Deriving Shape Functions and Verified for One Dimensional Hermite Polynomials by Taking Natural Coordinate System 0 To 1

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***Abstract* —** In *this paper, I derived shape functions for one dimensional Hermite Polynomials by taking natural coordinate system 0 to 1 and also I verified three verification conditions for shape functions. First verification condition is at node 1 is* $N_1 = 1$ and $N_2 = 0,\ N_3 = 0, N_4 = 0$ *and also* $\frac{\partial N_2}{\partial x} = 1$ and $\frac{\partial N_1}{\partial x} = 0,\ \frac{\partial N_3}{\partial x} = 0,\ \frac{\partial N_4}{\partial x} = 0$, *Second Verification condition is at node 2* $N_3 = 1$ and $N_1 = 0,\ N_2 = 0, N_4 = 0$ *and also* $\frac{\partial N_4}{\partial x} = 1$ and $\frac{\partial N_1}{\partial x} = 0,\ \frac{\partial N_2}{\partial x} = 0,\ \frac{\partial N_3}{\partial x} = 0$. *Third Verification condition is* $N_1 + N_3 = 1$. *For computational purpose I used Mathematica 9 Software [2].*

Keywords — *Hermite Polynomials, Natural Coordinate System, Shape functions*.

I. INTRODUCTION

In Finite Element Analysis any domain of geometry can be split into finite number of domains. Each domain has a particular shape of geometry for example like Rectangular shape, Triangle shape, Circular shape. To study the analysis of these geometries first we need shape functions.

II. GEOMETRICAL DESCRIPTION

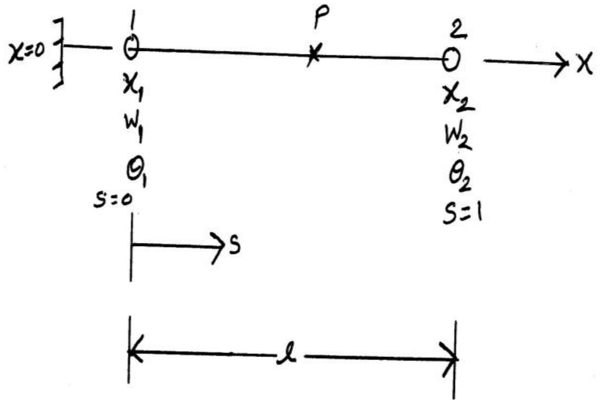

*Figure.*1 Beam element with natural coordinates varying from 0 to 1

A two noded beam element shown in Figure.1 in which nodal unknowns are the displacement W and Slope $\frac{\partial W}{\partial x}$.

III. DERIVING SHAPE FUNCTIONS FOR ONE DIMENSIONAL HERMITE POLYNOMIALS

Since the element in figure.1 has four degrees of freedom,We have to select the polynomial with only 4 constants. In this polynomial after boundary conditions we get shape functions this we can take as first order (cubic) Hermitian Polynomials as shape functions.

$$W(x) = A_1 + A_2 x + A_3 x^2 + A_4 x^3 \qquad (1)$$

Where W is the transverse displacement and A_1, A_2, A_3, A_4 are polynomial Coefficients

Differentiating eq(1) w.r.t. 'x'

$$(1) \Rightarrow \frac{\partial W}{\partial x} = 0 + A_2(1) + A_3(2x) + A_4(3x^2)$$

$$\frac{\partial W}{\partial x} = A_2 + 2A_3x + 3A_4x^2 \qquad (2)$$

Applying the nodal conditions such that

$W=W_1$ and $\frac{\partial W}{\partial x} = \theta_1$ at x= 0

and $W=W_2$ and $\frac{\partial W}{\partial x} = \theta_2$ at $x=l$

in equations (1) and (2) , we get

When $W=W_1$ and $x = 0$

$$(1) \Rightarrow W_1 = A_1 + A_2(0) + A_3(0)^2 + A_4(0)^3$$

$$W_1 = A_1 + 0 + 0 + 0$$

$$W_1 = A_1 \qquad (3)$$

When $\frac{\partial W}{\partial x} = \theta_1$ and $x = 0$

$$(2) \Rightarrow \theta_1 = A_2 + 2A_3(0) + 3A_4(0)^2$$

$$\theta_1 = A_2 + 0 + 0$$

$$\theta_1 = A_2 \qquad (4)$$

When $W=W_2$ and $x = l$

$$(1) \Rightarrow W_2 = A_1 + A_2(l) + A_3l^2 + A_4l^3 \qquad (5)$$

When $\frac{\partial W}{\partial x} = \theta_2$ and $x = l$

$$(2) \Rightarrow \theta_2 = A_2 + 2A_3l + 3A_4l^2 \qquad (6)$$

Using Mathematica 9 Software Solving (3),(4),(5) and (6) we get A_1, A_2, A_3, A_4

