2-Colored and 3-Colored Diagrams of Posets in Cover-incomparability Graphs

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Abstract: *The notion of forbidden ⊲ - preserving 2-colored and 3-colored diagrams is introduced here as part of the study of cover-incomparability graphs of posets. Posets whose C-I graphs are chordal, which are characterized here using 2-colored and 3-colored diagrams.*

Keywords: *cover-incomparability graph, cograph, poset.*

1. Introduction

Cover-incomparability graphs of posets, or shortly C-I graphs, were introduced in [3] as the underlying graphs of the standard interval function or transit function on posets (for more on transit functions in discrete structures cf. [3,4, 5, 6, 10]). On the other hand, C-I graphs can be defined as the edge-union of the covering and incomparability graph of a poset; in fact, they present the only non-trivial way to obtain an associated graph as unions and/or intersections of the edge sets of the three standard associated graphs (i.e. covering, comparability and incomparability graph). In the paper that followed [8], it was shown that the complexity of recognizing whether a given graph is the C-I graph of some poset is in general NP-complete. In [1] the problem was investigated for the classes of split graphs and block graphs, and the C-I graphs within these two classes of graphs were characterized. This resulted in a linear-time recognition algorithms for C-I block and C-I split graphs. It was also shown in [1] that whenever a C-I graph is a chordal graph, it is necessarily an interval graph, however a structural characterization of C-I interval graphs (and thus C-I chordal graphs) is still open. Recently C-I distance-hereditary graphs have been characterized and shown to be efficiently recognizable [9].

Let P = (V;_≤) be a poset. If u ≤ v but u ≠ v, then we write u < v. For u, v ϵ V we say that v *covers* u in P if u < v and there is no w in V with u < w < v. If u ≤ v we will sometimes say that u is *below* v, and that v is *above* u. Also, we will write u ⊲ v if v covers u; and u ⊲⊲ v if u is below v but not covered by v. By u ∥ v we denote that u and v are incomparable. Let V' be a nonempty subset of V. Then there is a natural poset Q = (V';≤ ′), where u ≤ ′ v if and only if u ≤ v for any u,v ϵ V'. The poset Q is called a *subposet* of P and its notation is simplified to Q = (V';≤). If, in addition, together with any two comparable elements u and v of Q, a chain of shortest length between u and v of P is also in Q, we say that Q is an isometric subposet of P. Recall that a poset P is *dual* to a poset Q if for any x, y ϵ P the following holds: x ≤ y in P if and only if y ≤ x in Q. Given a poset P, its cover-incomparability graph G_P has V as its vertex set, and uv is an edge of G_P if u ⊲ v, v ⊲ u, or u and v are incomparable. A graph that is a cover-incomparability graph of some poset P will be called a C-I graph.

2. 2-colored and 3-colored diagrams

2-coloured diagram $\mathscr{P}$; in [12] we describe the family $\mathscr{P}$ by the Hasse diagram of initial poset P using normal edges, added by the bold edges between u_i and v_j (u_i and v_j are incomparable pairs) for all i and j. It follows that if there is a bold edge between an incomparable pair of elements u_i and v_j in P then either u_i ⊲ v_j or v_j ⊲ u_i,which neither affect the covering nor the incomparability relation of any other pair of elements in P. Any subset of the set of bold edges can thus be chosen and removed arbitrarily to obtain one of the Hasse diagram of a poset from the family $\mathscr{P}$. Hence one drawing, using normal and bold edges, suffices to describe all posets of $\mathscr{P}$.

