Chromatic Strong(Weak) Excellent In $K_m \cup P_n$

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Abstract—*Let G be an simple undirected graphs. A subset D of V is said to be a chromatic strong(weak) dominating set if D is a strong(weak) dominating set and* $\chi(\langle D\rangle) = \chi(G)$. *The minimum cardinality of a chromatic strong(weak) dominating set in a graph G is called the chromatic strong(weak) dominating number and is denoted by* $\gamma_s^c(G)$ ($\gamma_w^c(G)$)*.A graph G is called a* $\gamma_s^c(\gamma_w^c)$ *-excellent if every vertex of G belongs to a* $\gamma_s^c(\gamma_w^c)$ *- set. We find that the necessary and sufficient condition for some particular graph, of the form* $K_m \cup P_n$, *is* γ_s^c*- excellent and* γ_w^c *- excellent.*

Keywords—*Chromatic strong domination, Chromatic weak domination, Chromatic strong excellent graph, Chromatic weak excellent graph.*

I. INTRODUCTION

Let G=(V,E) be a simple undirected graph with vertex set V and edge set E.A*n – coloring*of a graph G is an assignment of n colors to its vertices so that no two adjacent vertices have the same color. The *chromatic number* $\chi(G)$ is defined as the minimum n for which G has an n- coloring.The *diameter* of a connected graph G is defined by max{ d(u,v) : u,v∈ V(G)}and is denoted by diam(G). Note that diameter of a path P_n is the distance betweenthe its end vertices. That is diam(P_n)=n-1.For graph theoretic terminology, we refer to [2] and [4]. A subset D of V is a *dominating set* of G if every vertex in V-D is adjacent to atleast one vertex in D. The *domination number*γ(G) of G is the minimum cardinality of the dominating set of G.A dominating set with minimum cardinality is called a γ - set of G.A study of domination in graphs and its advanced topics are given in [6].Prof. E.Sampathkumar and L.Pushpalatha have defined strong(weak) domination in graphs shown in [8].A subset D of V is called *a strong(weak) dominating set* of G if for every vertex in V-D there exists u ϵ D such that uv ϵ E and deg u ≥deg v (deg v ≥deg u).The *strong(weak) domination number*γ_s(G) (γ_w(G)) of G is the minimum cardinality of a strong(weak) dominating set of G.A Strong(Weak) dominating set with minimum cardinality is called a γ_s -set (γ_w - set)of G.

Prof. T. N. Janakiramanand M. Poobalaranjani [7] introduced a new conditional dom chromatic setand Prof. S. Balamurugan et al [3] extended this dom chromatic set to chromatic strong (weak) dominating set.A subset D of V is said to be a *chromatic strong(weak) dominating set* if D is a strong(weak) dominating set and $\chi(\langle D\rangle)=\chi(G)$.The minimum cardinality of a chromatic strong(Weak) dominating set in a graph G is called the*chromatic strong(weak) dominating number* and is denoted by γ_s^c(G) (γ_w^c(G)).A chromatic strong dominating set with cardinality γ_s^c (γ_w^c) iscalled γ_s^c- set (γ_w^c - set) of G.

Prof N Sridharan and M. Yamuna [9] defined some new classes of excellent graphs with respect to γ - set. Prof CVRHarinarayanan et al [5] extended it to strong (weak) domination excellent graphs.We introduce a chromatic strong (weak) excellent in graphs and find the condition for chromatic strong (weak) very excellent caterpillar in [1].

II. CHROMATIC STRONG (WEAK) EXCELLENT

A. γ_s^c (γ_w^c) –*Excellent* [1]

A vertex u in G is said to be γ_s^c (γ_w^c)*-good* ifu belongs to some γ_s^c (γ_w^c) - set of G andγ_s^c (γ_w^c)*- bad* otherwise. A graph G is a called γ_s^c (γ_w^c)*-excellent*if every vertex of G is γ_s^c (γ_w^c)- good. Equivalently,A graph G is said to be excellent with respect to chromatic strong (Weak) dominationif each
u∈ V(G) is contained in some γ_s^c (γ_w^c)- set of G.

