Order Statistics based on Exponential and Weibull Distributions

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Abstract: *Order statistics are widely used in Statistical modeling and its Statistical inference, which describes the random variables that are arranged in order of magnitude, frequently with respect to time. Order statistics deals with the properties and applications of ordered random variables and their functions. This study particularly focuses on two different failure time structures with Exponential and Weibull distributions. In particular, the study uses one parameter Exponential and Three parameter Weibull distributions. Distributions of Ordered statistics based on samples from exponential distribution are derived. Also their Moments, Skewness and Kurtosis were considered. Data were simulated and using them the parameters and moments of the distributions are estimated. All the estimated parameters are found to be very close to the population parameter with which the simulation has been initiated. The parameters are estimated with better accuracy. On the whole the study uses effectively the concept of Order Statistics in the domain of Reliability with focus on estimation and testing of its crucial parameters.*

Key words: *Order Statistics, Moments, Simulation, Parameter Estimation.*

I. Introduction

Order statistics and its functions on statistics play an important role in numerous practical applications. The important applications of order statistics is on many diverse areas, it includes, life testing and reliability, statistical quality control, filtering theory, radar target detection, signal processing, robustness studies, and image processing. The development on order statistics fall on two main categories, they are Statistical estimation, and the testing of statistical hypotheses. Order statistics deals with the properties and applications of ordered random variables and their functions. The study of applications of order statistics on survival analysis provides the way on future possibilities in the recurrence of extreme situations.

Let $X_1,X_2,...,X_n$ be n random variables. Define $X_{1:n}$ as $\min(X_1,X_2,...,X_n)$, $X_{2:n}$the second smallest of $(X_1,X_2,...,X_n)$, $X_{i:n}$ the ith smallest of $(X_1,X_2,...,X_n)$, $X_{n:n}$ the nth smallest (the largest) of $(X_1,X_2,...,X_n)$ or the $\max(X_1,X_2,...,X_n)$. Then we have $X_1{:}n \leq X_{2:n} \leq \cdots \leq X_{n:n}$ and these ordered random variables i.e. $X_{1:n},X_{2:n},...,X_{n:n}$ are known as the order statistics of the given set of random variables. In particular $X_{i:n}$ is called the ith order statistic.

The effects of ordering can be impressive in terms of both the value and the relative relation, and it explains the aspects of sample behavior which are employed in the effectiveness and efficiency of the resulting inferences.

The minimum and maximum are examples of extreme order statistics and they are defined by the following notation.

$$\text{Min}\{X_n\} = X_{1:n}$$
$$\text{Max}\{X_n\} = X_{n:n}$$

In random sampling theory, the unordered X, are assumed to be statistically independent and identically distributed, because, the inequality relations among the order statistics. $X_{i:n}$ are necessarily dependent. There is a difference between the random variables $X_{i:n}$ and the corresponding sample observations X_i.

II. Materials and Methods:

Distributions of Order Statistics

The distribution function of order statistics for maximum $X_{n,n}$ is

$$F_{n;n}(x) = F^n(x)$$
$$F_{1:n} = 1-\{1-F(x)\}^n$$

Now the distribution function of $X_{r:n;}$ the r^{th} order statistics; is denoted as $F_{r:n}(x)$and is given as

$$F_{r:n}(x) = P\{X_{r:n} \leq x\}$$
$$= P\{\text{atleast r of } X_i \text{ are less than or equal to x}\}$$
$$= \frac{1}{B(r,n-r+1)}\int_0^{F(z)} t^{r-1}(1-t)^{n-r}\,dt$$
$$= I_{F(x)}(r,n-r+1)$$

Moments

The moment of the rth order statistics is $\mu_{r:n}$ .knowledge of the means, variances, and co variances of the order statistics are used to find the expected values and variances of the linear function, which are readily used to compute the estimators.

$$\mu_{r:n} = E(X_{r:n}) = C_{r,n}\int_{-\infty}^{\infty} xF^{r-1}(x)[1-F(x)]^{n-r} f(x).dx$$

where,

$$C_{r,n} = \frac{n!}{(n-r)!(r-1)!},$$

and $\mu_{r:n}$ *holds for any cdf* $F(x)$

Order Statistics for Exponential Distribution

The exponential family of distributions is very rich class of distributions with extensive domain of applicability. The univariate exponential distribution is the most commonly used distribution in modeling reliability and life testing analysis. The exponential distribution is used to model the failure times of manufactured items in production.

