Some Specific Operators and a Poset on BE-Algebras

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Abstract

Since the introduction of the concepts of BCK and BCI algebras by K. Iseki in 1966, some more systems of similar type have been introduced and studied by a number of authors in the last two decades. K. H. Kim and Y. H. Yon studied dual BCK algebra[1] and M.V. algebra in 2007[4]. H. S. Kim and Y. H. Kim in 2006 have introduced the concept of BE-algebra as a generalization of dual BCK- algebra. Here we want to introduce some specific operators and their properties and a poset on BE-algebras.

Key words: *BCK-algebra, BCI-algebra, BE-algebra, M.V. algebra, Operator.*

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I. Preliminaries:

Definition 1.1. : Let $(X; *, 1)$ be a system of type (2, 0) consisting of a non-empty set X, a binary operation " $*$ " and a fixed element 1. The system $(X; *, 1)$ is called a BE- algebra ([2,3]) if the following conditions are satisfied:

(BE 1) $x * x = 1$

(BE 2) $x * 1 = 1$

(BE 3) $1 * x = x$

(BE 4) $x * (y * z) = y * (x * z)$, $\forall\, x, y, z \in X$.

Note 1.1. : In any BE-algebra one can define a binary relation " $\le$ " as $x \le y$ if and only if

$x * y = 1$, $\forall\, x, y, \in X$.

Example 1.1. : First of all we present a simplest example of a BE-algebra which is of much importance. Let X = {0, 1} and the binary operation $*$ is defined on X by the following Cayley table

$*$	0	1
0	1	1
1	0	1

Then $(X; *, 1)$ is a BE-algebra.

Example 1.2. : Let X be a non empty set having two or more elements and let A be a non empty subset of X .We consider the collection $T = \{X, A, A^c, \phi\}$ with binary operation $*$ defined as

$$A * B = (X - A) \cup (A \cap B).$$

Then Cayley table for this operation is given by

$*$	X	A	B	O
X	X	A	B	O
A	X	X	B	B
B	X	A	X	A
O	X	X	X	X

where B = A^c and O = ϕ. Here X = 1 and (T ; $*$, 1) is a BE – algebra.

Example 1.3. : Let X be a non - empty set and let Y = P (X), the power set of X. For A, B $\in$ Y, we define

$$A * B = A^c \cup B.$$

Then for A, B, C $\in$ Y, we have

(i) $A * A = A^c \cup A = X$;
(ii) $X * A = X^c \cup A = A$;
(iii) $A * (B * C) = A * (B^c \cup C)$
$= A^c \cup (B^c \cup C)$
$= (A^c \cup B^c) \cup C$
$= (B^c \cup A^c) \cup C$
$= B^c \cup (A^c \cup C)$
$= B^c \cup (A * C)$
$= B * (A * C)$
(iv) $A * X = A^c \cup X = X$.

Thus we see that (Y ; $*$, 1) is a BE – algebra where 1 denote the set X.

II. A specific poset:

Theorem 2.1. : Let (X; $*$, 1) be a system consisting of a non – empty set X, a binary operation ‘‘$*$’’ and a distinct element 1. Let Y = X x X =$\{(x_1, x_2) : x_1, x_2 \in X\}$. For u, v $\in$ Y with $u = (x_1, x_2)$, $v = (y_1, y_2)$, we define an operation ‘‘ Θ ’’ in Y as

$$u \Theta v = (x_1 * y_1, x_2 * y_2)$$

Then (Y; Θ, (1, 1)) is a BE – algebra iff (X; $*$, 1) is a BE – algebra[5,6].

Example 2.1. : We recall BE – algebra (X; $*$, 1) considered in example (1.1). Let

$$Y = \frac{X\,x\,X \ldots\ldots\ldots\ldots x\,X}{8 times}$$

Then Y is the set of all bytes considered in computer. Thus each y $\in$ Y is expressible as $y = (y_1, y_2, \ldots\ldots\ldots., y_8)$, where each y_i is either 0 or 1. The set Y contains 256 elements. Also Y is a BE – algebra by theorem (2.1). Here the unit element is 1 = (1, 1, 1, 1, 1, 1, 1, 1). This BE –algebra is a BE- algebra with zero element 0 = (0, 0, 0, 0, 0, 0, 0, 0) because $0 \Theta x = 1$ for all x $\in$ Y.

Now we see that Y is partially ordered w. r. t. ordering defined in note (1.1). We have,

(i) Since $y \Theta y = 1$ for all y $\in$ Y, i. e. $y \le y$, so $\le$ is reflexive.
(ii) Let $x \le y$ and $y \le x$. Then
$x \Theta y = 1$ .....(A) and $y \Theta x = 1$ ...(B).

