Common Fixed Point Theorems for Four Maps in d-Complete Topological Spaces

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Abstract - *The purpose of this paper is to prove some common fixed point theorems for four self maps satisfying contractive conditions in d-complete topological spaces using w-continuity. Our results are generalization and enhancement of the results of Cho and Lee and also of Hicks and Rhoades.*

Keywords - *Commuting maps, d-complete topological spaces, Common fixed point, w-continuous function.*

I. INTRODUCTION

In 1975, d-complete topological space was introduced by Kasahara[8] as a generalization of complete metric space**.** After that several fixed point theorems were proved by Hicks[4], Hicks and Rhoades[5,6], Cho and Lee[2] and many other researchers in a large class of non-metric spaces so called d-complete topological spaces. The basic idea of d-complete topological space can understand from Kasahara[8], Iseki[7], Hicks[3] and their L-spaces. Several fixed point results were proved in many directions[1,9].

Let (X, t) be a topological space and d: $X \times X \to [0,\infty)$ satisfying

(i) $d(x, y) = 0$ *if and only if* $x = y$,

(ii) *for any sequence* $<x_n>$ *in X,*
$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty \Rightarrow <x_n>$ *is convergent in (X, t)*

Then (X, t, d) is called a d-complete topological space.

In 2010, Cho and Lee[2] proved fixed point theorems for three self maps in d-complete topological spaces by generalizing fixed point theorems of Hicks and Rhoades[6] for two self maps.

In the present work, we generalize the results of Cho and Lee[2] to four self maps.

A function f : X→X is **w-continuous** at $x \in X$ if $\lim_{n\to\infty} x_n = x$ implies $\lim_{n\to\infty} fx_n = fx$.
Denote $\wedge$ as the family of all nondecreasing and continuous functions $\phi: R^+ \to R^+$ such that
(i) $\phi(0) = 0$ and $0 < \phi(t) < t$ for all $t > 0$,
(ii) $\sum_{n=0}^{\infty} \phi^n(t) < \infty$ for all $t > 0$, where $\phi^n(t)$ is n^{th} iteration of $\phi(t)$.
Note that $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t > 0$.
Now, before proving the theorem, we state a lemma.
Lemma 1.1[2] Let (X, t) be a d-complete topological space and $<x_n>$ be a sequence in X. If $d(x_n, x_{n+1}) \le \phi(d(x_{n-1}, x_n))$ for $n = 1,2,3,\ldots$, and $\phi \in \wedge$ then the sequence $<x_n>$ converges to a point x in (X, t).

II. MAIN RESULTS

Theorem 2.1 For a Hausdorff d-complete topological space (X, t); A, B and S are w-continuous functions from X to itself. The mappings A, B and S have a common fixed point in X if and only if there exists a function $\phi \in \wedge$ and w-continuous function $T : X \to X$ such that for all $x, y \in X$,

a) T commute with A, B and S,

b) $T(X) \subset A(X) \cap B(X) \cap S(X)$,

c) $d(Tx, Ty) \le \phi(\max\{d(Ax, Sy), d(Ax, By), d(Ax, Tx), d(Sy, Ty), d(By,Ty)\})$.

Also, if above three conditions hold then A, B, S and T have a unique common fixed point in X.

Proof . Suppose $u \in X$ is a common fixed point of A, B and S. Put T(x) = u for all $x \in X$. Then, T commute with A, B and S. Also $T(x) \subset A(X) \cap B(X) \cap S(X)$ and T is w-continuous function.
$d(Tx, Ty) = d(u , u) = 0 \le \phi(\max\{d(Ax, Sy), d(Ax, By), d(Ax, Tx), d(Sy, Ty), d(By, Ty)\})$, for $\phi \in \wedge$ and $x, y \in X$.
Suppose that there exist a w-continuous function $T : X \to X$ and $\phi \in \wedge$ such that (a), (b), (c) are satisfied.
Let $x_0 \in X$ and $< x_n >$ be such that $Tx_n = Ax_{n+1} = Bx_{n+1} = Sx_{n+1}$.
If $d(Tx_{n-1}, Tx_n) < d(Tx_n,Tx_{n+1})$ for some $n \in N$, then

$$d(Tx_n,Tx_{n+1}) \le \phi(\max\{d(Ax_n, Sx_{n+1}), d(Ax_n, Bx_{n+1}), d(Ax_n, Tx_n), d(Sx_{n+1}, Tx_{n+1}), d(Bx_{n+1}, Tx_{n+1})\})$$
$$\le \phi(\max\{d(Tx_{n-1}, Tx_n), d(Tx_{n-1}, Tx_n), d(Tx_{n-1}, Tx_n), d(Tx_n, Tx_{n+1}), d(Tx_n, Tx_{n+1})\})$$