If X has cumulative distribution function $F(x) = 1-e^{-\lambda x}$, $x > 0$, $\lambda > 0$ then $1 - F_{1:n}(x) = P(X_{1:n} \geq x) = e^{-n\lambda x}$ and $F_{n:n}(x) = P(X_{n:n} \leq x) = (1-e^{-\lambda x})^n$

The probability density function of two parameter exponential distribution is given by

$$f(x;\theta;\eta) = \begin{cases} \frac{1}{\theta}e^{-\frac{(x-\eta)}{\theta}} & ; x > \eta, 0 < \theta < \infty, 0 < \eta < \infty \\ 0 & ; otherwise \end{cases}$$

i.e. $X \sim Exp(\theta,\eta)$

θ *is scale parameter*

η *is location parameter*

Density of the r^th order statistics

Let $x_{1:n}, x_{2:n}, \ldots, x_{n:n}$ be the order statistics of a random sample of size n from the exponential distribution with the following pdf,

$f(x) = e^{-x}$, $x > 0$, the cumulative distributive function of X is $F(x) = 1-e^{-x}$, then the probability density function of $x_{r:n}$ is

$$f_{r:n} = \frac{n!}{(r-1)!(n-r)!}(1-e^{-x})^{r-1}e^{-(n-r+1)x}, \quad x \geq 0$$

When r=1, The minimum order density on exponential distribution is

$$f_{1:n}(x) = ne^{-nx}, \quad x > 0$$

When r = n,

$$f_{n:n}(x) = n(1-e^{-x})^{n-1}.e^{-x}, \quad x>0.$$

The Probability density function of first order statistics is also exponential distribution with parameters $\frac{\theta}{n}$ *and* η

Theorem: Suppose $Y = \frac{(x-\eta)}{\theta} \sim Exp(1, 0)$, and consider the differences of the consecutive order statistics

$\mathbf{W}_1 = \mathbf{Y}_{1:n}$, $\mathbf{W}_2 = Y_{2:n} - \mathbf{Y}_{1:n}$, ..., $W_n = Y_{n:n} - Y_{n-1:n}$ *then,*

i. $W_1, \mathbf{W}_2, \mathbf{W}_3, \ldots, W_n$ *are mutually independent.*

ii. $W_i \sim Exp\left(\frac{1}{n-i+1}\right)$

iii. $Y_{k:n} = \sum_{j=1}^{k}\frac{Z_j}{(n-j+1)}$, *where* $Z_j \sim Exp(1,0)$ *and the* Z_j *are mutually independent.*

iv. $E(Y_{k:n}) = \sum_{j=1}^{k}(n-j+1)^{-1}$ *and* $Var(Y_{k:n}) = \sum_{j=1}^{k}(n-j+1)^{-2}$

Central Moments of order Statistics

For exponential order Statistics

$Y_{1:n}, Y_{2:n}, \ldots, Y_{n:n}$,

$$(Y_{1:n}, Y_{2:n}, \ldots, Y_{n:n}) \overset{d}{\approx} \left(\frac{z_1}{n}, \frac{z_1}{n}+\frac{z_2}{n-1}, \ldots, \frac{z_1}{n}+\frac{z_2}{n-1}+\ldots+\frac{z_{n-1}}{2}+z_n\right)$$

For Third central moment of kth order statistics

$$E[Y_{k:n} - E(Y_{k:n})]^3 = \sum_{r=1}^{k}\frac{1}{(n-r+1)^3}E(z-1)^3$$

where z has the standard e(1) exponential distribution

Generally,

$$E(z^r) = \Gamma(r+1)$$

$$E(z-1)^3 = \Gamma(4) - 3\Gamma(3) + 3\Gamma(2) - 1 = 2$$

$$E[Y_{k:n} - E(Y_{k:n})]^3 = 2\sum_{r=1}^{k}\frac{1}{(n-r+1)^3}$$

similarly,

$$E[Y_{k:n} - E(Y_{k:n})]^4 = \sum_{r=1}^{k}\frac{1}{(n-r+1)^4}E(z-1)^4$$

$$E(z-1)^4 = E(z^4) - 4E(z^3) + 6E(z^2) - 4E(z) + 1 = \Gamma(5) - 4\Gamma(4) + 6\Gamma(3) - 4\Gamma(2) + 1 = 9$$

$$\therefore \quad \mu_4 = 9\sum_{r=1}^{k}\frac{1}{(n-r+1)^4}$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