Now if $x_i = 1$, $1 \le i \le 8$, then condition (A) implies that $y_i = 1$. Again if $x_i = 0$, then $y_i = 0$ or 1. If possible, suppose $y_i = 1$. Then condition (B) implies that $x_i = 1$ which is a contradiction. So we see that $x_i = 1 \Rightarrow y_i = 1$ and $x_i = 0 \Rightarrow y_i = 0$. This proves that x = y. So the relation $\le$ is anti symmetric.

(iii) Let $x \le y$ and $y \le z$. Then
$x \Theta y = 1$ and $y \Theta z = 1$

So $x_i = 1 \Rightarrow y_i = 1 \Rightarrow z_i = 1 \quad (1 \le i \le 8)$

Again $x_i = 0 \Rightarrow y_i = 0$ or $1 \Rightarrow z_i = 0$ or 1

So in all the cases $x \Theta z = 1$, i.e. $x \le z$

and the relation $\le$ is transitive .

Hence Y is partially ordered w. r. t. the relation $\le$.

III. Some specific operators:

Definition 3.1. : Let $(X; *, 1)$ and $(Y; o, e)$ be BE – algebras and let $f : X \to Y$ be a mapping. Then f is called a homomorphism[7] if

$$f(x * y) = f(x) \, o \, f(y)$$

for all $x, y \in X$.

Proposition 3.1. : Let $f : (X; *, 1) \to (Y; o, e)$ be a homomorphism. Then

(a) $f(1) = e$

and (b) $x \le y \Rightarrow f(x) \le f(y)$.

Proof : (a) We see that $1 * 1 = 1 \Rightarrow f(1 * 1) = f(1)$.

$$\Rightarrow f(1) \, o \, f(1) = f(1)$$

$$\Rightarrow e = f(1).$$

(b) Again $x \le y \Rightarrow x * y = 1$

$$\Rightarrow f(x * y) = f(1) = e$$

$$\Rightarrow f(x) \, o \, f(y) = e$$

$$\Rightarrow f(x) \le f(y).$$

Definition 3.2. : Let $(X; *, 1)$ be a BE – algebra and let $Y = X^n$ be the Cartesian product of X with itself upto n times. Then theorem (2.1) implies that Y is a BE – algebra under the binary operation Θ and fixed element $1^n = (1,1,....,1)$.

The mappings P_k and P_{ij} defined on X^n into itself as

$$P_k(x_1, ..., x_k, ..., x_n) = (1,1,......., x_k, ...,1)$$

$$P_{ij}(x_1, ..., x_i, ..., x_j, ..., x_n) = (1,1,....., x_i, 1, ..., x_j, ...,1)$$

are called dual projection maps.

Theorem 3.1. : P_k and P_{ij} are homomorphism on X^n.

Proof : Let $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ be elements of X^n . Then

$$\begin{aligned} P_k(x \Theta y) &= P_k(x_1 * y_1, ..., x_k * y_k, ..., x_n * y_n) \\ &= (1, ..., x_k * y_k, ..., 1) \\ &= (1, ..., x_k, ..., 1) \Theta (1, ..., y_k, ..., 1) \\ &= P_k(x) \Theta P_k(y). \end{aligned}$$

This implies that P_k is a homomorphism.

Definition 3.3. : Let $(X; *, 1)$ be a BE – algebra and let $Y = X^n$. Then forward shift with replacement 1 and backward shift with replacement 1, denoted as (F S 1) and (B S 1) respectively, are defined as

$$(F\ S\ 1)(x) = (1, x_1, x_2, ..., x_{n-1})$$

$$(B\ S\ 1)(x) = (x_2, x_3, ..., x_n, 1)$$

for all $x = (x_1, x_2, \ldots\ldots, x_n) \in Y$.

Theorem 3.2. : (F S 1) and (B S 1) are homomorphism on Y.

Proof : Let $u, v \in Y$. Then $u = (x_1, \ldots\ldots, x_n)$ and $v = (y_1, \ldots\ldots, y_n)$. We have

$$(F\,S\,1)(u \,\Theta\, v) = (1, x_1 * y_1, \ldots\ldots, x_{n-1} * y_{n-1})$$
$$= (1, x_1, \ldots\ldots, x_{n-1}) \,\Theta\, (1, y_1, \ldots\ldots, y_{n-1})$$
$$= ((F\,S\,1)(u)) \,\Theta\, ((F\,S\,1)(v)).$$

Also $$(B\,S\,1)(u \,\Theta\, v) = (x_2 * y_2, \ldots\ldots, x_n * y_n, 1)$$
$$= (x_2, \ldots\ldots, x_n, 1) \,\Theta\, (y_2, \ldots\ldots, y_n, 1)$$
$$= ((B\,S\,1)(u)) \,\Theta\, ((B\,S\,1)(v)).$$

Hence (F S 1) and (B S 1) are homomorphism.

Note 3.1. : If we consider (F S 0) and (B S 0) on Y then (F S 0) and (B S 0) are not homomorphism on Y, since $0 * 0 = 1 \neq 0$.

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