Deriving Shape Functions for 2,3,4,5 Noded Line Element by Polynomial and Verified

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Abstract* — In this paper, I derived shape functions for 2,3,4,5 noded line element by polynomial functions, by taking natural coordinate system and also I verified two verification conditions for shape functions. First verification condition is sum of all the shape functions is equal to one and second verification condition is each shape function has a value of one at its own node and zero at the other nodes. For computational purpose I used Mathematica 9 Software[2].

Keywords — *Line element, Polynomial functions, Shape functions.*

I. INTRODUCTION

Initially shape functions were derived interms of Cartesian coordinates. Polynomial functions were used for this. After natural coordinates were identified and its advantage was noticed researchers started deriving shape functions interms of natural coordinates. By this approach more elements could be developed [1]. By increasing more number of nodes we can minimize error in Finite Element Method [3]. In this computation one of the important observation is one dimensional polynomial changes to Lagrange interpolation formula.

II. GEOMETRICAL DESCREPTION

Line element with 2,3,4,5 nodes is shown in figures.1, 2, 3, 4.

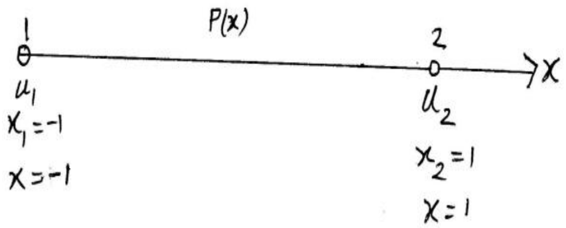

Figure.1: 2 noded line element

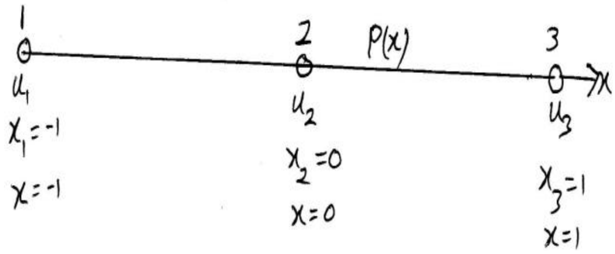

Figure.2: 3 noded line element

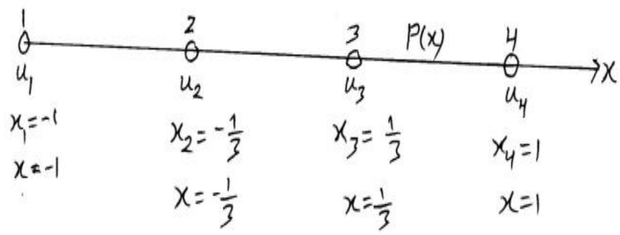

Figure.3: 4 noded line element

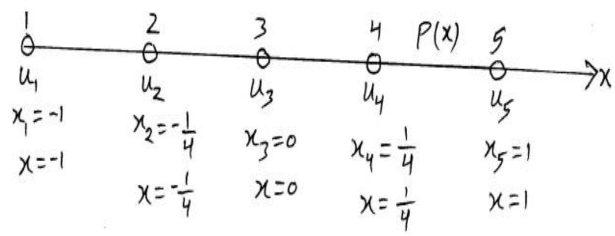

Figure.4: 5 noded line element

III. DERIVING SHAPE FUNCTIONS FOR 2,3,4,5 NODED LINE ELEMENT

(i) The typical 2 noded element is shown in Figure.1.

Shape function for node 1 is N_1 and for node 2 is N_2

The typical bar element in the natural coordinate x varying from -1 to 1 is shown in Figure.1.In figure.1 nodal unknowns are displacements u_1 and u_2 along x-axis. For this element we have to select polynomial with only two constants to represent displacement at any point in the elements.Since there are only two nodal values, a linear polynomial is to be selected. Let displacement at any point P(x) be

$$u = \alpha_1 + \alpha_2 x$$

$$u = \begin{bmatrix} 1 & x \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

$$A := \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix}$$

$b :=$ *Inverse*$[A]$ // *Matrix*Form

B

Output

$$\begin{pmatrix} \frac{x_2}{-x_1 + x_2} & -\frac{x_1}{-x_1 + x_2} \\ -\frac{1}{-x_1 + x_2} & \frac{1}{-x_1 + x_2} \end{pmatrix}$$

$$d := \begin{pmatrix} \frac{x_2}{-x_1 + x_2} & -\frac{x_1}{-x_1 + x_2} \\ -\frac{1}{-x_1 + x_2} & \frac{1}{-x_1 + x_2} \end{pmatrix}$$

