Generalized Prefunctions and Applications

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Abstract *: The aim of this paper is to introduce the General Prefunctions, Laplace transform of general prefunctions and its applications.*

Keywords: *Prefunctions, Extended Prefunctions and Laplace Transforms.*

I. Introduction

It isrenowned that some elementary functions such as polynomial functions, exponential functions, logarithmic functions and trigonometric functions have played a significant role in development of Mathematical, Physical as well as Engineering Sciences. Deo and Howell in his paper [1] introduced an alternative method and studied trigonometric and trigonometric like functions. This new technique is lucid and easily applicable in the study of differential equations. Khandepaekar, Deo and Dhaigude [2] have defined new kind of Exponential, Trigonometric and Hyperbolic functions. They are coined as Prefunctions and Extended Prefunctions. It is shown that these prefunctions and extended prefunctions satisfy many interesting properties and relations. Some properties and Laplace Transforms of these functions are studied in the papers [2], [3], [4], [5] & [6].

The main aim of this paper is to introduce the generalized form of prefunctions and their Laplace transforms. Also we are showing theseprefunctions are the solutions of Initial Valued Problems (IVPs) of first order, second order and third order differential equations.

II. Definitions of prefunctions and extended prefunctions

In this section we introduce the notion prefunctions and its Laplace transforms.

Definition: The pre- exponential functions is denoted by pexp (t,∝) and is defined as

$$pexp(t,\propto) = 1 + \sum_{n=1}^{\infty} \frac{t^{n+\propto}}{\Gamma(n+1+\propto)}, t \in \mathbb{R} \ and \ \propto \geq 0.$$

Also,

$$pexp(-t,\propto) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+\propto} t^{n+\propto}}{\Gamma(n+1+\propto)}, t \in \mathbb{R} \ and \ \propto \geq 0.$$

The pre-cosine function is denoted by pcos (t,∝) and is defined as

$$pcos(t,\propto) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n t^{2n+\propto}}{\Gamma(2n+1+\propto)}, t \in \mathbb{R} \ and \ \propto \geq 0.$$

The pre-sine function is denoted by psin (t,∝) and is defined as

$$psin(t,\propto) = 1 + \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+\propto}}{\Gamma(2n+2+\propto)}, t \in \mathbb{R} \ and \ \propto \geq 0.$$

The extended pre-function is denoted by ρM_{32} and is defined as

$$\rho M_{32}(t,\propto) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{3n+2+\propto}}{\Gamma(3n+3+\propto)}, t \in \mathbb{R} \ and \ \propto \geq 0.$$

Laplace Transforms of Pre-functions and extended Pre-functions:

1. $L[pexp(at,\propto)] = \frac{1}{s} + \frac{a^{1+\propto}}{(s-a)^{1+\propto}}.$
2. $L[pexp(-at,\propto)] = \frac{1}{s} - \frac{(-1)^{\propto} a^{1+\propto}}{(s+1)^{1+\propto}}.$
3. $L[pcos(at,\propto)] = \frac{1}{s} - \frac{a^{\propto+2}}{(S^2+a^2)S^{1+\propto}}.$

4. $L[psin(at,\propto)] = \frac{a^{1+\propto}}{(S^2+a^2)S^{\propto}}$.
5. $L[pcosh(at,\propto)] = \frac{1}{s} + \frac{a^{\propto+2}}{(S^2-a^2)S^{1+\propto}}$.
6. $L[psinh(at,\propto)] = \frac{a^{1+\propto}}{(S^2-a^2)S^{\propto}}$.
7. $L[\rho M_{3,2}(at,\propto)] = \frac{a^{2+\propto}}{(S^3+a^3)S^{\propto}}$.

III. Main Results:

In this section we discuss some main results which play a key role in the development of Laplace Transforms of General pre-functions and extended pre-functions.

Theorem 3.1 *First Shifting Theorem*:

$$If\ L\{f(t,\propto)\} = F(s,\propto),$$
$$then\ L\{e^{-at}f(t,\propto)\} = F(s+a,\propto).$$

Proof: By definition of Laplace Transforms, we have

$L\{e^{-at}f(t,\propto)\} = \int_0^\infty e^{-st}e^{-at}f(t,\propto)dt = \int_0^\infty e^{-(s+a)t}f(t,\propto)dt.$ (3.1.1)

Put $s+a=p$ in the equation (3.1.1), we have

$$L\{e^{-at}f(t,\propto)\} = \int_0^\infty e^{-pt}f(t,\propto)dt = F(s+a,\propto).$$

Thus,

$$L\{e^{-at}f(t,\propto)\} = F(s+a,\propto).$$

This completes the proof.

Theorem 3.2 (*Second Shifting Theorem*)

$$If\ L\{f(t,\propto)\} = F(s,\propto),$$
$$then\ L\{e^{at}f(t,\propto)\} = F(s-a,\propto).$$

Proof: By definition of Laplace Transforms,

$L\{e^{at}f(t,\propto)\} = \int_0^\infty e^{-st}e^{at}f(t,\propto)dt = \int_0^\infty e^{-(s-a)t}f(t,\propto)dt.$ (3.1.2)

Put $s-a=p$ in the equation (3.1.2), we have

$$L\{e^{at}f(t,\propto)\} = \int_0^\infty e^{-pt}f(t,\propto)dt = F(s-a,\propto).$$

Thus,

$$L\{e^{-at}f(t,\propto)\} = F(s-a,\propto).$$

This completes the proof.

