Deriving Shape Functions and Verified for Two Dimensional Hermite Polynomials by Taking Natural Coordinate System -1 To 1

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Abstract* — *In this paper, I derived shape functions for two dimensional Hermite Polynomials by taking natural coordinate system -1 to 1 and also I verified three verification conditions for shape functions. First verification condition at node 1 is $N_1 = 1$ and $N_2 = 0,\ N_3 = 0, N_4 = 0$ *and also* $\frac{\partial N_2}{\partial x} = 1$ and $\frac{\partial N_1}{\partial x} = 0,\ \frac{\partial N_3}{\partial x} = 0,\ \frac{\partial N_4}{\partial x} = 0$ *, Second Verification condition is at node 2* $N_3 = 1$ and $N_1 = 0,\ N_2 = 0, N_4 = 0$ *and also* $\frac{\partial N_4}{\partial x} = 1$ and $\frac{\partial N_1}{\partial x} = 0,\ \frac{\partial N_2}{\partial x} = 0, \frac{\partial N_3}{\partial x} = 0.$ *Third verification condition is* $N_1 + N_3 = 1$. *For computational purpose I used Mathematica 9 Software [2].*

Keywords — *Hermite Polynomials, Natural Coordinate System -1 to 1, Shape functions.*

I. INTRODUCTION

Hermite Polynomials is used for c^n continuity elements. Notation for Hermite Polynomials in one dimension is $H^n(x)$. $H^1(x)$ is first order polynomial and which is 3rd order in x. $H^2(x)$ is second order polynomial and which is 5th order in x. $H^3(x)$ is third order polynomial and which is 7th order in x. $H^4(x)$ is fourth order polynomial and which is 9th order in x. $H^5(x)$ is fifth order polynomial and which is 11th order in x. In general $H^n(x)$ is nth order polynomial and which is 2n+1 order in x. In the notation $H^n_{mi}(x)$, m denotes order of derivative, i denotes node number and n denotes order of Hermitian function.

II. GEOMETRICAL DESCRIPTION

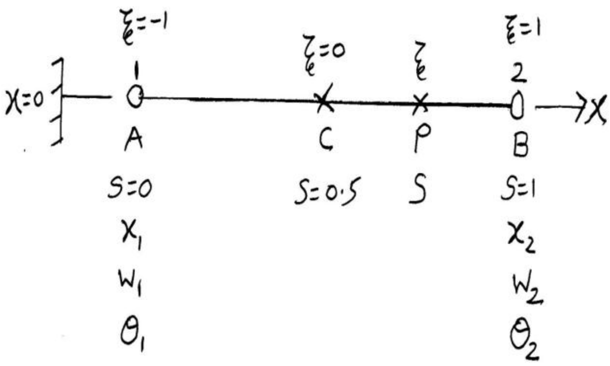

*Figure.*1 Beam element with natural coordinates ξ varying from -1 to 1

$$\xi = \frac{PC}{\frac{l}{2}} = \frac{2}{l}(AP - AC) = 2(s - 0.5) = 2s - 1$$

A two noded beam element shown in Figure.1 in which nodal unknowns are the displacement W and Slope $\frac{\partial W}{\partial x}$.

III. DERIVING SHAPE FUNCTIONS FOR ONE DIMENSIONAL HERMITE POLYNOMIALS

Since the element has four degrees of freedom, We have to select the polynomial with only 4 constants. In this polynomial after boundary conditions we get shape functions this we can take as first order (cubic) Hermitian Polynomials as shape functions.

$$W(\xi) = A_1 + A_2\xi + A_3\xi^2 + A_4\xi^3 \qquad (1)$$

Where W is the transverse displacement and A_1, A_2, A_3, A_4 are polynomial Coefficients

Differentiating eq(1) w.r.t. 'ξ'

$$(1) \Rightarrow \frac{\partial W}{\partial \xi} = 0 + A_2(1) + A_3(2\xi) + A_4(3\xi^2)$$

