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ON THE PROPERTIES OF IDEMPOTENTS OF THE MATRIX RING $M_3(\mathbb{Z}_n[x])$

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Abstract. In this paper, we find all the idempotents of 3×3 upper (lower) triangular matrices over the commutative ring $\mathbb{Z}_p$, i.e., $U_3(\mathbb{Z}_p[x])$ ($L_3(\mathbb{Z}_p[x])$) for any prime p. We also show that for the ring of upper (lower) triangular matrices over $\mathbb{Z}_n[x]$, i.e., $U_3(\mathbb{Z}_n[x])$ ($L_3(\mathbb{Z}_n[x])$), every diagonal entry of any idempotent matrix in $U_3(\mathbb{Z}_n[x])$ ($L_3(\mathbb{Z}_n[x])$) must be an idempotent of $\mathbb{Z}_n$ for every n.

Keywords: Idempotents, Upper triangular matrices, Lower triangular matrices, Commutative rings, Polynomial Rings

1. Introduction

Idempotents play a very important role in the study of rings as well as modules. An element a of a ring is said to be an idempotent if $a^2 = a$. Because of their importance in various fields, the idempotents have drawn interest of many researchers. In [1], various conditions are given for an element to be an idempotent in $M_2(\mathbb{Z}_{2p}[x])$ and $M_2(\mathbb{Z}_{3p}[x])$ for any prime $p > 3$. So, in this article we find out all the idempotents of 3×3 upper (lower) triangular matrices over the commutative ring $\mathbb{Z}_p[x]$ for any prime p. Moreover, we also give the condition on the diagonal entries of the idempotents of $m \times m$ upper triangular matrices over $\mathbb{Z}_n[x]$ for any m and n. While finding the idempotents of $U_3(\mathbb{Z}_p[x])$, i.e., the ring of 3×3 upper triangular matrices over $\mathbb{Z}_p[x]$, we conclude via suitable example that the result already known for the idempotents of $M_2(R)$, for a commutative ring R in [1], does not hold in 3×3 case.

2. Main result

Theorem 2.1. *For any commutative ring F, set of all the idempotents of the polynomial ring $F[x]$ and F are same.*

Proof. For proof , see ([2], Lemma 1]. $\square$

Theorem 2.2. *Diagonal entries of any idempotent of $U_3(\mathbb{Z}_n[x])$, the ring of all 3×3 upper triangular matrices, are idempotents of $\mathbb{Z}_n$ for all $n \in \mathbb{N}$.*

Proof. Let G be any arbitrary idempotent of $U_3(\mathbb{Z}_n[x])$. Then

$$G = \begin{pmatrix} p(x) & q(x) & r(x) \\ 0 & s(x) & t(x) \\ 0 & 0 & u(x) \end{pmatrix}$$

where $p(x), q(x), r(x), s(x), t(x), u(x) \in Z_n[x]$. Being an idempotent, G must satisfy $G^2 = G$ which implies that

$$\begin{pmatrix} (p(x))^2 & q(x)(p(x)+s(x)) & r(x)(u(x)+p(x))+q(x)t(x) \\ 0 & (s(x))^2 & (s(x)+u(x))t(x) \\ 0 & 0 & (u(x))^2 \end{pmatrix} = \begin{pmatrix} p(x) & q(x) & r(x) \\ 0 & s(x) & t(x) \\ 0 & 0 & u(x) \end{pmatrix}$$

Comparing diagonal entries, we get

$$(p(x))^2 = p(x), \quad (q(x))^2 = q(x), \quad (r(x))^2 = r(x)$$

Thus $p(x), q(x)$ and $r(x)$ are idempotents of $Z_n[x]$ and so, by theorem 2.1, these must be elements of Z_n. Thus, idempotents of $U_3(\mathbb{Z}_n[x])$ must have idempotents of $\mathbb{Z}_n$ on diagonal. □

Theorem 2.3. *All the idempotents of $U_3(\mathbb{Z}_p[x])$, where p is a prime can be classified into 8 classes of 3×3 upper triangular matrices.*