A 3-coloured diagram Q; we consider normal edges to represent vertices in a covering relation and red edges to represent incomparable vertices or vertices in a covering relation and dashed lines to represent a chain of length three and thus constitute the 3-colors and hence the name 3-*colored* diagram. The idea of 3-colored diagrams is explained as follows. Let G be a C-I graph and H be an induced subgraph of G. We note that there can be different ⊲- preserving subposets Q_i of some posets with G_{Q_i} isomorphic to the subgraph H. Let u,v,w be an induced path in the direction from u to v in H. There are four possibilities in which u, v and w can be related in the ⊲ - preserving subposets. It is possible to have u ⊲ v, u ∥ v, v ⊲ w and v ∥ w. Each case will appear as a ⊲ - preserving subposet of four different posets. If u ⊲ v and v ⊲ w in a subposet, then u ⊲ v ⊲ w is a chain in the subposet and u,v,w is an induced path in H. If there is either u ∥ v or v ∥ w in a subposet Q, then there should be another chain from u to w in Q in order to have u, v,w an induced path in H. We try to capture this situation using the idea of 3-colored diagram. Suppose in ⊲ - preserving subposet Q of a poset P, there exists two elements u, v which is always connected by some chain of length three in Q. Let w be an element in Q such that either both uw and vw are red edges or any one of them is a red edge. Then in order to have a chain between u and v, there must exist an element x in Q so that u, x, v form a chain in Q. When both edges are normal, then we have the chain u,w, v in Q and hence the chain u, x, v is not required in this case. We denote the chain u, x, v by dashed lines between ux and xv in order to specify that it is possible to have the presence or absence of the chain u, x, v in Q. The presence of the chain u, x, v implies that either both of the edges uw and wv are red edges or one of them is a red edge. The absence of the chain implies that both uw and vw are normal edges in Q. We call posets having the above mentioned diagrams as 3-colored diagrams. Thus a 3-colored diagram contains normal edges, red edges and dashed lines, in which the dashed line between elements u and v will vanish, when there is a chain between u and v using normal or red edges. We can define 3-colored subposets in a similar way as discussed above. All subposets of the poset P that we consider in this paper are 3-colored diagrams. Thus by a single 3-colored diagram, we represent a collection of ⊲ - preserving subposets to be forbidden for a poset. We sometimes use the term 3-colored subposets instead of 3-colored diagrams in this paper. In a similar way the dual of a 3-colored diagram is also meaningful and represents a collection of ⊲- preserving dual subposets.

Theorem 1: (Theorem 1,[7]): Let $\mathcal{G}$ be a class of graphs with a forbidden induced subgraphs characterization. Let $\mathscr{P}$ = {P | P is a poset with G_{T_P} ϵ $\mathcal{G}$ }. Then $\mathscr{P}$ has a characterization by forbidden ⊲ - preserving subposets.

3. ⊲- preserving posets yielding C-I cograph

Theorem 2: (Theorem 4.1,[11]) Let P be a poset. Then G_P is cograph if and only if P contains none of T_1,. T_7, depicted in Figure 1, and no duals of T_2 and T_5 as ⊲- preserving subposet.

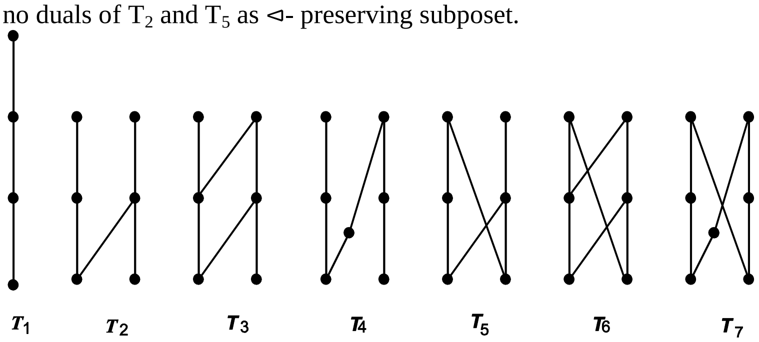
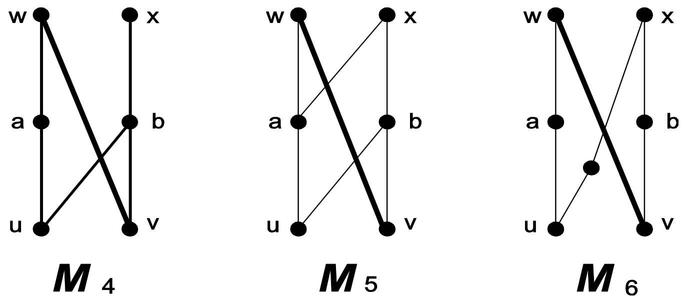


Figure 1: Forbidden ⊲ - preserving subposets for C-I cographs

Theorem 3: If P is a poset, then G_P is cograph if and only if P does not contain T_1 from Figure 1 and no 2-colored diagrams M_4, M_5 and M_6 from Figure 2 are ⊲ -preserving subposets .



Forbidden ⊲ - preserving 2-colored subposets for C-I cographs

Theorem 4: If P is a poset, then G_P is cograph if and only if P does not contain T_1 from Figure 1 and no 3-colored diagram Q_C from Figure 3 and its dual are ⊲ - preserving subposets .

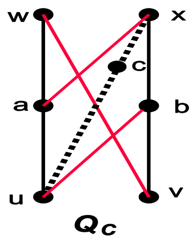


Figure 3: Forbidden ⊲ - preserving 3-colored subposets for C-I cographs

Remarks

The number of forbidden ⊲ - preserving subposets of a poset P is such that its C-I graph G_P belongs to a graph possessing a forbidden induced subgraph characterization as instances of the Theorem 1 is in general very large compared to the number of forbidden induced subgraphs. The idea of 2- colored and 3-colored diagrams is introduced to shorten the list of forbidden ⊲ - preserving subposets.

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