$$e := \begin{pmatrix} 1 & x \end{pmatrix}$$

$$f := e.d$$

FullSimplify$[f]$ // *MatrixForm*

$$\begin{pmatrix} \frac{x - x_2}{x_1 - x_2} & \frac{-x + x_1}{x_1 - x_2} \end{pmatrix}$$

$$x_1 := -1 \qquad x_2 := 1$$

$$a := \begin{pmatrix} \frac{x - x_2}{x_1 - x_2} & \frac{-x + x_1}{x_1 - x_2} \end{pmatrix}$$

a

Output

$$\left\{\left\{\frac{1-x}{2}, \frac{1+x}{2}\right\}\right\}$$

$\therefore$ *For* 2 noded element shape functions are $N_1 = \frac{1-x}{2}$

and $N_2 = \frac{1+x}{2}$

Verification

I^{st} Condition : Sum of all the shape functions is equal to one

$$N_1 := \frac{1-x}{2}, \quad N_2 := \frac{1+x}{2}$$

FullSimplify$[N_1 + N_2]$

Output

1

II^{nd} Condition: Each shape function has a value of one at its own node and zero at the other nodes.

At Node 1 x = -1 then we get $N_1 = 1, \ N_2 = 0$

At Node 2 x = 1 then we get $N_1 = 0, \ N_2 = 1$

(ii) The typical 3 noded element is shown in Figure.2. Shape function for node 1 is N_1 , for node 2 is N_2 and for node 3 is N_3

The typical bar element in the natural coordinate x varying from -1 to 1 is shown in Figure.2. Since there are only three nodal values, a quadratic polynomial is to be selected. Let displacement at any point P(x) be

$$u=\alpha_1+\alpha_2 x+\alpha_3 x^2$$

$$u=\begin{pmatrix}1 & x & x^2\end{pmatrix}\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{Bmatrix}$$

$$\begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix}=\begin{pmatrix}1 & x_1 & (x_1)^2\\ 1 & x_2 & (x_2)^2\\ 1 & x_3 & (x_3)^2\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}$$

$$A:=\begin{pmatrix}1 & x_1 & (x_1)^2\\ 1 & x_2 & (x_2)^2\\ 1 & x_3 & (x_3)^2\end{pmatrix}$$

$b := Inverse[A];$

$e := \begin{pmatrix}1 & x & x^2\end{pmatrix};$

$f := e.b;$

$FullSimplify[f]//MatrixForm$

Output

$$\left(\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} \quad -\frac{(x-x_1)(x-x_3)}{(x_1-x_2)(x_2-x_3)} \quad -\frac{(x-x_1)(x-x_2)}{(x_1-x_3)(-x_2+x_3)}\right)$$

$$x_1:=-1 \qquad x_2:=0 \qquad x_3:=1$$

$$a:=\left(\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} \quad -\frac{(x-x_1)(x-x_3)}{(x_1-x_2)(x_2-x_3)} \quad -\frac{(x-x_1)(x-x_2)}{(x_1-x_3)(-x_2+x_3)}\right)$$

a

Output

$$\left\{\left\{\frac{1}{2}(-1+x)\,x, -(-1+x)(1+x), \frac{1}{2}x(1+x)\right\}\right\}$$

$\therefore$ *For* 3 noded element shape functions are

$$N_1 := \frac{1}{2}(-1+x)\,x \quad N_2 := -(-1+x)(1+x)$$

$$N_3 := \frac{1}{2}x(1+x)$$

Verification

I^{st} Condition : Sum of all the shape functions is equal to one

$FullSimplify[N_1+N_2+N_3]$

Output

1

II^{nd} Condition: Each shape function has a value of one at its own node and zero at the other nodes.

At Node 1 x = -1 then we get $N_1=1$, $N_2=0, N_3=0$

At Node 2 x = 0 then we get $N_1=0$, $N_2=1, N_3=0$

At Node 3 x = 1 then we get $N_1=0$, $N_2=0$, $N_3=1$

(iii) The typical 4 noded element is shown in Figure.3.

