Construction of Some (r,λ)-designs from Generalized Row Orthogonal Matrices

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Abstract: *Some series of (r,λ)-designs have been constructed from Generalized Row Orthogonal Matrices and Balanced Incomplete Block Designs. It is shown that a Generalized Row Orthogonal Matrix is the incidence matrix of an (r, λ)-design under certain conditions.*
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I. INTRODUCTION

We recall the following definitions from Dey [2]

1.1 Block Design

Let V={1,2,3,…,v} be a non-empty set and β={$\beta_1,\beta_2,\ldots,\beta_b$} be a multiset of subsets of V. Then (V,β) is a block design. The elements of V are called treatments and the elements of β are called blocks.

1.2Balanced Incomplete Block Design (BIBD):

Let V={1,2,3,…,v} be a non-empty set and β={$\beta_1,\beta_2,\ldots,\beta_b$} be a multiset of subsets of V. The elements of V are called treatments and the elements of β are called blocks. A BIBD is an arrangement of v treatments into b blocks such that each block contains k treatments, each treatment belongs to r blocks and each pair of treatments belongs to λ blocks. v,b,r,k,λ are called parameters of the BIBD. These parameters are not all independent but are related by the following relations:

(i) vr=bk (ii) r(k-1)=λ(v-1).

A BIBD for which v=b is called a Symmetric BIBD (SBIBD).

1.3 (r,λ)-Design

An (r,λ)-design is a block design (V,β) such that

(i) Every element of V occurs in precisely r blocks.

(ii) Every pair of distinct elements of V occurs in precisely λ blocks.

1.4Circulant Matrix:Ann × n matrix

$C = [c_{ij}]_{0\le i,j\le n-1}$ where $c_{ij} = c_{j-i(\mathrm{mod}\, n)}$ is a circulant matrix of order n.

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{pmatrix}$$

$$= \mathrm{circ}(c_0, c_1, \ldots, c_{n-1})$$

Deza et al[3], Vanstone[11] and so on contributed in the constructions of an (r,λ)-design. BIBDs have been studied byBose[1],Kageyama[4],Rao[5],Shrikhande and Raghvarao[6],Yalavigi[12] and so on. Singh and Prasad [7] defined Generalized Orthogonal Combinatorial matrix (GOCM). Saurabh and Singh [9] defined Generalized Row Orthogonal Matrices (GROM) and Generalized Row Orthogonal Constant Column Matrices (GROCM) and have shown that a GROCM is in general an incidence matrix of at most three class association scheme. In this paper we have constructed some (r,λ)-designs from GROM and BIBD.

For convenience,I_n denotes the identity matrix of order n, $J_{t\times u}$ denotes the $t \times u$ matrix all of whose entries are 0,$K_{t\times u} = J_{t\times u} - I_{t\times u}$.$e_{t\times 1}$denotes the $t \times 1$ matrix with all its entries 1.$A \otimes B$denotes Kronecker product of matrices A and B.$\alpha^i = \mathrm{circ.}(0,0,0,\ldots 1,\ldots,0)$is a circulant matrix of order n with 1 at (i+1)-th position such that $\alpha^n = I_n$.

II. GROM AND ITS REDUCTION TO AN INCIDENCE MATRIX OF AN(r,λ)-DESIGN

We recall the definition of GROM. Let N=[N_{ij}], i, j∊{1,2,.. .,m } where N_{ij} are {0,1} matrices oforder $n \times s_j$. Let $R_i = (N_{i1}, N_{i2},...,N_{im})$ be the ith row of blocks. We define inner product of two rows of blocks R_i and R_j as $R_i \circ R_j = R_iR_j^T = \sum_{k=1}^{m} N_{ik}N_{jk}^T$.

N is called a Generalized Row Orthogonal Matrices (GROM) if there exists fixed positive integer r and fixed non-negative integers $\lambda_1, \lambda_2, \lambda_3$ such that

$$R_i \circ R_j = R_iR_j^T = \sum_{k=1}^{m} N_{ik}N_{jk}^T = \begin{cases} rI_n + \lambda_1 K_n \text{ if } i = j \\ \lambda_2 I_n + \lambda_3 K_n \text{ if } i \neq j. \end{cases}$$

Clearly $v = mn, b = m(s_1 + s_2 + \cdots + s_m)$.$v, b, r, s_1, s_2, \dots, s_m, \lambda_1, \lambda_2, \lambda_3, m, n$are called the parameters of the GROM.$\lambda_1, \lambda_2, \lambda_3$will be also called concurrences of the GROM.