$$\frac{\partial W}{\partial \xi} = A_2 + 2A_3\xi + 3A_4\xi^2 \qquad (2)$$

Applying the nodal coditions such that

$W = W_1$ and $\frac{\partial W}{\partial \xi} = \theta_1$ at $\xi = -1$

and $W = W_2$ and $\frac{\partial W}{\partial \xi} = \theta_2$ at $\xi = 1$

in equations (1) and (2) , we get

When $W = W_1$ and $\xi = -1$

$$(1) \Rightarrow W_1 = A_1 + A_2(-1) + A_3(-1)^2 + A_4(-1)^3$$

$$W_1 = A_1 - A_2 + A_3 - A_4 \qquad (3)$$

When $\frac{\partial W}{\partial \xi} = \theta_1$ and $\xi = -1$

$$(2) \Rightarrow \theta_1 = A_2 + 2A_3(-1) + 3A_4(-1)^2$$

$$\theta_1 = A_2 - 2A_3 + 3A_4 \qquad (4)$$

When $W = W_2$ and $\xi = 1$

$$(1) \Rightarrow W_2 = A_1 + A_2(1) + A_3(1)^2 + A_4(1)^3$$

$$W_2 = A_1 + A_2 + A_3 + A_4 \qquad (5)$$

When $\frac{\partial W}{\partial \xi} = \theta_2$ and $\xi = 1$

$$(2) \Rightarrow \theta_2 = A_2 + 2A_3(1) + 3A_4(1)^2$$

$$\theta_1 = A_2 + 2A_3 + 3A_4 \qquad (6)$$

Using Mathematica 9 Software Solving (3),(4),(5) and (6) we get A_1, A_2, A_3, A_4

Input

$Solve[A_1 - A_2 + A_3 - A_4 - W_1 == 0 \&\& A_2 - 2*A_3 + 3*A_4 - \theta_1 == 0 \&\& A_1 + A_2 + A_3 + A_4 - W_2 == 0 \&\& A_2 + 2*A_3 + 3*A_4 - \theta_2 == 0, \{A_1, A_2, A_3, A_4\}]$

Output

$$\{\{A_1 -> \frac{1}{4}(2W_1 + 2W_2 + \theta_1 - \theta_2),$$

$$A_2 -> \frac{-3W_1}{4} + \frac{3W_2}{4} - \frac{\theta_1}{4} - \frac{\theta_2}{4},$$

$$A_3 -> \frac{1}{4}(-\theta_1 + \theta_2), A_4 -> \frac{W_1}{4} - \frac{W_2}{4} + \frac{\theta_1}{4} + \frac{\theta_2}{4}\}\}$$

Substituting A_1, A_2, A_3, A_4 in eq(1)

$$A_1 := \frac{1}{4}(2W_1 + 2W_2 + \theta_1 - \theta_2)$$

$$A_2 := \frac{-3W_1}{4} + \frac{3W_2}{4} - \frac{\theta_1}{4} - \frac{\theta_2}{4}$$

$$A_3 := \frac{1}{4}(-\theta_1 + \theta_2)$$

$$A_4 := \frac{W_1}{4} - \frac{W_2}{4} + \frac{\theta_1}{4} + \frac{\theta_2}{4}$$

$$W(\xi) := A_1 + A_2*\xi + A_3*\xi^2 + A_4*\xi^3$$

$Expand[W(\xi)]$

Output

$$\frac{W_1}{2}-\frac{3\xi W_1}{4}+\frac{\xi^3 W_1}{4}+\frac{W_2}{2}+\frac{3\xi W_2}{4}-\frac{\xi^3 W_2}{4}$$
$$+\frac{\theta_1}{4}-\frac{\xi\theta_1}{4}-\frac{\xi^2\theta_1}{4}+\frac{\xi^3\theta_1}{4}-\frac{\theta_2}{4}-\frac{\xi\theta_2}{4}+$$
$$\frac{\xi^2\theta_2}{4}+\frac{\xi^3\theta_2}{4}$$

$$W(\xi)=W_1\left(\frac{1}{2}-\frac{3\xi}{4}+\frac{\xi^3}{4}\right)+$$
$$\theta_1\left(\frac{1}{4}-\frac{\xi}{4}-\frac{\xi^2}{4}+\frac{\xi^3}{4}\right)+W_2\left(\frac{1}{2}+\frac{3\xi}{4}-\frac{\xi^3}{4}\right)+$$
$$\theta_2\left(-\frac{1}{4}-\frac{\xi}{4}+\frac{\xi^2}{4}+\frac{\xi^3}{4}\right) \tag{7}$$

$$\left.\begin{aligned} \text{i.e., } W&=N_1W_1+N_2\theta_1+N_3W_2+N_4\theta_2\\ &=N_1\delta_1+N_2\delta_2+N_3\delta_3+N_4\delta_4\end{aligned}\right\} \tag{8}$$

Where N_1, N_2, N_3, N_4 are shape functions for the beam elements and $\delta_1,\delta_2,\delta_3,\delta_4$ are the nodal displacements

$$\text{i.e., } \{\delta\}=\begin{Bmatrix}\delta_1\\ \delta_2\\ \delta_3\\ \delta_4\end{Bmatrix}=\begin{Bmatrix}W_1\\ \theta_1\\ W_2\\ \theta_2\end{Bmatrix}$$