Proof. From theorem 2.2, if G is any idempotent of $U_3(\mathbb{Z}_p[x])$, then all its diagonal elements must be the idempotents of $\mathbb{Z}_p$. As the only idempotents of $\mathbb{Z}_p$ are 0 and 1, so diagonal entries of G must be 0 or 1. Hence

$$G = \begin{pmatrix} \delta_1 & q(x) & r(x) \\ 0 & \delta_2 & t(x) \\ 0 & 0 & \delta_3 \end{pmatrix}$$

where $q(x), r(x), t(x) \in \mathbb{Z}_p[x]$ and $\delta_1, \delta_2, \delta_3 \in \{0, 1\}$. Now $G^2 = G$ implies (writing q, r, t in place of $q(x), r(x), t(x)$ for our convenience), we get

$$\begin{pmatrix} \delta_1 & (\delta_1+\delta_2)q & \delta_1 r + qt + \delta_3 r \\ 0 & \delta_2 & (\delta_2+\delta_3)t \\ 0 & 0 & \delta_3 \end{pmatrix} = \begin{pmatrix} \delta_1 & q & r \\ 0 & \delta_2 & t \\ 0 & 0 & \delta_3 \end{pmatrix} \tag{1}$$

Now since each δ_i for $1 \le i \le 3$ has 2 choices, either 0 or 1 and thus the triad $(\delta_1, \delta_2, \delta_3)$ can take 8 values in all. We now consider every possible value of δ_i for each i.

Case-1: If $(\delta_1, \delta_2, \delta_3) = (0, 0, 0)$, then from (1), we get $t = 0, q = 0, r = 0$. Thus G is zero matrix in this case.

Case-2: If $(\delta_1, \delta_2, \delta_3) = (0, 0, 1)$, then from (1), we get t, r arbitrary and $q = 0$. Thus any idempotent of this type is of the form

$$G = \begin{pmatrix} 0 & 0 & r(x) \\ 0 & 0 & t(x) \\ 0 & 0 & 1 \end{pmatrix}$$

where $r(x), t(x)$ are arbitrary polynomials in $\mathbb{Z}_p[x]$.

Case-3: If $(\delta_1, \delta_2, \delta_3) = (0, 1, 0)$, then from (1), we get q, t arbitrary and $r = qt$. Thus any idempotent of this type is of the form

$$G = \begin{pmatrix} 0 & q(x) & q(x)t(x) \\ 0 & 1 & t(x) \\ 0 & 0 & 0 \end{pmatrix}$$

where $q(x), t(x)$ are arbitrary polynomials in $\mathbb{Z}_p[x]$.

Case-4: If $(\delta_1, \delta_2, \delta_3) = (0, 1, 1)$, then from (1), we get $t = 0$ and r, q arbitrary. Thus any idempotent of this type is of the form

$$G = \begin{pmatrix} 0 & q(x) & r(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $r(x), q(x)$ are arbitrary polynomials in $\mathbb{Z}_p[x]$.

Case-5: If $(\delta_1, \delta_2, \delta_3) = (1, 0, 0)$, then from (1), we get q, r arbitrary and $t = 0$. Thus any idempotent of this type is of the form

$$G = \begin{pmatrix} 1 & q(x) & r(x) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $q(x), r(x)$ are arbitrary polynomials in $\mathbb{Z}_p[x]$.

Case-6: If $(\delta_1, \delta_2, \delta_3) = (1, 0, 1)$, then from (1), we get q, r arbitrary and $t = 0$. Thus any idempotent of this type is of the form

$$G = \begin{pmatrix} 1 & q(x) & -q(x)t(x) \\ 0 & 0 & t(x) \\ 0 & 0 & 1 \end{pmatrix}$$

where $q(x), t(x)$ are arbitrary polynomials in $\mathbb{Z}_p[x]$.