Input

$Solve[A_1 - W_1 == 0 \,\&\&\, A_2 - \theta_1 == 0$
$\&\& A_1 + (A_2 * l) + (A_3 * l^2) + (A_4 * l^3) - W_2 == 0$
$\&\& A_2 + (2 * A_3 * l) + (3 * A_4 * l^2) - \theta_2 == 0,$
$\{A_1, A_2, A_3, A_4\}]$

Output

$\{\{A_1 -> W_1, A_2 -> \theta_1,$

$$A_3 -> -\frac{3W_1 - 3W_2 + 2l\theta_1 + l\theta_2}{l^2},$$

$$A_4 -> -\frac{-2W_1 + 2W_2 - l\theta_1 - l\theta_2}{l^3}\}\}$$

Substituting A_1, A_2, A_3, A_4 in eq(1)

$$A_1 := W_1$$

$$A_2 := \theta_1$$

$$A_3 := -\frac{3W_1 - 3W_2 + 2l\theta_1 + l\theta_2}{l^2}$$

$$A_4 := -\frac{-2W_1 + 2W_2 - l\theta_1 - l\theta_2}{l^3}$$

$$W(\xi) := A_1 + A_2 * x + A_3 * x^2 + A_4 * x^3$$

Expand$[W(x)]$

Output

$$W_1 - \frac{3x^2W_1}{l^2} + \frac{2x^3W_1}{l^3} + \frac{3x^2W_2}{l^2} - \frac{2x^3W_2}{l^3} + x\theta_1 - \frac{2x^2\theta_1}{l} + \frac{x^3\theta_1}{l^2} - \frac{x^2\theta_2}{l} + \frac{x^3\theta_2}{l^2}$$

$$W(x)=W_1\left(1-\frac{3x^2}{l^2}+\frac{2x^3}{l^3}\right)$$
$$+\theta_1\left(x-\frac{2x^2}{l}+\frac{x^3}{l^2}\right)$$
$$+W_2\left(\frac{3x^2}{l^2}-\frac{2x^3}{l^3}\right)+\theta_2\left(-\frac{x^2}{l}+\frac{x^3}{l^2}\right) \quad (7)$$

$$\left.\begin{aligned} \text{i.e., } W &= N_1W_1+N_2\theta_1+N_3W_2+N_4\theta_2 \\ &= N_1\delta_1+N_2\delta_2+N_3\delta_3+N_4\delta_4 \end{aligned}\right\} \quad (8)$$

Where N_1, N_2, N_3, N_4 are shape functions for the beam elements and $\delta_1, \delta_2, \delta_3, \delta_4$ are the nodal displacements

$$\text{i.e., } \{\delta\}=\begin{Bmatrix}\delta_1\\ \delta_2\\ \delta_3\\ \delta_4\end{Bmatrix}=\begin{Bmatrix}W_1\\ \theta_1\\ W_2\\ \theta_2\end{Bmatrix}$$

Comparing (7) and (8) we get

$$N_1=1-\frac{3x^2}{l^2}+\frac{2x^3}{l^3} \quad (9)$$

$$N_2=x-\frac{2x^2}{l}+\frac{x^3}{l^2} \quad (10)$$

$$N_3=\frac{3x^2}{l^2}-\frac{2x^3}{l^3} \quad (11)$$

$$N_4=-\frac{x^2}{l}+\frac{x^3}{l^2} \quad (12)$$

Substituting length $l = 1-0 = 1$ *and* $x = s$ *in eqs (9),(10),(11), and (12) in general we get*

$$N_1=H_{01}^1(s)=1-3s^2+2s^3 \quad (13)$$

$$N_2=H_{11}^1(s)=s-2s^2+s^3$$
$$=ls(s-1)^2$$

(Including length of beam element)

$$N_2=H_{11}^1(s)=s(s-1)^2 \quad (\because l=1) \quad (14)$$

$$N_3=H_{02}^1(s)=3s^2-2s^3$$

$$N_3=H_{02}^1(s)=s^2(3-2s) \quad (15)$$

$$N_4=H_{12}^1(s)=-s^2+s^3$$
$$=ls^2(s-1)$$

(Including length l of beam element)

$$=s^2(s-1) \quad (\because l=1) \quad (16)$$

In $H_{01}^1(s)$, 0 represents Zero[th] order derivative, 1 represents node number one and power 1 represents first order Hermitian function.

In $H_{11}^1(s)$, 1 represents first order derivative,1 represents node number one and power 1 represents first order Hermitian function.

In $H_{02}^1(s)$, 0 represents Zero[th] order derivative, 2 represents node number two and power 1 represents first order Hermitian function.

In $H_{12}^1(s)$, 1 represents first order derivative, 2 represents node number two and power 1 represents first order Hermitian function.