Comparing (7) and (8) we get

$$N_1=\frac{1}{2}-\frac{3\xi}{4}+\frac{\xi^3}{4} \tag{9}$$

$$N_2=\frac{1}{4}-\frac{\xi}{4}-\frac{\xi^2}{4}+\frac{\xi^3}{4} \tag{10}$$

$$N_3=\frac{1}{2}+\frac{3\xi}{4}-\frac{\xi^3}{4} \tag{11}$$

$$N_4=-\frac{1}{4}-\frac{\xi}{4}+\frac{\xi^2}{4}+\frac{\xi^3}{4} \tag{12}$$

Eqations (9),(10),(11) and (12) can be taken as first order (cubic) Hermitian Polynomials as shape functions.

$$\begin{aligned}N_1=H_{01}^1&=\frac{1}{2}-\frac{3\xi}{4}+\frac{\xi^3}{4}\\ &=\frac{2\text{-}3\xi+\xi^3}{4}\end{aligned} \tag{13}$$

$$\begin{aligned}N_2=H_{11}^1&=\frac{1}{4}-\frac{\xi}{4}-\frac{\xi^2}{4}+\frac{\xi^3}{4}\\ &=\frac{1}{4}\left(1-\xi-\xi^2+\xi^3\right)\\ &=\frac{l}{8}\left(1-\xi-\xi^2+\xi^3\right)\end{aligned}$$

(Including length of beam element)

Substituting length $l=1\text{-}(\text{-}1)=2$

$$=\frac{2}{8}\left(1-\xi-\xi^2+\xi^3\right)$$
$$=\frac{1}{4}\left(1-\xi-\xi^2+\xi^3\right)$$
$$N_2=H_{11}^1==\frac{1}{4}-\frac{\xi}{4}-\frac{\xi^2}{4}+\frac{\xi^3}{4} \tag{14}$$

$$\begin{aligned}N_3=H_{02}^1&=\frac{1}{2}+\frac{3\xi}{4}-\frac{\xi^3}{4}\\ &=\frac{2+3\xi-\xi^3}{4}\end{aligned} \tag{15}$$

$$\begin{aligned}N_4=H_{12}^1&=-\frac{1}{4}-\frac{\xi}{4}+\frac{\xi^2}{4}+\frac{\xi^3}{4}\\ &=\frac{1}{4}\left(-1-\xi+\xi^2+\xi^3\right)\\ &=\frac{l}{8}\left(-1-\xi+\xi^2+\xi^3\right)\end{aligned}$$

(Including length of beam element)

Substituting length $l=1\text{-}(\text{-}1)=2$

$$=\frac{2}{8}\left(-1-\xi+\xi^2+\xi^3\right)$$

$$=\frac{1}{4}\left(-1-\xi+\xi^2+\xi^3\right)$$

$$N_4=H_{12}^1=-\frac{1}{4}-\frac{\xi}{4}+\frac{\xi^2}{4}+\frac{\xi^3}{4} \qquad (16)$$

Substituting ξ=2s-1 in equations (13),(14),(15) and (16) we get

Whenever you are changing ξ into s, length also changes ξ into s.

$\xi := 2s-1$ and length l=1-0 $\Rightarrow l$=1 ($\because$ s varies from s=0 to s=1)

$$N_1 := \frac{1}{2}-\frac{3\xi}{4}+\frac{\xi^3}{4}$$

$$N_2 := \frac{1}{8}-\frac{\xi}{8}-\frac{\xi^2}{8}+\frac{\xi^3}{8}$$

$$N_3 := \frac{1}{2}+\frac{3\xi}{4}-\frac{\xi^3}{4}$$

$$N_4 := -\frac{1}{8}-\frac{\xi}{8}+\frac{\xi^2}{8}+\frac{\xi^3}{8}$$

Expand[*FullSimplify*[N_1]]

FullSimplify[N_2]

FullSimplify[N_3]

FullSimplify[N_4]

$1-3s^2+2s^3$

$(-1+s)^2 s$

$(3-2s)s^2$

$(-1+s)s^2$

$$N_1=H_{01}^1(s)=1-3s^2+2s^3$$

$$N_2=H_{11}^1(s)=s(s-1)^2$$

$$N_3=H_{02}^1(s)=s^2(3-2s)$$

$$N_4=H_{12}^1(s)=s^2(s-1)$$

In $H_{01}^1(s)$, 0 represents Zeroth order derivative, 1 represents node number one and power 1 represents first order Hermitian function.

In $H_{11}^1(s)$, 1 represents first order derivative,1 represents node number one and power 1 represents first order Hermitian function.

In $H_{02}^1(s)$, 0 represents Zeroth order derivative, 2 represents node number two and power 1 represents first order Hermitian function.

In $H_{12}^1(s)$, 1 represents first order derivative, 2 represents node number two and power 1 represents first order Hermitian function.