$= \phi(\max\{d(Tx_n, Tx_{n+1})\})$
$< d(Tx_n, Tx_{n+1})$,
which is a contradiction. Thus we have $d(Tx_n,Tx_{n+1}) \leq d(Tx_{n-1}, Tx_n)$, for all n.
Then, we have
$d(Tx_n,Tx_{n+1}) \leq \phi(\max\{d(Ax_n , Sx_{n+1}), d(Ax_n, Bx_{n+1}), d(Ax_n, Tx_n), d(Sx_{n+1}, Tx_{n+1}), d(Bx_{n+1}, Tx_{n+1})\})$
$\leq \phi(\max\{d(Tx_{n-1}, Tx_n), d(Tx_{n-1}, Tx_n), d(Tx_{n-1}, Tx_n), d(Tx_n, Tx_{n+1}), d(Tx_n, Tx_{n+1})\})$
$= \phi(d(Tx_{n-1}, Tx_n))$.
By lemma 1.1, there exists a $p \in X$ such that $Tx_n \to p$ as $n \to\infty$. Then $Ax_n \to p$, $Bx_n \to p$, $Sx_n \to p$ as $n \to \infty$. A, B, S and T are w-continuous, therefore $ATx_n \to Ap$, $TAx_n \to Tp$, $TBx_n \to Tp$, $BTx_n \to Bp$, $STx_n \to Sp$, $TSx_n \to Tp$ as $n \to \infty$.
From (a), Ap = Bp = Sp = Tp since (X, t) is a Hausdorff.
Thus, u = Ap = Bp = Sp = Tp.
From (a), we have STp = TTp = BTp.
$d(u,Tu) = d(Tp,TTp)$
$\leq \phi(\max\{d(Ap, STp), d(Ap, BTp), d(Ap, Tp),d(STp,TTp),d(BTp, TTp)\})$
$= \phi(\max\{d(Tp, TTp), d(Tp, TTp), d(Tp, Tp), d(TTp, TTp), d(TTp, TTp)\})$
$= \phi(d(Tp, TTp))$
$= \phi(d(u, Tu))$,
which implies that d(u,Tu) = 0 and therefore u = Tu. Thus, we find u = Tu = Au = Bu = Su.
Now, for the uniqueness, let z = Tz = Az = Bz = Sz.
Then,
$d(u, z) = d(Tu, Tz)$
$\leq \phi (\max\{d(Au, Sz), d(Au, Bz), d(Au, Tu), d(Sz,Tz) d(Bz, Tz)\})$
$= \phi (\max\{d(u, z), d(u, z), d(u, u), d(z, z), d(z, z)\})$
$= \phi (d(u,z))$.
Thus, we find z = u.
This completes the proof.

If we have $\phi(t) = kt$, $k \in (0,1)$, $t \geq 0$ in theorem 2.1, then we obtain the following theorem proved by Cho and Lee[2].

Corollary 2.1 For a Hausdorff d-complete topological space (X, t); A, B and S are w-continuous functions from X to itself. The mappings A, B and S have a common fixed point in X if and only if there exists $k \in (0,1)$ and w-continuous function $T : X \to X$ such that for all $x, y \in X$,

a) T commute with A, B and S,
b) $T(X) \subset A(X)\cap B(X)\cap S(X)$,
c) $d(Tx, Ty) \leq k \max\{d(Ax, Sy), d(Ax, By,) d(Ax, Tx), d(Sy, Ty), d(By,Ty)\}$.

Also, if above three conditions hold then A, B, S and T have a unique common fixed point in X.

If we take B = S in theorem 2.1, then we obtain the following theorem proved by Cho and Lee[2].

Corollary 2.2 For a Hausdorff d-complete topological space (X, t); A and B are w-continuous functions from X to itself. The mappings A and B have a common fixed point in X if and only if there exists a function $\phi \in \wedge$ and w-continuous function $T : X \to X$ such that for all $x, y \in X$,

a) T commute with A and B,
b) $T(X) \subset A(X)\cap B(X)$,
c) $d(Tx, Ty) \leq \phi(\max\{d(Ax, By,) d(Ax, Tx), d(By,Ty)\}$.

Also, if above three conditions hold then A, B and T have a unique common fixed point in X.