The typical bar element in the natural coordinate x varying from -1 to 1 is shown in Figure.3.Since there are only four nodal values, Hence a polynomial with 4 generalized coordinates is to be selected. Let displacement at any point P(x) be

$$u=\alpha_1+\alpha_2 x+\alpha_3 x^2+\alpha_4 x^3$$

$$u=\begin{pmatrix}1 & x & x^2 & x^3\end{pmatrix}\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{Bmatrix}$$

$$\begin{pmatrix}u_1\\ u_2\\ u_3\\ u_4\end{pmatrix}=\begin{pmatrix}1 & x_1 & (x_1)^2 & (x_1)^3\\ 1 & x_2 & (x_2)^2 & (x_2)^3\\ 1 & x_3 & (x_3)^2 & (x_3)^3\\ 1 & x_4 & (x_4)^2 & (x_4)^3\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{pmatrix}$$

$$A:=\begin{pmatrix}1 & x_1 & (x_1)^2 & (x_1)^3\\ 1 & x_2 & (x_2)^2 & (x_2)^3\\ 1 & x_3 & (x_3)^2 & (x_3)^3\\ 1 & x_4 & (x_4)^2 & (x_4)^3\end{pmatrix}$$

$b := Inverse[A];$

$e := \begin{pmatrix}1 & x & x^2 & x^3\end{pmatrix};$

$f := e.b;$

FullSimplify[f] // MatrixForm

Output

$$\left(\frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}\right.$$
$$-\frac{(x-x_1)(x-x_3)(x-x_4)}{(x_1-x_2)(x_2-x_3)(x_2-x_4)}$$
$$-\frac{(x-x_1)(x-x_2)(x-x_4)}{(x_1-x_3)(-x_2+x_3)(x_3-x_4)}$$
$$\left.-\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_1-x_4)(-x_2+x_4)(-x_3+x_4)}\right)$$

$$x_1:=-1 \quad x_2:=-\frac{1}{3} \quad x_3:=\frac{1}{3} \quad x_4:=1$$

$$a:=\left(\frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}\right.$$
$$-\frac{(x-x_1)(x-x_3)(x-x_4)}{(x_1-x_2)(x_2-x_3)(x_2-x_4)}-$$
$$\frac{(x-x_1)(x-x_2)(x-x_4)}{(x_1-x_3)(-x_2+x_3)(x_3-x_4)}$$
$$\left.-\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_1-x_4)(-x_2+x_4)(-x_3+x_4)}\right)$$

a

Output

$$\left\{\left\{-\frac{9}{16}(-1+x)\left(-\frac{1}{3}+x\right)\left(\frac{1}{3}+x\right),\right.\right.$$
$$\frac{27}{16}(-1+x)\left(-\frac{1}{3}+x\right)(1+x),$$
$$-\frac{27}{16}(-1+x)\left(\frac{1}{3}+x\right)(1+x),$$
$$\left.\left.\frac{9}{16}\left(-\frac{1}{3}+x\right)\left(\frac{1}{3}+x\right)(1+x)\right\}\right\}$$

$\therefore$ *For* 4 noded element shape functions are

$$N_1:=-\frac{9}{16}(-1+x)\left(-\frac{1}{3}+x\right)\left(\frac{1}{3}+x\right)$$

$$N_2:=\frac{27}{16}(-1+x)\left(-\frac{1}{3}+x\right)(1+x)$$

$$N_3:=-\frac{27}{16}(-1+x)\left(\frac{1}{3}+x\right)(1+x)$$

$$N_4:=\frac{9}{16}\left(-\frac{1}{3}+x\right)\left(\frac{1}{3}+x\right)(1+x)$$

Verification

I^{st} Condition : Sum of all the shape functions is equal to one

FullSimplify$[N_1 + N_2 + N_3 + N_4]$

Output

1

II^{nd} Condition: Each shape function has a value of one at its own node and zero at the other nodes.

At Node 1 x = -1 then we get $N_1 = 1$, $N_2 = 0, N_3 = 0, N_4 = 0$

At Node 2 x = $-\frac{1}{3}$ then we get $N_1 = 0$, $N_2 = 1, N_3 = 0, N_4 = 0$

At Node 3, x = $\frac{1}{3}$ then we get $N_1 = 0$, $N_2 = 0, N_3 = 1, N_4 = 0$

At Node 4, x = 1 then we get $N_1 = 0$, $N_2 = 0, N_3 = 0, N_4 = 1$

(iv) The typical 5 noded element is shown in Figure.4.

The typical bar element in the natural coordinate x varying from -1 to 1 is shown in Figure.3. Since there are only four nodal values, Hence a polynomial with 4 generalized coordinates is to be selected. Let displacement at any point P(x) be

$$u = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 + \alpha_5 x^4$$

$$u = \begin{pmatrix} 1 & x & x^2 & x^3 & x^4 \end{pmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{Bmatrix}$$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & (x_1)^2 & (x_1)^3 & (x_1)^4 \\ 1 & x_2 & (x_2)^2 & (x_2)^3 & (x_2)^4 \\ 1 & x_3 & (x_3)^2 & (x_3)^3 & (x_3)^4 \\ 1 & x_4 & (x_4)^2 & (x_4)^3 & (x_4)^4 \\ 1 & x_5 & (x_5)^2 & (x_5)^3 & (x_5)^4 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix}$$

$$A := \begin{pmatrix} 1 & x_1 & (x_1)^2 & (x_1)^3 & (x_1)^4 \\ 1 & x_2 & (x_2)^2 & (x_2)^3 & (x_2)^4 \\ 1 & x_3 & (x_3)^2 & (x_3)^3 & (x_3)^4 \\ 1 & x_4 & (x_4)^2 & (x_4)^3 & (x_4)^4 \\ 1 & x_5 & (x_5)^2 & (x_5)^3 & (x_5)^4 \end{pmatrix}$$