Theorem 2.1:A GROM with concurrences $\lambda_1, \lambda_2, \lambda_3$is an incidence matrix of an (r, λ)-design if$\lambda_1 = \lambda_2 = \lambda_3$.

proof: Let N be a GROM and let

$\lambda_1 = \lambda_2 = \lambda_3 = \lambda$. Then

$$NN^T = \begin{pmatrix} rI_n + \lambda_1 K_n & \cdots & \lambda_2 I_n + \lambda_3 K_n \\ \vdots & \ddots & \vdots \\ \lambda_2 I_n + \lambda_3 K_n & \cdots & rI_n + \lambda_1 K_n \end{pmatrix}$$

$$= r(I_m \otimes I_n) + \lambda_1(I_m \otimes K_n) + \lambda_2(K_m \otimes I_n) + \lambda_3(K_m \otimes K_n) = rI_{mn} + \lambda K_{mn}$$

HenceN is an incidence matrix of an (r,λ)-design with parameters

$v = mn, b = m(s_1 + s_2 + \cdots + s_m), r, \lambda$.

III. CONSTRUCTION THEOREMS

***Theorem 3.1*:Construction of an (r,λ)-design with p^2 treatments**

There exists an (r, λ)-design with parameters $v = p^2, b = p^2 + p, r = 2p - 1, \lambda = p - 1$.

Proof: Let K=J-I is a pxp matrix and e is a p-rowed all 1 matrix and α is a circulant matrix of order p such that $\alpha^p = I_p$. Consider block matrices

$$N_1 = \begin{bmatrix} I_p & I_p & \cdots & I_p & I_p \\ I_p & \alpha & \dots & \alpha^{p-2} & \alpha^{p-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I_p & \alpha^{p-1} & \cdots & (\alpha^{p-2})^{p-1} & (\alpha^{p-1})^{p-1} \end{bmatrix}, \quad K \otimes e = \begin{pmatrix} 0 & e & \cdots & e \\ e & 0 & \cdots & e \\ \vdots & \vdots & \vdots & \vdots \\ e & e & \cdots & e \\ e & e & \cdots & 0 \end{pmatrix}$$

Inner product of rows of N= $[N_1 \quad K \otimes e]$ is$R_i \circ R_j = \begin{cases} (2p-1)I_p + (p-1)K_p \text{if } i = j \\ (p-1)I_p + (p-1)K_p \text{ if } i \neq j \end{cases}$

Then N is the incidence matrix of an (r,λ)-design with the required parameters.

[vide theorem 2.1]

Example 3.1.1 There exists an (r,λ)-design with parameters

(a) v=25,b=30,r=9, λ=4.

(b) v=49,b=56,r=13, λ=6.

(c) v=121,b=132,r=21, λ=10 and so on.

Corollary 3.1.1 There exists an (r, λ)-design with parameters

$v = p^2 - lp, b = p^2 + p, r = 2p - 1, \lambda = p - 1$.

proof If we remove l rows of blocks from N then we obtain an incidence matrix of an (r, λ)-design with the required parameters. [vide theorem 2.1]

***Theorem 3.2*:**There exists an (r, λ)-design with parameters

$$v = p^2 - lp, b = sp(p + 1), r = s(p + 1), \lambda = s.$$

proof: Saurabh and Singh[10] have shown that a GROCM is an incidence matrix of a SBIBD D(let) with parameters

v=p^2, $b = sp(p + 1), r = s(p + 1), k = p, \lambda = s$.If we remove l rows of blocks from the incidence matrix of the SBIBD D then we obtain an incidence matrix N of an (r, λ)-design with the required parameters

[vide theorem 2.1].

Inner product of two rows of N is $R_i \circ R_j = \begin{cases} s(p + 1)I_p + sK_p \text{if } i = j \\ sI_p + sK_p \text{ if } i \neq j \end{cases}$

Theorem3.3: The existence of a (v,b,r,k,λ)-design implies the existence of an (r,λ)-design with parameters $(v^{(n+1)}, b^{(n+1)}, r^{(n+1)}, \lambda^{(n+1)})$ where$v^{(n+1)} = 4^n(4v), b^{(n+1)} = 4^n\big(16(r - \lambda)\big)$,$r^{(n+1)} = 4^n\big(4(r - \lambda) +$ 2r+s, λn+1=4n2(r+s).

Proof: Singh and Prasad[8] have shown that there exists an (r,λ)-design with parameters $v^{(1)} = 4v, b^{(1)} = 4(r-\lambda), r^{(1)} = 4(r-\lambda) + 2(r+s), \lambda^{(1)} = 2(r+s)$. Let us denote this design by $D^1(v^{(1)}, b^{(1)}, r^{(1)}, \lambda^{(1)})$. On substituting 0 by I_4 and 1 by K_4 successively in the incidence matrix of $D^1(v^{(1)}, b^{(1)}, r^{(1)}, \lambda^{(1)})$, we obtain the(r,λ)-design with the required parameters.It is trivial and obvious for $v^{(n+1)}$ and $b^{(n+1)}$.In the construction of $D^1(v^{(1)}, b^{(1)}, r^{(1)}, \lambda^{(1)})$ from the BIBD (v,b,r,k,λ),we have
$r^{(1)} = 4(r-\lambda) + 2(r+s)$, s is the number of columns of K_4 and $\lambda^{(1)} = 2(r+s)$ and in the construction of $D^2(v^{(2)}, b^{(2)}, r^{(2)}, \lambda^{(2)})$ from $D^1(v^{(1)}, b^{(1)}, r^{(1)}, \lambda^{(1)})$ no more columns of K_4 are added i.e. s=0 so