Corollary:

If $X_{j:n} \sim \exp(\theta)$ then,

$$i)\ \ E(X_{r:n}) = \theta \sum_{j=1}^{r} \frac{1}{(n-j+1)}$$

$$ii)\ \ V(X_{r:n}) = \theta^2 \sum_{j=1}^{r} \frac{1}{(n-j+1)^2}$$

ORDER STATISTICS FOR WEIBULL DISTRIBUTION

Three parameter Density of Weibull Distribution

A random variable X has a three–parameter WEIBULL distribution with parameters a, b and c if its probability density functions is given by

$$f_X(x\,|\,a,b,c) = \frac{c}{b}\left(\frac{x-a}{b}\right)^{c-1} \exp\left\{-\left(\frac{x-a}{b}\right)^{c}\right\}, x \geq a$$

i.e.

X~ We(a,b,c)

$a \to$ *location parameter or failure free time*

$b \to$ *scale parameter*

$c \to$ *shape parameter* (*Rinne*, 2009)

The CDF of a WEIBULL random variable can be transformed to a straight line.

The density of the reduced Weibull distribution is

$$f_U(u\,|\,c) = c\,u^{c-1} \exp(-u^{c})\ ;\ c > 0,\ u \geq 0$$

where $u = \frac{x-a}{b}$

General formulae

Moments and cumulants are the expected values of certain functions of random variables. It describes numerically, with respect to given characteristics such as location, variation, skewness and kurtosis.

The moments about zero plays a key role to find all other of moments, For the random variable X = a +b U, the moments of the reduced Weibull random variable can be obtained from the transformed U- moments. The expected value of X_r is termed the r[th] moment about zero of the random variable X:

$$\mu_r'(X) = E(X^r)$$

Where.ris any real number, but for the most part r is taken as a non–negative integer. With regard to the reduced Weibull variable we get

$$\mu_r'(U) = E(U^r) = \int_0^\infty u^r f_U(u\,|\,c)du$$

Order Statistics for Weibull Distribution

Let X_i ~We (a, b, c);i=1,2,3…,n; with pdf and cdf

$$f(x) = \frac{c}{b}\left[\frac{(x-a)}{b}\right]^{c-1} \exp\left\{-\left(\frac{x-a}{b}\right)^{c}\right\},$$

$$F(x) = 1 - \exp\left\{-\left(\frac{x-a}{b}\right)^{c}\right\}$$

First order density on Weibull

The random variable $X_{1:n}$ is minimum and the density function of the minimum order statistics on Weibull is

$$f_{1:n}(x) = n.f(x)[1-F(x)]^{n-1}$$

$$= \frac{nc}{b}\left(\frac{(x-a)}{b}\right)^{c-1} \exp\left\{-n\left(\frac{(x-a)}{b}\right)^{c}\right\}$$

$$= \frac{c}{b*}\left(\frac{(x-a)}{b}\right)^{c-1} \exp\left\{\left(\frac{(x-a)}{b*}\right)^{c}\right\}$$

Density function of r[th] Order Statistics

The density function of the r [th] order statistics 1 ≤ r ≤ n, of a Weibull variate, U~ We(0,1,c) with the distribution function

X ~We (a, b, c) ,by using the linear transformation

$X_{r:n} = a + bU_{r:n}$,

The DF of $U_{r:n}$, 1 ≤ r ≤ n, is

$$f_{r:n}(u) = r\binom{n}{r} c\,u^{c-1} \exp\{-u^{(n-r+1)c}\}[1-\exp(-u^{c})]^{r-1}$$

Where,

$$f(u) = cu^{c-1}\exp\left\{-u^{c}\right\}$$

and CDF

$$F(u) = 1 - \exp\left\{-u^{c}\right\},$$

where,

$$u = \frac{(x-a)}{b}$$

.

