Local Landmarks in Circulant Graph

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Abstract — *Let G(V, E) be a graph with vertex set V and edge set E. Then a subset of V namely, W is said to be a local metric basis of G, if for any two adjacent vertices u, v belonging to V\W, there exists a vertex w belonging to W, such that the distance from u to w is distinct from the distance from v to w. The minimum cardinality of local metric basis is called the local metric dimension of G. In this paper, we investigate the local metric basis and local metric dimension of Circulant graph* $G(n, \pm\{1,2\})$.

Keywords — *metric basis, metric dimension, local metric basis, local metric dimension, circulant graph.*

I. INTRODUCTION

Graph theory is a delightful playground for the exploration of various techniques in discrete mathematics and its results have applications in many areas of the computing, social and natural sciences. The vertices of a graph G of order n are distinguished by distinct colors. The vertices of a connected graph G are distinguished by distinct distance between the vertices. Distinguishing adjacent vertices using distance between vertices gives the motivation to study local metric dimension of graphs.

The metric dimension problem is an application to network discovery, robotics, security coding and verification in the area of the telecommunication and traffic signals.

Definition 1.1

A metric basis of a graph G (V, E) is a subset $M \subseteq V$ such that for each pair of vertices $u, v \in V \setminus M$, there exists a vertex $w \in M$ such that the length of the shortest path from w to u is distinct from the length of the shortest path from w to v. The minimum cardinality of a metric basis is called the minimum metric dimension and is denoted by $\beta(G)$.

Definition 1.2

Let $G(V, E)$ be a graph then the subset W of the vertex set V is said to be a local metric basis if for any two adjacent vertices $u, v \in V \setminus W$ there exists a $w \in W$ such that $d(u,w) \neq d(v,w)$. The minimum cardinality of local metric basis is said to be the local metric dimension (lmd) of the graph G and is denoted by $\beta_l(G)$.

Definition 1.3

An undirected circulant graph, denoted by $G(n, \pm\{1,2,\ldots,j\}), 1 \leq j \leq \lfloor n/2 \rfloor, n \geq 3$ is defined as a graph consisting of the vertex set $V = \{1,2,\ldots,n\}$ and the edge set $E = \{(i,j) : |j-i| \equiv s \,(\mathrm{mod}\, n), \quad s \in \{1,2,\ldots,j\}\}$. Also $G(n, \pm 1)$ is an undirected cycle and $G(n, \pm\{1,2,\ldots,\lfloor n/2 \rfloor\})$ is a complete graph K_n.

An important class of interconnection networks in parallel and distributed computing are Circulant graphs [3]. The applications of circulant graphs in graph theory have appeared in coding theory, VLSI design, Ramsey theory, and many other areas [13].

II. LITERATURE SURVEY

Harary and Melter [2] were the first to study the problem of finding the minimum metric dimension of a graph and gave a characterization for the metric dimension of trees. Melter and Tomescu [10] studied the metric dimension problem for grids induced by lattice points in the plane. The generalized result on the metric dimension of d-dimensional grids was given by Khuller et al. [9]. The metric dimension of circulant graphs was studied by Muhammad Imran et al. [11]. The metric dimension of circulant and Harary graphs were studied by Cyriac Grigorious et al. [1].

In view of local metric basis, Okamoto et al. [12] has established sharp bounds for the local metric dimension of a graph in terms of well-known graphical parameters. The local metric dimension of Cyclic Split Graph $C_nK_r^k$, enhanced mesh and extended mesh were studied by J. A. Cynthia and Ramya [7], [8]. J. A. Cynthia and Fancy [4]–[6] gave local metric dimension of family of generalized Petersen graph, Kautz digraph, de Bruijn digraph, Benes and Butterfly networks.

III. RESULTS AND DISCUSSION ON CIRCULANT GRAPH

In this section we investigate the properties of equidistant pairs of adjacent vertices and the local metric dimension of undirected Circulant graph $G = G(n, \pm\{1,2\}),\ n \geq 6$.

A. Properties of Pairs of Adjacent Vertices Equidistant from a Vertex

Let u be any vertex of graph G where $1 \leq u \leq n$.

3.1.1 For all n, the pairs of adjacent vertices (x_1, x_2) and (y_1, y_2) belonging to G of the form $x_1 = (u + 2i - 1) \bmod n,\ x_2 = (u + 2i) \bmod n,\ y_1 = (u + n + 1 - 2i) \bmod n$ and $y_2 = (u + n - 2i) \bmod n$ where $i = 1,2,\ldots,\lfloor (n-2)/4 \rfloor$ are such that $d(u, x_1) = d(u, x_2) = d(u, y_1) = d(u, y_2) = i$.

3.1.2 For all n, the pair of adjacent vertices $v_1 = (u+1) \bmod n$ and $v_2 = (u + n - 1) \bmod n$ are such that $d(u, v_1) = d(u, v_2) = 1$.