B. γ_s^c (γ_w^c) **–*Just Excellent*** [1]

A graph G is said to be *just excellent* with respect to chromatic strong (Weak) domination if each u∈V(G) iscontained in a unique γ_s^c (γ_w^c) - set of G. We also says that G is γ_s^c (γ_w^c) *- just excellent graph.*

C. γ_s^c (γ_w^c) **–*Very Excellent*** [1]

A graph G is said to be *very excellent* with respect to chromatic strong (Weak) dominationif there is aγ_s^c (γ_w^c) - set of G such that to each vertex u∈ V-D, there exists a vertex v ∈ D such that (D - {v}) ∪ {u} is aγ_s^c (γ_w^c)- set of G. We also says that G isγ_s^c (γ_w^c)- *very excellent graph.*

D. γ_s^c (γ_w^c) **–*Rigid Very Excellent*** [1]

letG be a very excellent graph and D be a very excellentγ_s^c (γ_w^c)- set of G.To each u not in D, let E(u,D) be the set of vertices of D which are exchangeable with u.

ie., E(u,D)={ v ∈ D | (D-v)∪{u} is aγ_s^c (γ_w^c) - set of G}

If |E(u,D) |=1, for all unot in D,then D is said to be a *rigid very excellent*γ_s^c (γ_w^c) - set of G.If G has atleast one rigid very excellentγ_s^c (γ_w^c) - set then G is said to be *rigid very excellent.*

III. CHROMATIC STRONG EXCELLENT GRAPHS

A. *Theorem*

Let G = $K_m \cup P_n$ be a graph where K_m is the complete graph with m(>3) vertices andP_n(n≥ 2) is the path with the vertex set {1,2,3,… ,n}. Let X={x_1,x_2,x_3,… ,x_k}be a non empty set where x_i is the i^{th} vertex of $P_n$such that x_i∈V(K_m). then G is γ_s^c- excellent graph if and only if the following hold

- i. V(K_m) is a subset of every chromatic strong dominating set of G.
- ii. In P_n, for t∈ N,
  - a. d(1,x_1) = d(x_k,n) = 3t-1
  - b. d(x_i,x_{i+1})= 3t-2, for all i=1,2,… , k-1

Proof:

Given G=$K_m \cup P_n$. Clearly, the chromatic number of G is m. Let D beγ_s^c- set of G.If G is γ_s^c- excellent, thenClearly, V(K_m)⊂V(G) is a subset of every chromatic strong dominating set, D of G.Therefore (i) holds.

Case : 1

If k=1,ie., V(K_m) ∩ V(P_n)={x_1}.Then, we have to prove that both d(1,x_1) and d(x_1,n) is of the form 3t-1, t∈ N, in P_n.Suppose d(x_1,n)≠3t-1 if x_1≠ n.If d(x_1,n)=3t, thenthe vertices x_1+1,x_1+4, x_1+7, … , n-2,n belongs to no γ_s^c- set of G.Otherwise, |D|>γ_s^c. If d(x_1,n)= 3t+1, then D=V(K_m)∪{x_1+3,x_1+6,…,n-1} is a unique γ_s^c- set of G.Otherwise, |D|>γ_s^c. Since both sub cases lead to contradiction, d(x_1,n)=3t-1.Similarly, d(1,x_1)=3t-1 if x_1≠1

Case : 2

If k≠ 1,If x_1≠ 1, thenBy case : 1, the result,d(1,x_1)=3t-1 is true. Similarly, d(x_k, n)=3t-1 if x_k≠ n.Hence, (ii)-(a) holds.Let S_i be the set of all vertices lies between x_i and x_{i+1} in P_n. That is S_i={s∈ P_n| x_i<s<x_{i+1}}.Let S_i={s_{i1}, s_{i2},… ,$s_{i(ni)}$}. Suppose that d(x_i,x_{i+1}) ≠ 3t-2 in P_n.If d(x_i,x_{i+1}) =3t, thenD=V(K_m)∪{s_{i3},s_{i6}, … , $s_{i(3t)}$} is a unique γ_s^c- set of G.If d(x_i,x_{i+1})=3t-1, thenthe vertices s_{i1},s_{i4}, … , $s_{i(3t+1)}$ belong to no γ_s^c- set of G.Otherwise, |D|>γ_s^c. Since both sub cases lead to contradiction, d(x_i,x_{i+1}) = 3t-2 in P_n.Hence (ii)-(b) holds.