$$E(U_{r:n}^{k}) = r\binom{n}{r}\int_0^\infty u^k cu^{c-1}\exp\left\{-u^{(n-r+1)c}\right\}\left[1-\exp(-u^c)\right]^{r-1}du$$

$$= r\binom{n}{r}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\int_0^\infty u^k cu^{c-1}\exp\left\{-(n-r+i+1)u^c\right\}du$$

$$= r\binom{n}{r}\Gamma(1+\frac{k}{c})\sum_{i=0}^{r-1}\frac{(-1)^i\binom{r-1}{i}}{(n-r+i+1)^{1+\frac{k}{c}}}$$

$$\therefore\quad X_{1:n} \sim We(a,b^*,c)$$

When X ~ We(a,b,c) then $X_{1:n} \sim We(a,b^*,c)$

where $b^* = bn^{-\frac{1}{c}}$

III. Results and Discussion:

Application of exponential order statistics for simulated data

Data have been simulated from exponential distribution with parameters λ =1 and 11 and from Three parameter Weibull distribution with parameters 15, 30 and 2.5. From each distribution a sample of size 10 to 50 with constant difference 10 is taken and simulated. Using the generated data the mean, variance, the first four moments and the corresponding skewness and Kurtosis are also estimated to study the nature of the distribution. Theoretical mean and variance are also estimated

Table I: Theoretical and sample moments of exponential distribution with parameter λ=1 for order samples of size 10 to 50

	X k:n	**Theoretical .mean**	**Sample .mean**			**Theoretical variance**	**Sample .variance**			**m3**	**m4**	**b1**	**b2**
n	k		s=10	s=50	s=100		s=10	s=50	s=100				
10	1	0.1000	0.1286	0.0899	0.0980	0.0100	0.0135	0.0109	0.0074	0.0020	0.0009	4.0000	9.0000
	5	0.6456	0.8275	0.5399	0.6516	0.0862	0.1247	0.0666	0.0696	0.0237	0.0152	0.8812	2.0426
	10	2.9290	3.7416	2.8489	2.9408	1.5498	4.9384	1.5166	1.2864	2.3951	9.7383	1.5411	4.0546
20	1	0.0500	0.0227	0.0552	0.0637	0.0025	0.0007	0.0025	0.0046	0.0003	0.0001	4.0000	9.0000
	10	0.6688	0.7322	0.6540	0.6483	0.0464	0.0673	0.0470	0.0452	0.0067	0.0022	0.4457	1.0369
	20	3.5977	3.6785	3.6873	3.5236	1.5962	1.1531	2.4314	1.4093	2.4017	9.7406	1.4185	3.8232
30	1	0.0333	0.0263	0.0314	0.0302	0.0011	0.0003	0.0009	0.0013	0.0001	0.0000	4.0000	9.0000
	15	0.6768	0.6995	0.7000	0.6829	0.0317	0.0168	0.0382	0.0402	0.0031	0.0007	0.2982	0.6944
	30	3.9950	4.1283	4.0935	4.1539	1.6122	1.3931	1.2451	1.6530	2.4030	9.7408	1.3782	3.7479
40	1	0.0250	0.0317	0.0286	0.0282	0.0006	0.0016	0.0007	0.0008	0.0000	0.0000	4.0000	9.0000
	20	0.6808	0.6097	0.6887	0.6845	0.0241	0.0175	0.0206	0.0252	0.0018	0.0003	0.2240	0.5219
	40	4.2785	4.1284	4.3502	4.4903	1.6202	1.5437	1.3831	1.7848	2.4035	9.7409	1.3582	3.7105
50	1	0.0200	0.0244	0.0186	0.0174	0.0004	0.0008	0.0002	0.0003	0.0000	0.0000	4.0000	9.0000
	25	0.6832	0.6721	0.7185	0.7038	0.0194	0.0222	0.0202	0.0206	0.0011	0.0002	0.1794	0.4181
	50	4.4992	4.4689	4.5252	4.3036	1.6251	0.9117	1.9305	1.1616	2.4037	9.7409	1.3462	3.6883