3.1.3 For $n \equiv 0 (\bmod 4)$, there are three diametrically opposite vertices to u namely z_1, z_2 and z_3 where $z_1 = \left(u + \frac{n}{2} - 1\right) \bmod n$, $z_2 = \left(u + \frac{n}{2}\right) \bmod n$ and $z_3 = \left(u + \frac{n}{2} + 1\right) \bmod n$, such that $d(u, z_1) = d(u, z_2) = d(u, z_3) = \frac{n}{4}$. Also $\{(z_1, z_2), (z_2, z_3), (z_1, z_3)\}$ are the pairs of adjacent vertices.

3.1.4 For $n \equiv 1 (\bmod 4)$, there are four diametrically opposite vertices to u namely t_1, t_2, t_3 and t_4 where $t_1 = \left(u + \frac{n-3}{2}\right) \bmod n$, $t_2 = \left(u + \frac{n-1}{2}\right) \bmod n$, $t_3 = \left(u + \frac{n+1}{2}\right) \bmod n$ and $t_4 = \left(u + \frac{n+3}{2}\right) \bmod n$, such that $d(u,t_1) = d(u,t_2) = d(u,t_3) = d(u,t_4) = \frac{n-1}{4}$. Also $\{(t_1,t_2), (t_2,t_3), (t_3,t_4),\ (t_1,t_3), (t_2,t_4)\}$ are the pairs of adjacent vertices.

3.1.5 For $n \equiv 2\ (\bmod 4)$, the diametrically opposite vertex to u is $v = \left(u + \frac{n}{2}\right) \bmod n$ such that $d(u,v) = \left\lceil \frac{n}{4} \right\rceil$. Also there exist a pair of adjacent vertices (l_1, l_2) where $l_1 = \left(u + \frac{n}{2} - 1\right) \bmod n$ and $l_2 = \left(u + \frac{n}{2} + 1\right) \bmod n$ are such that $d(u, l_1) = d(u, l_2) = \left\lfloor \frac{n}{4} \right\rfloor$.

3.1.6 For $n \equiv 3 (\bmod 4)$, there are two diametrically opposite vertices to u namely r_1 and r_2 where $r_1 = \left(u + \frac{n-1}{2}\right) \bmod n$ and $r_2 = \left(u + \frac{n+1}{2}\right) \bmod n$ such that $d(u, r_1) = d(u, r_2) = \frac{n+1}{4}$ and r_1, r_2 are adjacent vertices.

Note: $n \bmod n$ is taken to be n.

B. Local Metric Dimension of Circulant Graph with Even Order

Theorem 3.1: Let G be the undirected Circulant graph $G(n, \pm\{1,2\}),\ n \geq 6$ and n is even. Then

$$\beta_l(G)=\begin{cases}2, & n\equiv 2\,(\mathrm{mod}\,4)\\ 3, & n\equiv 0\,(\mathrm{mod}\,4)\end{cases}.$$

Proof: Let G be the undirected Circulant graph $G(n,\pm\{1,2\})$, $n\geq 6$ and n is even. Let a vertex u be a member of the local metric basis such that $1\leq u\leq n$. Then by Property 3.1.1, there exists pairs of adjacent vertices (x_1,x_2) and (y_1,y_2) such that $d(u,x_1)=d(u,x_2)=d(u,y_1)=d(u,y_2)=i$ where $i=1,2,\ldots,\lfloor (n-2)/4\rfloor$. Also by Property 3.1.2, there exist a pair of adjacent vertices (v_1,v_2) such that $d(u,v_1)=d(u,v_2)=1$.

Case (i): When $n\equiv 2\,(\mathrm{mod}\,4)$

By Property 3.1.5 there exists a pair of adjacent vertices (l_1,l_2) such that $d(u,l_1)=d(u,l_2)=\left\lfloor\frac{n}{4}\right\rfloor$. Let $w'=(u+1)\,\mathrm{mod}\,n$ be another member of the local metric basis where $1\leq w'\leq n$. Then we have $d(w',x_1)\neq d(w',x_2)$, $d(w',y_1)\neq d(w',y_2)$, $d(w',v_1)\neq d(w',v_2)$ and $d(w',l_1)\neq d(w',l_2)$.
Thus, the local metric basis is given by $\{u,w'\}$ where $w'=(u+1)\,\mathrm{mod}\,n$ such that $|u-w'|=1$.

Case (ii): When $n\equiv 0\,(\mathrm{mod}\,4)$

By Property 3.1.3 there exists pairs of adjacent vertices $\{(z_1,z_2),(z_2,z_3),(z_1,z_3)\}$ such that $d(u,z_1)=d(u,z_2)=d(u,z_3)=\frac{n}{4}$. Let $u'=z_2=\left(u+\frac{n}{2}\right)\mathrm{mod}\,n$ be another member of the local metric basis where $1\leq u'\leq n$. Then we have $d(u',x_1)\neq d(u',x_2)$ and $d(u',y_1)\neq d(u',y_2)$. But $d(u',v_1)=d(u',v_2)$ and $d(u',z_1)=d(u',z_3)$. In order to resolve this, let $u''=z_1=\left(u+\frac{n}{2}-1\right)\mathrm{mod}\,n$ be the third member of the local metric basis where $1\leq u''\leq n$. Then we have $d(u'',v_1)\neq d(u'',v_2)$.