Case-7: If $(\delta_1, \delta_2, \delta_3) = (1, 1, 0)$, then from (1), we get $q = 0$, r, t arbitrary. Thus any idempotent of this type is of the form

$$G = \begin{pmatrix} 1 & 0 & r(x) \\ 0 & 1 & t(x) \\ 0 & 0 & 0 \end{pmatrix}$$

where $r(x), t(x)$ are arbitrary polynomials in $\mathbb{Z}_p[x]$.

Case-8: If $(\delta_1, \delta_2, \delta_3) = (1, 1, 1)$, then from (1), we get $q, t, r = 0$. Thus G is identity in this case.

So from 8 cases, we get 8 different classes of idempotent elements.

$\square$

Theorem 2.4. *Diagonal entries of idempotents of $L_3(\mathbb{Z}_n[x])$, the group of all 3×3 lower triangular matrices, must be the idempotents of $\mathbb{Z}_n$ for all $n \in \mathbb{N}$.*

Proof. This can be proved exactly on the similar lines of theorem 2.2. □

Theorem 2.5. *All the idempotents of $L_3(\mathbb{Z}_p[x])$ for any p prime can be classified into 8 classes of 3×3 lower triangular matrices.*

Proof. Proof can be done similarly on the lines of theorem 2.3.

□

Example 2.1. *With the help of theorem 2.2 or 2.3 we easily conclude that all the idempotents of 3×3 diagonal matrices over $Z_{10}[x]$ are of the form*

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

where $a, b, c \in \{0, 1, 5, 6\}$. Thus this ring has total 3^4 idempotent elements.

Theorems 2.2 and 2.3 can also be proved for any ring of $n \times n$ matrices, not only 3×3.

Theorem 2.6. *For any idempotent of $U_m(\mathbb{Z}_n[x])$, the ring of all $m \times m$ upper triangular matrices over $\mathbb{Z}_n[x]$, diagonal entries must be the idempotents of $\mathbb{Z}_n$.*

Proof. Follows on the similar lines of theorem 2.1. □

3. Notable difference between idempotents of $M_2(R)$ and $M_3(R)$

In this section, we give two instances, where we can clearly notice that the results already known (cf. [1]), for the idempotents of $M_2(R)$ are not applicable for $M_3(R)$ when R is a commutative ring.

Case-1: From theorem 2.5 in [1], we know that for any commutative ring R, the trace of every non-trivial idempotent in $M_2(R)$ with determinant 0 is an idempotent. But above result does not hold in the case of 3×3 matrices in $U_3(\mathbb{Z}_n[x])$ and hence it will not hold for $M_3(R)$ as $U_3(\mathbb{Z}_n[x])$ is a subring of $M_3(R)$ when $R = \mathbb{Z}_n[x]$. To prove the above said, we give a counter example.

Example 3.1. *Take the matrix ring $M = M_3(\mathbb{Z}_3[x])$. Idempotent elements of $\mathbb{Z}_3$ are $0, 1$ only. Let $G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ be an element of M. Now $G^2 = G$ implies G is an idempotent. Also $\det G = 0$, but Trace $G = 2$, which is not an idempotent of $\mathbb{Z}_3[x]$, as only idempotents of $\mathbb{Z}_3[x]$ are 0 and 1.*

Case-2: From Proposition 2.6 in [1], an element of $M_2(\mathbb{Z}_p)$ with determinant 0 is an idempotent if and only if its trace is 1. But this fails for $M_3(\mathbb{Z}_p)$. For e.g. Consider $G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ be an element of M. For this matrix det $= 0$, but its trace is 2.

4. Conclusion

From the above theorems we can explicitly find all the idempotents of 3×3 upper (lower) triangular matrices over the ring $\mathbb{Z}_p[x]$. Also we can easily find all the idempotents of $n \times n$ diagonal matrices over the ring $\mathbb{Z}_n[x]$ for any n. Moreover from third section, we conclude that some results that hold for $M_2(R)$ does not hold good for $M_3(R)$ when R is a commutative ring.

References

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