Theorem 2.2 For a Hausdorff d-complete topological space (X, t); A, B and S are w-continuous functions from X to itself. The mappings A, B and S have a common fixed point in X if and only if there exist non-negative constants a_i $(1\leq i \leq 5)$ satisfying $a_1+a_2+a_3+a_4+a_5 < 1$ and w-continuous function $T : X \to X$ such that for all $x, y \in X$,

a) T commute with A, B and S,
b) $T(X) \subset A(X)\cap B(X)\cap S(X)$,
c) $d(Tx, Ty) \leq a_1d(Ax, Sy) + a_2d(Ax, By) + a_3d(Ax, Tx) + a_4d(Sy, Ty) + a_5d(By,Ty)$.

Also, if above three conditions hold, then A, B, S and T have a unique common fixed point in X.

Proof:

Let u = Au = Bu = Su and let Tx = u for all $x \in X$. Then, conditions (a), (b) and (c) are satisfied.

Assume that there exist non-negative constants a_i satisfying $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ and w-continuous function $T : X \to X$ such that (a), (b) and (c) are satisfied. Then, as in the proof of theorem 2.1, we have a sequence $<x_n>$ in X such that $Tx_n = Ax_{n+1} = Bx_{n+1} = Sx_{n+1}$.

Then, we have

$d(Tx_n, Tx_{n+1}) \leq a_1d(Ax_n, Sx_{n+1}) + a_2d(Ax_n, Bx_{n+1}) + a_3d(Ax_n, Tx_n) + a_4d(Sx_{n+1},Tx_{n+1}) + a_5d(Bx_{n+1},Tx_{n+1})$

$\leq a_1d(Tx_{n-1}, Tx_n)+a_2d(Tx_{n-1}, Tx_n) + a_3d(Tx_{n-1}, Tx_n) + a_4d(Tx_n, Tx_{n+1}) + a_5d(Tx_n, Tx_{n+1})$,

which implies

$d(Tx_n, Tx_{n+1}) \leq kd(Tx_{n-1}, Tx_n)$, where $k = \frac{a_1+a_2+a_3}{1-a_4-a_5}$.

By lemma 1.1, there exists a point $p \in X$ such that $Tx_n \to p$ as $n \to \infty$. Then $Ax_n \to p$, $Bx_n \to p$, $Sx_n \to p$ as $n \to \infty$.

As in the proof of theorem 2.1, u = Ap = Bp = Sp = Tp. From (a), we have STp = TTp = BTp . Thus, we obtain

d(u, Tu) = d(Tp, TTp)

$\leq a_1d(Ap, STp) + a_2d(Ap, BTp) + a_3d(Ap, Tp) + a_4d(STp,TTp) + a_5(BTp,TTp)$.

$= a_1d(Tp, TTp) + a_2d(Tp, TTp) + a_3d(Tp, Tp) + a_4d(TTp, TTp) + a_5d(TTp, TTp)$.

$= (a_1 + a_2)\, d(Tp, TTp)$

$= (a_1 + a_2)\, d(u, Tu)$.

This implies d(u, Tu) = 0 and so u = Tu. Thus we obtain u = Au = Bu = Su = Tu.

Now, for the uniqueness, assume that z = Az = Bz = Sz = Tz.

Then, we have

d(u, z) = d(Tu, Tz)

$\leq a_1d(Au, Sz)+ a_2\, d(Au, Bz)+ a_3\, d(Au, Tu)+ a_4\, d(Sz,Tz)+ a_5\, d(Bz, Tz)$

$= a_1\, d(u, z)+ a_1\, d(u, z)+ a_1\, d(u, u)+ a_1\, d(z, z)+ a_1\, d(z, z)$

$= (a_1 + a_2)\, d(u, z)$.

Thus, we obtain z = u. This completes the proof.

If we take B = S in theorem 2.2, then we obtain the following theorem proved by Cho and Lee[2].

Corollary 2.3: For a Hausdorff d-complete topological space (X,t); A and B are w-continuous functions from X to itself. The mappings A and B have a common fixed point in X if and only if there exist non-negative constants b_i ($1 \leq i \leq 3$) satisfying $b_1+b_2+b_3 <1$ and w-continuous function $T : X \to X$ such that for all $x, y \in X$,

a) T commute with A and B,
b) $T(X) \subset A(X) \cap B(X)$,
c) $d(Tx, Ty) \leq b_1d(Ax, By) + b_2d(Ax, Tx) + b_3d(By,Ty)$.

Also, if above three conditions hold then A, B and T have a unique common fixed point in X.

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