$$

$$\frac{\partial N_2}{\partial x}$$

$$\frac{\partial N_3}{\partial x}$$

$$\frac{\partial N_4}{\partial x}$$

$$\frac{\partial N_5}{\partial x}$$

$$\frac{\partial N_6}{\partial x}$$

Output

0

1

0

0

0

0

Finding second derivatives for (12), (13),(14) , (15),(16), and (17)

$$N_1 = H_{01}^2(x) := 1 - \frac{10x^3}{h^3} + \frac{15x^4}{h^4} - \frac{6x^5}{h^5}$$

$$N_2 = H_{11}^2(x) := x - \frac{6x^3}{h^2} + \frac{8x^4}{h^3} - \frac{3x^5}{h^4}$$

$$N_3 = H_{21}^2(x) := \frac{x^2}{2} - \frac{3x^3}{2h} + \frac{3x^4}{2h^2} - \frac{x^5}{2h^3}$$

$$N_4 = H_{02}^2(x) := \frac{10x^3}{h^3} - \frac{15x^4}{h^4} + \frac{6x^5}{h^5}$$

$$N_5 = H_{12}^2(x) := -\frac{4x^3\theta_2}{h^2} + \frac{7x^4}{h^3} - \frac{3x^5}{h^4}$$

$$N_6 = H_{22}^2(x) := \frac{x^3}{2h} - \frac{x^4}{h^2} + \frac{x^5}{2h^3}$$

$$\partial_{x,x}\left(N_1\right)$$

$$\partial_{x,x}\left(N_2\right)$$

$$\partial_{x,x}\left(N_3\right)$$

$$\partial_{x,x}\left(N_4\right)$$

$$\partial_{x,x}\left(N_5\right)$$

$$\partial_{x,x}\left(N_6\right)$$

Output

$$-\frac{60x}{h^3} + \frac{180x^2}{h^4} - \frac{120x^3}{h^5}$$

$$-\frac{36x}{h^2} + \frac{96x^2}{h^3} - \frac{60x^3}{h^4}$$

$$1 - \frac{9x}{h} + \frac{18x^2}{h^2} - \frac{10x^3}{h^3}$$

$$\frac{60x}{h^3} - \frac{180x^2}{h^4} + \frac{120x^3}{h^5}$$

$$-\frac{24x}{h^2} + \frac{84x^2}{h^3} - \frac{60x^3}{h^4}$$

$$\frac{3x}{h} - \frac{12x^2}{h^2} + \frac{10x^3}{h^3}$$

$$\therefore \partial_{x,x}\left(N_1\right) = -\frac{60x}{h^3} + \frac{180x^2}{h^4} - \frac{120x^3}{h^5}$$

$$\therefore \partial_{x,x}\left(N_2\right) = -\frac{36x}{h^2} + \frac{96x^2}{h^3} - \frac{60x^3}{h^4}$$

$$\therefore \partial_{x,x}\left(N_3\right) = 1 - \frac{9x}{h} + \frac{18x^2}{h^2} - \frac{10x^3}{h^3}$$

$$\therefore \partial_{x,x}\left(N_4\right) = \frac{60x}{h^3} - \frac{180x^2}{h^4} + \frac{120x^3}{h^5}$$

$$\therefore \partial_{x,x}\left(N_5\right) = -\frac{24x}{h^2} + \frac{84x^2}{h^3} - \frac{60x^3}{h^4}$$

$$\therefore \partial_{x,x}(N_6) = \frac{3x}{h} - \frac{12x^2}{h^2} + \frac{10x^3}{h^3}$$

Partial second derivative condition at node 1, x = 0

$$\frac{\partial^2 N_1}{\partial x^2} := -\frac{60x}{h^3} + \frac{180x^2}{h^4} - \frac{120x^3}{h^5}$$

$$\frac{\partial^2 N_2}{\partial x^2} := -\frac{36x}{h^2} + \frac{96x^2}{h^3} - \frac{60x^3}{h^4}$$

$$\frac{\partial^2 N_3}{\partial x^2} := 1 - \frac{9x}{h} + \frac{18x^2}{h^2} - \frac{10x^3}{h^3}$$

$$\frac{\partial^2 N_4}{\partial x^2} := \frac{60x}{h^3} - \frac{180x^2}{h^4} + \frac{120x^3}{h^5}$$

$$\frac{\partial^2 N_5}{\partial x^2} := -\frac{24x}{h^2} + \frac{84x^2}{h^3} - \frac{60x^3}{h^4}$$

$$\frac{\partial^2 N_6}{\partial x^2} := \frac{3x}{h} - \frac{12x^2}{h^2} + \frac{10x^3}{h^3}$$