$b := Inverse[A];$

$e := \begin{pmatrix} 1 & x & x^2 & x^3 & x^4 \end{pmatrix};$

$f := e.b;$

FullSimplify$[f]$// *MatrixForm*

Output

$$\left(\frac{(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)} \quad -\frac{(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_2)(x_2-x_3)(x_2-x_4)(x_2-x_5)} \quad -\frac{(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_1-x_3)(-x_2+x_3)(x_3-x_4)(x_3-x_5)} \quad -\frac{(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_1-x_4)(-x_2+x_4)(-x_3+x_4)(x_4-x_5)} \quad -\frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_5)(-x_2+x_5)(-x_3+x_5)(-x_4+x_5)} \right)$$

$x_1 := -1 \quad x_2 := -\frac{1}{4} \quad x_3 := 0 \quad x_4 := \frac{1}{4} \quad x_5 := 1$

$$a := \left(\frac{(x-x_2)(x-x_3)(x-x_4)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)} - \frac{(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_2)(x_2-x_3)(x_2-x_4)(x_2-x_5)} - \frac{(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_1-x_3)(-x_2+x_3)(x_3-x_4)(x_3-x_5)} - \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_1-x_4)(-x_2+x_4)(-x_3+x_4)(x_4-x_5)} - \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_5)(-x_2+x_5)(-x_3+x_5)(-x_4+x_5)} \right)$$

a

Output

$$\left\{\left\{\frac{8}{15}(-1+x)\left(-\frac{1}{4}+x\right)x\left(\frac{1}{4}+x\right), -\frac{128}{15}(-1+x)\left(-\frac{1}{4}+x\right)x(1+x), 16(-1+x)\left(-\frac{1}{4}+x\right)\left(\frac{1}{4}+x\right)(1+x) -\frac{128}{15}(-1+x)x\left(\frac{1}{4}+x\right)(1+x), \frac{8}{15}\left(-\frac{1}{4}+x\right)x\left(\frac{1}{4}+x\right)(1+x)\right\}\right\}$$

$\therefore$ *For* 5 noded element shape functions are

$$N_1 := \frac{8}{15}(-1+x)\left(-\frac{1}{4}+x\right)x\left(\frac{1}{4}+x\right)$$

$$N_2 := -\frac{128}{15}(-1+x)\left(-\frac{1}{4}+x\right)x(1+x)$$

$$N_3 := 16(-1+x)\left(-\frac{1}{4}+x\right)\left(\frac{1}{4}+x\right)(1+x)$$

$$N_4 := -\frac{128}{15}(-1+x)\left(\frac{1}{4}+x\right)(1+x)$$

$$N_5 := \frac{8}{15}\left(-\frac{1}{4}+x\right)x\left(\frac{1}{4}+x\right)(1+x)$$

Verification

I^{st} Condition : Sum of all the shape functions is equal to one

FullSimplify$[N_1 + N_2 + N_3 + N_4 + N_5]$

Output

1

II^{nd} Condition: Each shape function has a value of one at its own node and zero at the other nodes.

At Node 1 x = -1 then we get $N_1 = 1$, $N_2 = 0, N_3 = 0, N_4 = 0, N_5 = 0$

At Node 2 x = $-\frac{1}{4}$ then we get $N_1 = 0$, $N_2 = 1, N_3 = 0, N_4 = 0, N_5 = 0$

At Node 3, x = 0 then we get $N_1 = 0$, $N_2 = 0, N_3 = 1, N_4 = 0, N_5 = 0$

At Node 4, x = $\frac{1}{4}$ then we get $N_1 = 0$, $N_2 = 0, N_3 = 0, N_4 = 1, N_5 = 0$

At Node 5, x = 1 then we get $N_1 = 0$, $N_2 = 0,\ N_3 = 0, N_4 = 0, N_5 = 1$

V. CONCLUSIONS

1. Derived Shape functions for 2,3,4,5 noded line element by polynomial.
2. Verified sum of all the shape functions is equal to one.
3. Verified each shape function has a value of one at its own node and zero at the other nodes.

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