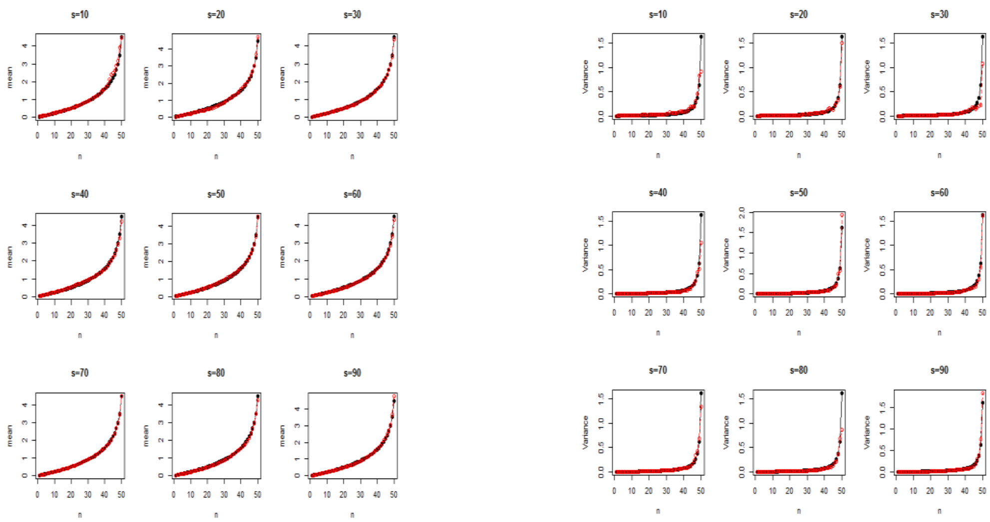

Fig. 1 Comparison of Theoretical and sample mean and variance for different iterations using exponential distribution with λ =1, n=50

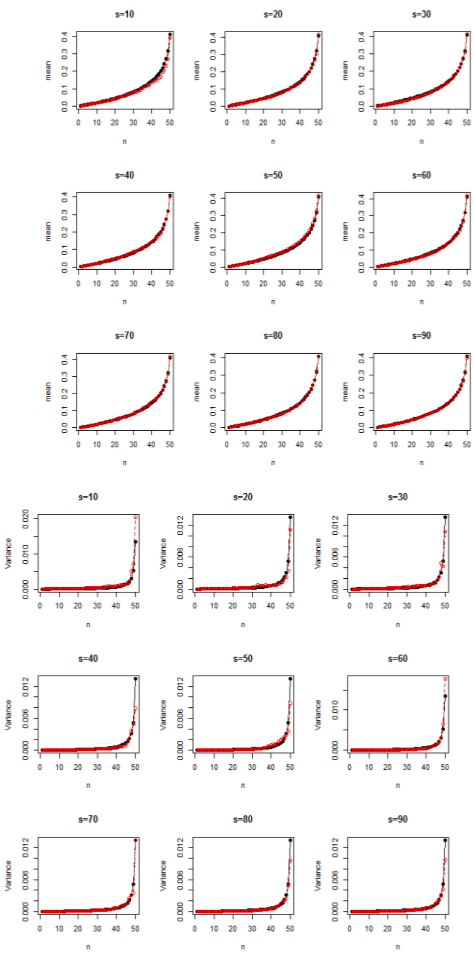

Fig. 2 Comparison of Theoretical and sample mean and variance for different iterations using exponential distribution with λ =11, n=50

Table II: Theoretical and sample moments of exponential distribution with parameter λ=11 for order samples of size 10 to 5

		lamda=11											
	X $_{k:n}$	**The.mean**	**Sample mean**			**Theoretical .variance**	**Sample variance**			**m3**	**m4**	**b1**	**b2**
n	k		s=10	s=50	s=100		s=10	s=50	s=100				
10	1	0.009091	0.00934	0.007594	0.009172	0.000083	0.000069	0.000055	0.000075	0.000002	0	4	9
	5	0.058694	0.05307	0.052218	0.060108	0.000712	0.000791	0.00046	0.000746	0.000018	0.000001	0.88124	2.04257
	10	0.26627	0.35489	0.24433	0.239466	0.012808	0.031276	0.008562	0.008011	0.001799	0.000665	1.54111	4.05463
20	1	0.004545	0.00484	0.005004	0.004014	0.000021	0.000013	0.000024	0.000016	0	0	4	9
	10	0.060797	0.05596	0.061804	0.058674	0.000383	0.000366	0.000258	0.000332	0.000005	0	0.4457	1.03693
	20	0.327067	0.35385	0.323152	0.341059	0.013191	0.024549	0.016477	0.012634	0.001804	0.000665	1.41847	3.82322
30	1	0.00303	0.00167	0.002668	0.00282	0.000009	0.000003	0.000009	0.000009	0	0	4	9
	15	0.061523	0.05337	0.064012	0.059749	0.000262	0.00027	0.000316	0.000234	0.000002	0	0.29816	0.69443
	30	0.363181	0.31583	0.390232	0.361854	0.013324	0.00788	0.019184	0.012374	0.001805	0.000665	1.37818	3.74786
40	1	0.002273	0.00207	0.002156	0.002167	0.000005	0.000004	0.000004	0.000005	2.35E-08	2.40E-10	4	9
	20	0.061891	0.06473	0.063722	0.062609	0.000199	0.000259	0.000207	0.000223	0.000001	2.07E-08	0.22399	0.52194
	40	0.388958	0.43721	0.385788	0.369881	0.01339	0.01405	0.010286	0.012913	0.001806	0.000665	1.35815	3.71054
50	1	0.001818	0.0029	0.001578	0.001993	0.000003	0.000014	0.000002	0.000004	0	0	4	9
	25	0.062113	0.05878	0.06742	0.063611	0.00016	0.000242	0.000163	0.000191	0.000001	0	0.17936	0.41807
	50	0.409019	0.38832	0.417082	0.423742	0.013431	0.02039	0.008872	0.016327	0.001806	0.000665	1.34618	3.68825