$x := 0$

$$\frac{\partial^2 N_1}{\partial x^2}$$

$$\frac{\partial^2 N_2}{\partial x^2}$$

$$\frac{\partial^2 N_3}{\partial x^2}$$

$$\frac{\partial^2 N_4}{\partial x^2}$$

$$\frac{\partial^2 N_5}{\partial x^2}$$

$$\frac{\partial^2 N_6}{\partial x^2}$$

Output

0

0

1

0

0

0

(II) 2nd VERIFICATION CONDITION

Second Verification condition at node 2 is

(iv) $N_4 = 1$ and $N_1 = 0,\ N_2 = 0, N_3 = 0, N_5 = 0, N_6 = 0$

(v) $\frac{\partial N_5}{\partial x} = 1$ and $\frac{\partial N_1}{\partial x} = 0, \frac{\partial N_2}{\partial x} = 0, \frac{\partial N_3}{\partial x} = 0, \frac{\partial N_4}{\partial x} = 0, \frac{\partial N_6}{\partial x} = 0$

(vi) $\frac{\partial^2 N_6}{\partial x^2} = 1$ and $\frac{\partial^2 N_1}{\partial x^2} = 0, \frac{\partial^2 N_2}{\partial x^2} = 0, \frac{\partial^2 N_3}{\partial x^2} = 0, \frac{\partial^2 N_4}{\partial x^2} = 0, \frac{\partial^2 N_5}{\partial x^2} = 0$

At Node 2, x = h

$$N_1 = H_{01}^2(x) := 1 - \frac{10x^3}{h^3} + \frac{15x^4}{h^4} - \frac{6x^5}{h^5}$$

$$N_2 = H_{11}^2(x) := x - \frac{6x^3}{h^2} + \frac{8x^4}{h^3} - \frac{3x^5}{h^4}$$

$$N_3 = H_{21}^2(x) := \frac{x^2}{2} - \frac{3x^3}{2h} + \frac{3x^4}{2h^2} - \frac{x^5}{2h^3}$$

$$N_4 = H_{02}^2(x) := \frac{10x^3}{h^3} - \frac{15x^4}{h^4} + \frac{6x^5}{h^5}$$

$$N_5 = H_{12}^2(x) := -\frac{4x^3\theta_2}{h^2} + \frac{7x^4}{h^3} - \frac{3x^5}{h^4}$$

$$N_6 = H_{22}^2(x) := \frac{x^3}{2h} - \frac{x^4}{h^2} + \frac{x^5}{2h^3}$$

$x := h$

N_1

N_2

N_3

N_4

N_5

N_6

Output

0

0

0

1

0

0

Partial first derivative condition at node 2, x = h

$$\frac{\partial N_1}{\partial x} := -\frac{30x^2}{h^3} + \frac{60x^3}{h^4} - \frac{30x^4}{h^5}$$

$$\frac{\partial N_2}{\partial x} := 1 - \frac{18x^2}{h^2} + \frac{32x^3}{h^3} - \frac{15x^4}{h^4}$$

$$\frac{\partial N_3}{\partial x} := x - \frac{9x^2}{2h} + \frac{6x^3}{h^2} - \frac{5x^4}{2h^3}$$

$$\frac{\partial N_4}{\partial x} := \frac{30x^2}{h^3} - \frac{60x^3}{h^4} + \frac{30x^4}{h^5}$$

$$\frac{\partial N_5}{\partial x} := -\frac{12x^2}{h^2} + \frac{28x^3}{h^3} - \frac{15x^4}{h^4}$$

$$\frac{\partial N_6}{\partial x} := \frac{3x^2}{2h} - \frac{4x^3}{h^2} + \frac{5x^4}{2h^3}$$