$$r^{(2)} = 4(r^{(1)} - \lambda^{(1)}) + 2r^{(1)}$$
$$= (2^4 + 2^3)(r-\lambda) + 2^2(r+s)$$
$$\lambda^{(2)} = 2r^{(1)} = 2^3(r-\lambda) + 2^2(r+s)$$
$$r^{(3)} = 4(r^{(2)} - \lambda^{(2)}) + 2r^{(2)}$$
$$= (2^6 + 2^5 + 2^4)(r-\lambda) + 2^3(r+s)$$
$$\lambda^{(3)} = 2r^{(2)} = (2^5 + 2^4)(r-\lambda) + 2^3(r+s)$$

Proceeding in the similar manner, we obtain

$$r^{(n)} = (2^{2n} + 2^{2n-1} + \cdots + 2^{n+1})(r-\lambda) + 2^n(r+s)$$

whichgives $r^{(n+1)} = 4^n(4(r-\lambda) + 2(r+s))$

$$\lambda^{(n)} = 2^{n+1}(2^{n-1} - 1)(r-\lambda) + 2^n(r+s)$$

which gives $\lambda^{(n+1)} = 4^n(2(r+s))$

Theorem 3.4: If R_i is an (r,λ)-design with parameters $(v_i, b_i, r_i, \lambda_i)$ with $b_i = 4(r_i - \lambda_i)$ and $A_i$and A_i^* are the incidence matrices of R_i and its complementary design R_i^*. Then with i=1,2
$N = A_1 \otimes A_2 + A_1^* \otimes A_2^*$is the matrix of an (r,λ)-design D with parameters (v, b, r, λ) given by$v = v_1 v_2, b = b_1 b_2, r = r_1 r_2 + (b_1 - r_1)(b_2 - r_2), \lambda = b_1 b_2 - 2(r_1 b_2 + r_2 b_1) + 6 r_1 r_2 - 4\lambda_1\lambda_2$, b=4(r-λ) also holds for design D.

proof: Since $A_i (i = 1,2)$ is an incidence matrix of R_i design,$A_i A_i^T = r_i I_{v_i} + \lambda_i K_{v_i}$ and $A_i J_{b_i} = r_i J_{v_i \times b_i}$ for i=1,2.
Clearly the number of rows and columns in NN^T are $v_1 v_2$ and $b_1 b_2$respectively.

$$NN^T = (A_1 \otimes A_2 + A_1^* \otimes A_2^*)(A_1 \otimes A_2 + A_1^* \otimes A_2^*)^T$$
$$= \left[A_1 \otimes A_2 + (J_{v_1 \times b_1} - A_1) \otimes (J_{v_2 \times b_2} - A_2)\right]\left[A_1 \otimes A_2 + (J_{v_1 \times b_1} - A_1) \otimes (J_{v_2 \times b_2} - A_2)\right]^T$$
$$=4(r_1 - \lambda_1)(r_2 - \lambda_2) I_v + (r_1 - \lambda_1)(4\lambda_2 + b_2 - 4r_2)(I_{v_1} \otimes J_{v_2}) +$$
$$(r_2 - \lambda_2)(4\lambda_1 + b_1 - 4r_1)(J_{v_1} \otimes I_{v_2}) + (b_1 b_2 - 2r_1 b_2 - 2r_2 b_1 + 6r_1 r_2 - 4\lambda_1\lambda_1) J_v$$
$$=4(r_1 - \lambda_1)(r_2 - \lambda_2) I_v + (b_1 b_2 - 2r_1 b_2 - 2r_2 b_1 + 6r_1 r_2 - 4\lambda_1\lambda_2) J_v$$
$$= [r_1 r_2 + (b_1 - r_1)(b_2 - r_2)] I_v + (b_1 b_2 - 2r_1 b_2 - 2r_2 b_1 + 6r_1 r_2 - 4\lambda_1\lambda_2) K_v$$

(since$b_i = 4(r_i - \lambda_i$ for $i = 1,2)$
Also $N_{v \times b} J_{b \times v} = rJ_v$. Hence N is the incidence matrix of an (r,λ)-design D with the required parameters [vide theorem 2.1].

Also $r - \lambda = -4r_1 r_2 + r_1 b_2 + r_2 b_1 + 6r_1 r_2 + 4\lambda_1\lambda_2$
$$= -4r_1 r_2 + r_2(4r_1 - 4\lambda_1) + r(4r_2 - 4\lambda_2) + 4\lambda_1\lambda_2$$
$$=4(r_1 - \lambda_1)(r_2 - \lambda_2) = \frac{b}{4}$$

IV. CONCLUSION

A combinatorial structure GROM has been used to construct some series of (r,λ)-designs. A GROM can also be used to construct some more combinatorial designs.

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