The convergence of estimated moments and that of the theoretical moments greatly depend on the number of iterations. As this number increases indefinitely the difference between the actual and estimates of the parameters (moments, skewness and kurtosis) narrow down and it vanishes after a considerable number of iterations. This number of iterations is dependent on the number of samples, from which the order statistics are estimated. As the sample size increases the stability in the probability distribution also increases, with increased number of iterations. i.e. for a sample of size n_1 the number of iterations required namely, s is a function of n, in order to achieve sufficient closeness between the actual and estimated values. i.e., s=f(n). Going by the graphical output of the estimated values, it is seen that this function is monotonic increasing.

Table III: Theoretical parameter of Weibull distribution with parameters a=15, b=30 and c=2.5 for order samples of sizes 10.20,30,40 and 50

		Theoretical .mean	**Sample .mean**			**Theoretical .variance**	**Sample variance**			**m3**	**m4**	**b1**	**b2**
n			s=10	s=50	s=100		s=10	s=50	s=100				
	$x_{1:10}$	25.596	25.837442	25.868875	25.139533	20.564784	21.921067	21.044477	16.109595	33.11846	1216.09	0.12612	2.87553
	$x_{5:10}$	39.567	40.451782	39.328521	39.590028	20.522511	8.052504	18.998655	21.262834	14.26493	1069.085	0.02354	2.53835
10	$x_{10:10}$	60.186	60.076414	60.828304	60.96414	56.035404	60.666947	71.11183	60.340262	179.3258	9939.449	0.18277	3.16546
	$x_{1:20}$	23.031	22.76057	22.159379	22.726682	11.823039	6.154272	15.525073	8.84258	15.27243	332.23	0.14113	2.37673
	$x_{10:20}$	40.224	38.499665	40.238202	40.194758	10.759824	12.728628	11.792716	9.979965	1.208159	379.1757	0.00117	3.27514
20	$x_{20:20}$	64.38	62.362653	63.54724	64.798312	45.1656	65.960874	30.176174	59.816946	139.2463	7829.013	0.21045	3.83788
	$x_{1:30}$	21.828	22.949529	21.246454	21.379601	8.548416	8.764717	7.591909	7.665981	8.762071	217.8741	0.1229	2.9815
	$x_{15:30}$	40.449	41.217566	40.103374	40.435942	7.278399	7.147552	9.775903	6.653037	1.930478	-2.31178	0.00967	-0.0436
30	$x_{30:30}$	66.624	61.905227	65.192632	65.390143	40.002624	27.779069	37.546575	41.470551	148.3241	4835.519	0.34368	3.0218

	$x_{1:40}$	21.087	21.756693	21.520428	21.796698	6.778431	9.351857	4.910249	8.31396	7.086171	106.73812	0.161226	2.323064
	$x_{20:40}$	45.777	40.23948	40.12369	40.297232	-217.323729	4.804921	6.809261	4.479061	8822.1125	-300979.3	-7.5827	-6.37268
40	$x_{40:40}$	68.133	69.73842	69.098811	67.973435	37.084311	42.300614	31.266623	36.319881	140.09201	3848.9708	0.384819	2.798751
	$x_{1:50}$	20.568	21.029398	20.499706	20.479494	5.627376	4.398537	4.773967	6.205172	6.0777009	43.694218	0.207282	1.379787
	$x_{25:50}$	-84958.07	39.747296	41.216237	40.495275	-7220166157	4.535407	3.828967	3.878593	-1.23E+15	-1.56E+20	-4	-3
50	$x_{50:50}$	69.69	67.758563	68.980392	70.120236	-10.1061	21.046002	45.600466	34.295147	3714.7311	-256415.4	-13369.2	-2510.6