Thus the local metric basis is given by $\{u,u',u''\}$ where $u'=\left(u+\frac{n}{2}\right)\mathrm{mod}\,n$ and $u''=\left(u+\frac{n}{2}-1\right)\mathrm{mod}\,n$ such that $|u-u'|=|u-u''|=\frac{n}{4}$ and $|u''-u'|=1$. □

The following figure gives the local metric basis of Circulant graph $G(n,\pm\{1,2\})$, $n\geq 6$ where $n\equiv 0\,(\mathrm{mod}\,4)$.

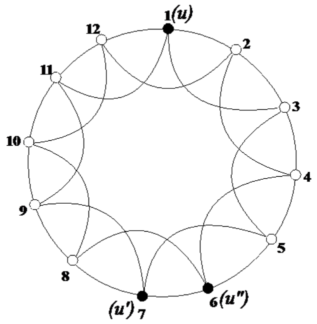

Fig. 1 The Circulant graph $G(12,\pm\{1,2\})$ with local metric basis $\{u,u',u''\}$

C. Local Metric Dimension of Circulant Graph with Odd Order

Theorem 3.2: Let G be the undirected circulant graph $G(n,\pm\{1,2\})$, $n\geq 6$ and $n\equiv 3(\mathrm{mod}\,4)$. Then $\beta_l(G)=2$.

Proof: Let G be the undirected Circulant graph $G(n, \pm\{1,2\})$, $n \geq 6$ and $n \equiv 3 (\bmod 4)$. Let a vertex u be a member of the local metric basis such that $1 \leq u \leq n$. Then by Property 3.1.1, 3.1.2, 3.1.6, there exists pairs of adjacent vertices $\{(x_1, x_2), (y_1, y_2), (v_1, v_2), (r_1, r_2)\}$ such that $d(u, x_1) = d(u, x_2) = d(u, y_1) = d(u, y_2) = i$, where $i = 1,2,\ldots,\lfloor (n-2)/4 \rfloor$, $d(u, v_1) = d(u, v_2) = 1$ and $d(u, r_1) = d(u, r_2) = \frac{n+1}{4}$.

But for a vertex w' we have $d(w', x_1) \neq d(w', x_2)$, $d(w', y_1) \neq d(w', y_2)$, $d(w', v_1) \neq d(w', v_2)$ and $d(w', r_1) \neq d(w', r_2)$, where $w' = (u+1) \bmod n$ and $1 \leq w' \leq n$.

Thus, the local metric basis is given by $\{u, w'\}$ where $d(u, w') = 1$. □

Theorem 3.3: Let G be the undirected circulant graph $G(n, \pm\{1,2\}), n \geq 6$ and $n \equiv 1 (\bmod 4)$. Then $\beta_l(G) = 4$.

Proof: Let G be the undirected Circulant graph $G(n, \pm\{1,2\})$, $n \geq 6$ and $n \equiv 1 (\bmod 4)$. Let a vertex u be a member of the local metric basis such that $1 \leq u \leq n$. Then by Property 3.1.1, 3.1.2 and 3.1.4, there exists pairs of adjacent vertices (x_1, x_2), (y_1, y_2), (v_1, v_2), (t_1, t_2), (t_2, t_3), (t_3, t_4), (t_1, t_3) and (t_2, t_4) such that the vertices in each pair are equidistant from the vertex u.

But for any vertex $v_1 = (u+1) \bmod n$, $1 \leq v_1 \leq n$, the pairs of adjacent vertices (x_1, x_2), (y_1, y_2), (v_1, v_2), (t_1, t_2) and (t_1, t_3) are such that the vertices in each pair are at distinct distances from v_1. Whereas, the vertices in each pair (t_2, t_3), (t_3, t_4) and (t_2, t_4) are equidistant from the vertex v_1.

Consider the vertex $t_4 = \left(u + \frac{n+3}{2}\right) \bmod n$, $1 \leq t_4 \leq n$. The vertices in each pair (t_3, t_4) and (t_2, t_4) are at distinct distances from the vertex t_4 but the adjacent pair (t_2, t_3) is such that $d(t_4, t_2) = d(t_4, t_3)$. To resolve we consider the vertex $t_3 = \left(u + \frac{n+1}{2}\right) \bmod n$ where $1 \leq t_3 \leq n$ to be a member of the metric basis.

Thus, the local metric basis is given by $\{u, v_1, t_4, t_3\}$ and the basis contains two adjacent pairs of vertices (u, v_1) and (t_3, t_4) such that the distance between the adjacent pairs is $\frac{n-1}{4}$. □

Conclusion

In this paper we have found the local metric dimension of undirected circulant graph $G(n, \pm\{1,2\}), n \geq 6$. Also, we would further study the applications of local metric dimension in interconnection networks.

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