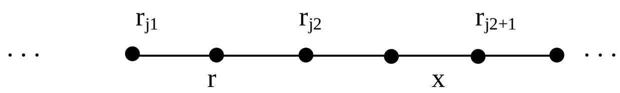
Conversely,

Now, we assume that the given graph G=$K_m \cup P_n$ satisfies the condition (i) and (ii)Suppose G is not a γ_s^cexcellent. Let D be any γ_s^c- set of G.Then there exists a vertex x in V(G) such that no γ_s^c- set,D of G containing x. Since by (i), xnotin V(K_m). Hence x∈ V(P_n)-V(K_m).

Case : 1

If x lies between 1 and x_1 then,Let Sbe the set of all vertices lies between x_1 and 1 including x_1 and 1. Let R=S∩D ={r_1, r_2, …. ,r_q},(say).Clearly, x_1∈R and x not in R.If d(r_i,r_{i+1})=3, for all 1 ≤ i < q, then, d(x_1,1)=3t+1,(t∈ N), contradicts (ii)-(a).Otherwise, If d(r_j,r_{j+1})=2, for a unique j, then d(x_1,1)=3t,(t∈ N), contradicts (ii)-(a).If

$d(r_j,r_{j+1})=2$, for any two j, j=1,2, … ,q-1ie., $d(r_{j1},r_{j1+1})=d(r_{j2},r_{j2+1})=2$ then,in particular, let j2=j1+1 and r_{j2+1} is adjacent to x.Let r be the adjacent vertex of r_{j2+1} other than x as shown in the following figure.



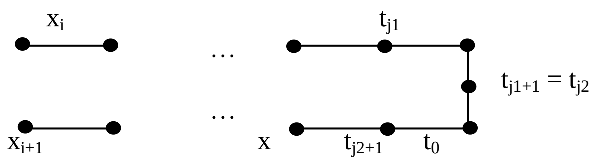
Then, clearly, D-$\{r_{j2+1}, r_{j2}\}\cup\{r,x\}$ is a γ_s^c- set of G containing x.which is contradiction. If $d(r_j,r_{j+1})=2$, for more than two j. then,$|D|>\gamma_s^c$.Hence G is γ_s^c- excellent.

Case :2

If x lies between x_k and n.It is similar to case : 1.Hence, by case :1, G is γ_s^c- excellent.

Case : 3

If x lies between x_i and x_{i+1} then,Let S be the set of all vertices lies between x_i and x_{i+1} including x_i and x_{i+1}andlet $T=S\cap D=\{t_1,t_2, \dots ,t_p\}$(say).Clearly, $x_i\in T$, $x_{i+1}\in T$ and x not in T. If $d(t_i,t_{i+1})=3$, for all $1\le i<p$, then, $d(x_i,x_{i+1})=3t,(t\in N)$. Which is contradiction to (ii)-(b). Otherwise, If $d(t_j,t_{j+1})=2$, for a unique j, then $d(x_i,x_{i+1})=3t-1,(t\in N)$.Which is contradiction to (ii)-(b). If $d(t_j,t_{j+1})=2$, for any two j,j=1,2, … ,p-1.ie.,$d(t_{j1},t_{j1+1})=d(t_{j2},t_{j2+1})=2$ thenin particular, let j2=j1+1 and t_{j2+1} is adjacent to x.Let t_0 be the adjacent vertex of t_{j2+1} other than x as shown in the following figure.



Then, clearly, D-$\{t_{j2+1},t_{j2}\}\cup\{t_0, x\}$ is a γ_s^c - set of G containing x.which is contradiction. Suppose, if $d(t_j,t_{j+1})=2$, for more than two j. then,$|D|>\gamma_s^c$.Hence G isγ_s^c- excellent.