x = h

$\frac{\partial N_1}{\partial x}$

$\frac{\partial N_2}{\partial x}$

$\frac{\partial N_3}{\partial x}$

$\frac{\partial N_4}{\partial x}$

$\frac{\partial N_5}{\partial x}$

$\frac{\partial N_6}{\partial x}$

Output

0

0

0

0

1

0

Partial second derivative condition at node 2, x = h

$$\frac{\partial^2 N_1}{\partial x^2} := -\frac{60x}{h^3} + \frac{180x^2}{h^4} - \frac{120x^3}{h^5}$$

$$\frac{\partial^2 N_2}{\partial x^2} := -\frac{36x}{h^2} + \frac{96x^2}{h^3} - \frac{60x^3}{h^4}$$

$$\frac{\partial^2 N_3}{\partial x^2} := 1 - \frac{9x}{h} + \frac{18x^2}{h^2} - \frac{10x^3}{h^3}$$

$$\frac{\partial^2 N_4}{\partial x^2} := \frac{60x}{h^3} - \frac{180x^2}{h^4} + \frac{120x^3}{h^5}$$

$$\frac{\partial^2 N_5}{\partial x^2} := -\frac{24x}{h^2} + \frac{84x^2}{h^3} - \frac{60x^3}{h^4}$$

$$\frac{\partial^2 N_6}{\partial x^2} := \frac{3x}{h} - \frac{12x^2}{h^2} + \frac{10x^3}{h^3}$$

$x := h$

$$\frac{\partial^2 N_1}{\partial x^2}$$

$$\frac{\partial^2 N_2}{\partial x^2}$$

$$\frac{\partial^2 N_3}{\partial x^2}$$

$$\frac{\partial^2 N_4}{\partial x^2}$$

$$\frac{\partial^2 N_5}{\partial x^2}$$

$$\frac{\partial^2 N_6}{\partial x^2}$$

Output

0

0

0

0

0

1

Third Verification Condition

3^{rd} verification condition is $N_1 + N_4 = 1$

$$N_1 = H_{01}^2(x) := 1 - \frac{10x^3}{h^3} + \frac{15x^4}{h^4} - \frac{6x^5}{h^5}$$

$$N_4 = H_{02}^2(x) := \frac{10x^3}{h^3} - \frac{15x^4}{h^4} + \frac{6x^5}{h^5}$$

$FullSimplify[N_1 + N_4]$

Output

1

V. CONCLUSIONS

1. Derived Shape Functions for a two node element with three primary variables at each node.
2. Verified First verification condition at node 1

   (*i*) $N_1 = 1$ and $N_2 = 0,\ N_3 = 0, N_4 = 0, N_5 = 0, N_6 = 0$

   (*ii*) $\frac{\partial N_2}{\partial x} = 1$ and $\frac{\partial N_1}{\partial x} = 0, \frac{\partial N_3}{\partial x} = 0, \frac{\partial N_4}{\partial x} = 0, \frac{\partial N_5}{\partial x} = 0, \frac{\partial N_6}{\partial x} = 0$

   (*iii*) $\frac{\partial^2 N_3}{\partial x^2} = 1$ and $\frac{\partial^2 N_1}{\partial x^2} = 0, \frac{\partial^2 N_2}{\partial x^2} = 0, \frac{\partial^2 N_4}{\partial x^2} = 0, \frac{\partial^2 N_5}{\partial x^2} = 0, \frac{\partial^2 N_6}{\partial x^2} = 0$

3. Verified Second Verification condition at node 2

$$(iv)\ N_4 = 1 \text{ and } N_1 = 0,\ N_2 = 0, N_3 = 0, N_5 = 0, N_6 = 0$$

$$(v)\ \frac{\partial N_5}{\partial x} = 1 \text{ and } \frac{\partial N_1}{\partial x} = 0, \frac{\partial N_2}{\partial x} = 0, \frac{\partial N_3}{\partial x} = 0, \frac{\partial N_4}{\partial x} = 0, \frac{\partial N_6}{\partial x} = 0$$

$$(vi)\ \frac{\partial^2 N_6}{\partial x^2} = 1 \text{ and } \frac{\partial^2 N_1}{\partial x^2} = 0, \frac{\partial^2 N_2}{\partial x^2} = 0, \frac{\partial^2 N_3}{\partial x^2} = 0, \frac{\partial^2 N_4}{\partial x^2} = 0, \frac{\partial^2 N_5}{\partial x^2} = 0$$

4. Verified Third verification condition

$$\mathrm{N}_1 + N_4 = 1.$$

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