IV. VERIFICATION

(i). 1st VERIFICATION CONDITION

First verification condition at node 1 is

$N_1=1$ and $N_2=0,\ N_3=0, N_4=0$ and also

$$\frac{\partial N_2}{\partial s}=1 \text{ and } \frac{\partial N_1}{\partial s}=0, \frac{\partial N_3}{\partial s}=0, \frac{\partial N_4}{\partial s}=0$$

At Node 1 s = 0

$$N_1 := 1 - 3*s^2 + 2*s^3 \quad (17)$$

$$N_2 := s(s-1)^2 \quad (18)$$

$$N_3 := s^2(3-2s) \quad (19)$$

$$N_4 := s^2(s-1) \quad (20)$$

$s := 0$

N_1

N_2

N_3

N_4

Output

1

0

0

0

Finding first derivatives for (17),(18),(19) and (20)

$N_1 := 1 - 3*s^2 + 2*s^3$

$N_2 := s(s-1)^2$

$N_3 := s^2(3-2s)$

$N_4 := s^2(s-1)$

$\partial_s(N_1)$

$\partial_s(N_2)$

$\partial_s(N_3)$

$\partial_s(N_4)$

Output

$-6s + s^2$

$(-1+s)^2 + 2(-1+s)s$

$2(3-2s)s - 2s^2$

$2(-1+s)s + s^2$

$$\therefore \partial_s(N_1) = -6s + s^2 \quad (21)$$

$$\therefore \partial_s(N_2) = (-1+s)^2 + 2(-1+s)s \quad (22)$$

$$\therefore \partial_s(N_3) = 2(3-2s)s - 2s^2 \quad (23)$$

$$\therefore \partial_s(N_4) = 2(-1+s)s + s^2 \quad (24)$$

Partial derivative condition at node 1, s = 0

$\frac{\partial N_1}{\partial s} := -6s + s^2$

$\frac{\partial N_2}{\partial s} := (-1+s)^2 + 2(-1+s)s$

$\frac{\partial N_3}{\partial s} := 2(3-2s)s - 2s^2$

$\frac{\partial N_4}{\partial s} := 2(-1+s)s + s^2$

$s = 0$

$\frac{\partial N_1}{\partial s}$

$\frac{\partial N_2}{\partial s}$

$\frac{\partial N_3}{\partial s}$

$\frac{\partial N_4}{\partial s}$

Output

0

1

0

0

(ii) 2^{nd} VERIFICATION CONDITION

Second Verification condition is at node 2

$N_3 = 1$ and $N_1 = 0,\ N_2 = 0, N_4 = 0$ and also $\frac{\partial N_4}{\partial x} = 1$ and $\frac{\partial N_1}{\partial x} = 0, \frac{\partial N_2}{\partial x} = 0, \frac{\partial N_3}{\partial x} = 0$.

At Node 2, s = 1

$N_1 := 1 - 3*s^2 + 2*s^3$

$N_2 := s(s-1)^2$

$N_3 := s^2(3-2s)$

$N_4 := s^2(s-1)$

$s := 1$

N_1

N_2

N_3

N_4

Output

0

0

1

0

Partial derivative condition at node 2, s = 1

$\frac{\partial N_1}{\partial s} := -6s + s^2$

$\frac{\partial N_2}{\partial s} := (-1+s)^2 + 2(-1+s)s$

$\frac{\partial N_3}{\partial s} := 2(3-2s)s - 2s^2$

$\frac{\partial N_4}{\partial s} := 2(-1+s)s + s^2$

$s: = 1$

$\frac{\partial N_1}{\partial s}$

$\frac{\partial N_2}{\partial s}$

$\frac{\partial N_3}{\partial s}$

$\frac{\partial N_4}{\partial s}$

Output

0

0

0

1

Third Verification Condition

3^{rd} verification condition is $N_1 + N_3 = 1$

$$N_1 := 1 - 3 * s^2 + 2 * s^3$$

$$N_3 := s^2(3 - 2s)$$

$FullSimplify[N_1 + N_3]$

Output

1

V. CONCLUSIONS

1. Derived Shape Functions for Hermite Polynomials.
2. Verified First verification condition at node 1, $N_1 = 1$ and $N_2 = 0,\ N_3 = 0, N_4 = 0$ and also $\frac{\partial N_2}{\partial x} = 1$ and $\frac{\partial N_1}{\partial x} = 0, \frac{\partial N_3}{\partial x} = 0, \frac{\partial N_4}{\partial x} = 0$
3. Verified Second Verification condition at node 2 $N_3 = 1$ and $N_1 = 0,\ N_2 = 0, N_4 = 0$ and also $\frac{\partial N_4}{\partial x} = 1$ and $\frac{\partial N_1}{\partial x} = 0, \frac{\partial N_2}{\partial x} = 0, \frac{\partial N_3}{\partial x} = 0$.
4. Verified Third verification condition $N_1 + N_3 = 1$.

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