B. ***Corollary***

Let $G = K_m\cup P$ be a graph where K_m is the complete graph with m(>3) vertices andP is the union of disjoint paths $P_j(j\ge2)$with the vertex set $\{1^{(j)},2^{(j)},3^{(j)},\dots ,n_j^{(j)}\}$. Let $X^{(j)}=\{x_1^{(j)},x_2^{(j)},x_3^{(j)}, \dots ,x_k^{(j)}\}$ be a non empty set, where $x_i^{(j)}$ is the i^{th} vertex of $P_j$such that $x_i^{(j)}\in V(K_m)$. then G is γ_s^c- excellent graph if and only if the following hold

i. $V(K_m)$ is a subset of every chromatic strong dominating set of G.
ii. In P_j, for each j and for $t\in N$
    a. $d(1,x_1^{(j)})=d(x_k^{(j)},n)=3t-1$
    b. $d(x_i^{(j)},x_{i+1}^{(j)})=3t-2$, for all i=1,2, … , k-1

C. *Theorem*

Let $G = K_m\cup C_n$ be a connected graph where K_m is the complete graph with m(>3) vertices andC_n is the cycle with the $(n\ge 3)$ vertices. Let $H=\langle\{V(G)-V(K_m)\}\rangle$ and let $H_1,H_2, \dots , H_p$ be a components of H Then G is γ_s^c- excellent graph if and only if the following hold

i. $V(K_m)$ is a subset of every chromatic strong dominating set of G.
ii. $diam(H_i) = 3t-1$, $t\in N$, for each i=1,2, … ,p

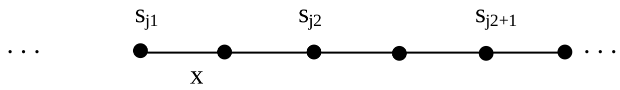
Proof:

Given $G=K_m\cup C_n$ is a connected graph.Then H is a disjoint union of paths. ie., each H_i is a path.Let $H_i=P_{ki}$ be a path with the vertex set $\{h_1, h_2,\dots ,h_{ki}\}$. It is clear that the chromatic number and clique number of the graph G is m.Let D be γ_s^c- set of G.If G is γ_s^c- excellent,Clearly, $V(K_m)\subset V(G)$ is a subset of every chromatic strong dominating set, D of G.Therefore (i) holds. Now we have to prove thatdiam$(H_i) = 3t-1$, $t\in N$, for each i=1,2, …,p.It is enough to prove that ki=3t, for each i.It is clear that the end vertex of P_{ki} is adjacent to vertex of D, since $V(K_m) \subset D$. Suppose $k_i \neq 3t$. If $k_i=3t+1$, thenthe vertices $h_1,h_4,h_7, \ldots ,h_{3t+1}$ belongs to no γ_s^c- set of G. Otherwise, $|D|>\gamma_s^c$.If $k_i=3t-1$, then $D=V(K_m)\cup\{h_3,h_6, \ldots , h_{3t}\}$ is a unique γ_s^c- set of G.Since both cases lead to contradiction, $k_i=3t$.

Hence (ii) holds.

Conversely,

Now, we assume that the given graph $G=K_m\cup C_n$ satisfies the condition (i) and (ii). Suppose G is not aγ_s^cexcellent. Let D be any γ_s^c- set of G.Then there exists a vertex x in V(G) such that no γ_s^c- set,D of G containing x. Since by (i), x not in $V(K_m)$. Hence $x\in H$ implies $x \in H_i$ for some i.Let p and q be the vertices of $V(K_m)$ which is also adjacent to the pendant vertex of H_i. Let $S= (H_i\cap D)\cup\{p,q\}$ and let $S=\{p=s_1,s_2, \ldots ,s_r=q\}$(say).Clearly, x not in S.If $d(s_i,s_{i+1})=3$, for all $1\leq i < r$, then, $d(p,q)=3t,(t\in N)$ impliesdiam $(H_i)=3t-2$, $(t\in N)$,contradicts (ii). Otherwise, If $d(s_j,s_{j+1})=2$, for a unique j, then $d(p,q)=3t-1,(t\in N)$ impliesdiam $(H_i)=3t$, $(t\in N)$ contradicts (2). If $d(s_j,s_{j+1})=2$, for any two j, $j=1,2, \ldots ,r-1$. ie., $d(s_{j1},s_{j1+1})=d(s_{j2},s_{j2+1})=2$ thenin particular, let $j2=j1+1$ and x is adjacent to both s_{j1} and s_{j1+1} as shown in the following figure.