IV. VERIFICATION

(i). 1[st] VERIFICATION CONDITION

First verification condition at node 1 is

$N_1 = 1$ and $N_2 = 0,\ N_3 = 0, N_4 = 0$ and also

$\frac{\partial N_2}{\partial s} = 1$ and $\frac{\partial N_1}{\partial s} = 0, \frac{\partial N_3}{\partial s} = 0, \frac{\partial N_4}{\partial s} = 0$

At Node 1 s = 0

$$N_1 := 1 - 3 * s^2 + 2 * s^3 \tag{17}$$

$$N_2 := s(s-1)^2 \tag{18}$$

$$N_3 := s^2(3-2s) \tag{19}$$

$$N_4 := s^2(s-1) \tag{20}$$

$s := 0$

N_1

N_2

N_3

N_4

Output

1

0

0

0

Finding first derivatives for (17),(18), (19) and (20)

$N_1 := 1 - 3 * s^2 + 2 * s^3$

$N_2 := s(s-1)^2$

$N_3 := s^2(3-2s)$

$N_4 := s^2(s-1)$

$\partial_s(N_1)$

$\partial_s(N_2)$

$\partial_s(N_3)$

$\partial_s(N_4)$

Output

$-6s + s^2$

$(-1+s)^2 + 2(-1+s)s$

$2(3-2s)s - 2s^2$

$2(-1+s)s + s^2$

$$\therefore \partial_s(N_1) = -6s + s^2 \tag{21}$$

$$\therefore \partial_s(N_2) = (-1+s)^2 + 2(-1+s)s \tag{22}$$

$$\therefore \partial_s(N_3) = 2(3-2s)s - 2s^2 \tag{23}$$

$$\therefore \partial_s(N_4) = 2(-1+s)s + s^2 \tag{24}$$

Partial derivative condition at node 1, s = 0

$\frac{\partial N_1}{\partial s} := -6s + s^2$

$\frac{\partial N_2}{\partial s} := (-1+s)^2 + 2(-1+s)s$

$\frac{\partial N_3}{\partial s} := 2(3-2s)s - 2s^2$

$\frac{\partial N_4}{\partial s} := 2(-1+s)s + s^2$

$s = 0$

$\frac{\partial N_1}{\partial s}$

$\frac{\partial N_2}{\partial s}$

$\frac{\partial N_3}{\partial s}$

$\frac{\partial N_4}{\partial s}$

Output

0

1

0

0

(ii) 2nd VERIFICATION CONDITION

Second Verification condition is at node 2

$N_3 = 1$ and $N_1 = 0,\ N_2 = 0, N_4 = 0$ and also

$\frac{\partial N_4}{\partial x} = 1$ and $\frac{\partial N_1}{\partial x} = 0, \frac{\partial N_2}{\partial x} = 0, \frac{\partial N_3}{\partial x} = 0$.

At Node 2, s = 1

$N_1 := 1 - 3 * s^2 + 2 * s^3$

$N_2 := s\left(s-1\right)^2$

$N_3 := s^2(3-2s)$

$N_4 := s^2(s-1)$

$s := 1$

N_1

N_2

N_3

N_4

Output

0

0

1

0

Partial derivative condition at node 2, s = 1

$\frac{\partial N_1}{\partial s} := -6s + s^2$

$\frac{\partial N_2}{\partial s} := (-1+s)^2 + 2(-1+s)s$

$\frac{\partial N_3}{\partial s} := 2(3-2s)s - 2s^2$

$\frac{\partial N_4}{\partial s} := 2(-1+s)s + s^2$

$s :\ = 1$

$\frac{\partial N_1}{\partial s}$

$\frac{\partial N_2}{\partial s}$

$\frac{\partial N_3}{\partial s}$

$\frac{\partial N_4}{\partial s}$

Output

0

0

0

1

Third Verification Condition

3^{rd} verification condition is $\mathrm{N}_1 + N_3 = 1$

$N_1 := 1 - 3 * s^2 + 2 * s^3$

$N_3 := s^2(3 - 2s)$

FullSimplify$[N_1 + N_3]$

Output

1

V. CONCLUSIONS

1. Derived Shape Functions for Hermite Polynomials.
2. Verified First verification condition at node 1, $N_1 = 1$ and $N_2 = 0,\ N_3 = 0, N_4 = 0$ and also $\frac{\partial N_2}{\partial x} = 1$ and $\frac{\partial N_1}{\partial x} = 0, \frac{\partial N_3}{\partial x} = 0, \frac{\partial N_4}{\partial x} = 0$
3. Verified Second Verification condition at node 2 $N_3 = 1$ and $N_1 = 0,\ N_2 = 0, N_4 = 0$ and also $\frac{\partial N_4}{\partial x} = 1$ and $\frac{\partial N_1}{\partial x} = 0, \frac{\partial N_2}{\partial x} = 0, \frac{\partial N_3}{\partial x} = 0$.
4. Verified Third verification condition $\mathrm{N}_1 + N_3 = 1$.

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