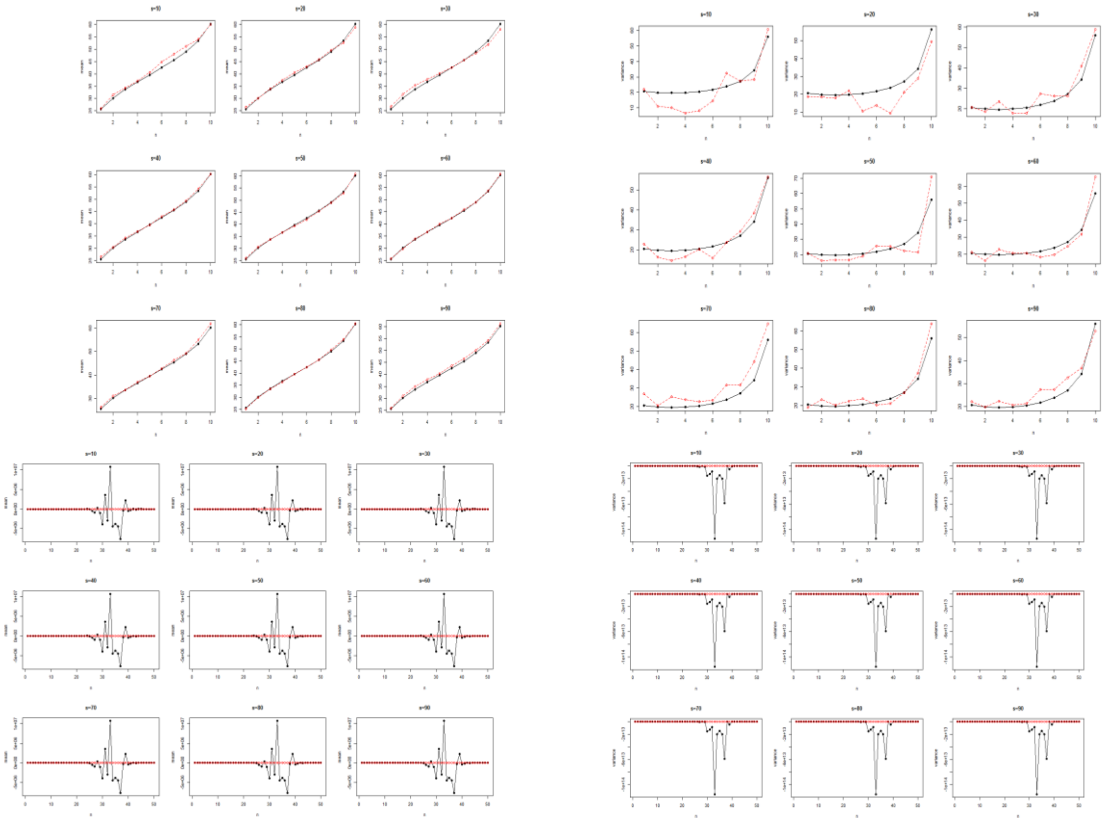

Fig. 3 Comparison of Theoretical and sample mean and variance for different iterations for Weibull three parameter distributions with parameters (15, 30, and 2.5) for different iterations for 10 and 50 samples

IV CONCLUSION

The simulation process exhibits certain patterns in the relationship among the number of samples used, the number of iterations used in the simulation and the order of the order statistics considered. Also, the comparison over exponential and Weibull distributions indicate the issue of dimensions, in that more samples are required to get consistent estimates of the population parameters (Mean). This pattern gets noticed on comparison between the exponential and Weibull distributions. In case of exponential distributions, we observe relatively less variance around the sample mean in exponential distribution, as it contains less number of parameters compared to its Weibull Generalization Model. The Graphical representation of the estimates, in case of Weibull distribution shows higher degree of oscillations in case of Mean estimation. This is mainly due to the increased number of parameters (THREE) compared to that of the exponential distributions (ONE Parameter).

We observe that the estimates of skewness and kurtosis remains unchanged in case of first order statistics, for varying sample size, simulation size and the order of the statistics. This is because the first order statistics behaves more like a simple individual sample from the parent population. The Higher order statistics were decided by complex combinatoric laws and thereby have variant estimates across different sample size, simulation size and the order of the statistics. These observations, the author has tested using continuous changes in the simulation size, but the results for selected values such as s=10,50 and 100 were presented in the table.

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