Then, clearly, $(S-s_{j1+1})\cup\{x\}$ is a γ_s^c - set of G containing x.which is also contradiction.If $d(s_j,s_{j+1})=2$, for more than two j. then,$|D|>\gamma_s^c$.

Hence G is γ_s^c- excellent.

D. *Corollary*

Let $G = K_m\cup C$ be a connected graph where K_m is the complete graph with m(>3) vertices andC is the union of disjoint cycles, $C_j,(n \geq 3)$. Let $H = \langle \{V(G)-V(K_m)\} \rangle$and let $H_1, H_2, \ldots , H_p$ be a components of H Then G is γ_s^c- excellent graph if and only if the following hold

i. $V(K_m)$ is a subset of every chromatic strong dominating set of G.
ii. $diam(H_i) = 3t-1,t\in N$, for each $i=1,2, \ldots ,p$

IV. Chromatic Weak Excellent Graphs

A. Theorem

Let $G = K_m\cup P_n$ be a graph where K_m is the complete graph with m(>3) vertices and$P_n(n\geq 2)$ is the path with the vertex set $\{1,2,3,\ldots ,n\}$. Let $X=\{x_1,x_2,x_3,\ldots ,x_k\}$is non empty, where x_i is the i^{th} vertex of $P_n$such that $x_i\in V(K_m)$. then G is γ_w^c- excellent graph if and only if the following hold

i. $V(K_m)$ is a subset of every chromatic weak dominating set of G.
ii. In P_n, for $t\in N$,
    a. $d(1,x_1)=d(x_k,n)=1$ or $3t$
    b. $d(x_i,x_{i+1})=3$ or$3t-1$, for all $i=1,2, \ldots , k-1$

B. Corollary

Let $G = K_m\cup P$ be a graph where K_m is the complete graph with m(>3) vertices andP is the union of disjoint paths $P_j(j\geq 2)$with the vertex set $\{1^{(j)},2^{(j)},3^{(j)},\ldots ,n_j^{(j)}\}$. Let $X^{(j)}=\{x_1^{(j)},x_2^{(j)},x_3^{(j)},\ldots,x_k^{(j)}\}$is non empty where $x_i^{(j)}$ is the i^{th} vertex of P_j such that $x_i^{(j)}\in V(K_m)$. thenG is γ_w^c- excellent graph if and only if the following hold

i. $V(K_m)$ is a subset of every chromatic weak dominating set of G.
ii. In P_j, for each j and for $t\in N$
    a. $d(1,x_1^{(j)})=d(x_k^{(j)},n)=1$ or $3t$
    b. $d(x_i^{(j)}, x_{i+1}^{(j)}) = 3$ or $3t-1$,
    for all $i=1,2,\ldots,k-1$

C. Theorem

Let G = $K_m \cup C_n$ be a connected graph where K_m is the complete graph with m(>3) vertices andC_n is the cycle with the (n ≥ 3) vertices. Let H=⟨ {V(G)-V(K_m)} ⟩ and let H_1, H_2, … , H_p be a components of H Then G is γ_w^c- excellent graph if and only if the following hold

1. V(K_m) is a subset of every chromatic weak dominating set of G.
2. diam(H_i) = 1or3t,t∈ N, for each i=1,2,…,p

D. Corollary

Let G = $K_m \cup$ C be a connected graph where K_m is the complete graph with m(>3) vertices andC is the union of disjoint cycles, C_n(n ≥ 3). Let H = ⟨ {V(G)-V(K_m)} ⟩ andlet H_1,H_2, … , H_p be a components of H Then G is γ_w^c- excellent graph if and only if the following hold

1. V(K_m) is a subset of every chromatic weak dominating set of G.
2. diam(H_i) = 1or3t,t∈ N, for each i=1,2, … ,p

V. References

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