3.3 General prefunctions and Laplace transform:

In this section, we introducesome applications of our main results.

1. **Laplace Transforms of** $f(t,\propto) = e^{at}pexp(t,\propto)$

We know that the$L[pexp(t,\propto)] = \frac{1}{s} + \frac{a^{1+\propto}}{(s-a)^{1+\propto}} = F(s,\propto)$.

By applying shifting theorem to more general prefunctions, we get

$$L\{e^{at}pexp(t,\propto)\} = F(s-a,\propto)$$

$=\frac{1}{s-a} + \frac{a^{1+\propto}}{[(s-a)-1](s-a)^{1+\propto}}$. (3.1.3)

And inverse Laplace Transforms of this function is

$L^{-1}\left\{\frac{1}{[(s-a)-1](s-a)^{\propto+1}} + \frac{1}{s-a}\right\} = e^{at}pexp(t,\propto)$.

2. **Laplace Transforms of** $f(t,\propto) = e^{at}pexp(-t,\propto)$

The Laplace Transforms of $pexp(-t,\propto)$ is given by

$L[pexp(-t,\propto)] = \frac{1}{s} - \frac{(-1)^{\propto}}{(s-1)^{1+\propto}} = F(s,\propto)$.

We can calculate the Laplace Transform of more general prefunctions $e^{at}pexp(-t,\propto)$.

By applying shifting theorem to more general prefunctions$e^{at}pexp(-t,\propto)$, we acquire

$$L\{e^{at}pexp(-t,\propto)\} = F(s-a,\propto).$$

$=\frac{1}{s-a} - \frac{a^{1+\propto}}{[(s-a)+1](s-a)^{1+\propto}}$. (3.1.4)

And Inverse Laplace Transform of this function is

$$L^{-1}\left\{-\frac{(-1)^{\propto}}{[(s-a)+1](s-a)^{\propto+1}}+\frac{1}{s-a}\right\}=e^{at}pexp(-t,\propto)$$

3. **Laplace Transforms of** $f(t,\propto)=e^{at}\,\frac{t^{\propto}}{\Gamma(\propto+1)}$.

The Laplace Transform of general prefunctions $f(t,\propto)=e^{at}\,\frac{t^{\propto}}{\Gamma(\propto+1)}$ is

$L\left\{e^{at}\,\frac{t^{\propto}}{\Gamma(\propto+1)}\right\}=F(s-a,\propto)$=$\frac{1}{(s-a)^{\propto+1}}$.

The Laplace Transform of this general prefunction is

$L^{-1}\left\{\frac{1}{(s-a)^{\propto+1}}=\right\}e^{at}\,\frac{t^{\propto}}{\Gamma(\propto+1)}$.

4. **Laplace Transforms of** $f(t,\propto)=e^{at}pcos(t,\propto)$.

The Laplace Transform of general prefunctions $f(t,\propto)=e^{at}pcos(t,\propto)$ is

$L\{e^{at}pcos(t,\propto)\}=F(s-a,\propto)$=$\frac{1}{s-a}-\frac{a^{1+\propto}}{[(s-a)^2+1](s-a)^{1+\propto}}$.

The inverse Laplace Transform of this general prefunction is

$$L^{-1}\left\{\frac{1}{s-a}-\frac{1}{[(s-a)^2+1](s-a)^{1+\propto}}\right\}=\ e^{at}pcos(t,\propto).$$

5. **Laplace Transforms of** $f(t,\propto)=e^{at}sin(t,\propto)$:

The Laplace Transform of general prefunctions $f(t,\propto)=e^{at}sin(t,\propto)$ is

$L\{e^{at}sin(t,\propto)\}=F(s-a,\propto)$=$\frac{1}{[(s-a)^2+1](s-a)^{\propto}}$.

The inverse Laplace Transform of this general prefunction is

$L^{-1}\left\{\frac{1}{[(s-a)^2+1](s-a)^{\propto}}\right\}\ =\ e^{at}sin(t,\propto)$.

6. **Laplace Transforms of** $f(t,\propto)=e^{at}sinh(t,\propto)$.

The Laplace Transform of general prefunctions $f(t,\propto)=e^{at}sinh(t,\propto)$ is

$L\{e^{at}sinh(t,\propto)\}=F(s-a,\propto)$=$\frac{1}{[(s-a)^2-1](s-a)^{\propto}}$.

The inverse Laplace Transform of this general prefunction is

$L^{-1}\left\{\frac{1}{[(s-a)^2-1](s-a)^{\propto}}\right\}=\ e^{at}sinh(t,\propto)$.

7. **Laplace Transforms of** $f(t,\propto)=e^{at}sinh(t,\propto)$.

The Laplace Transform of general prefunctions $f(t,\propto)=e^{at}pcosh(t,\propto)$ is

$L\{e^{at}pcosh(t,\propto)\}=F(s-a,\propto)$=$\frac{1}{s-a}+\frac{a^{1+\propto}}{[(s-a)^2-1](s-a)^{1+\propto}}$.

The inverse Laplace Transform of this general prefunction is

$L^{-1}\left\{\frac{1}{s-a}+\frac{a^{1+\propto}}{[(s-a)^2-1](s-a)^{1+\propto}}\right\}=\ e^{at}pcosh(t,\propto)$.

8. **Laplace Transforms of** $f(t,\propto)=e^{at}\rho M_{3,2}$

The Laplace Transform of general prefunction $e^{at}\rho M_{3,2}$ is

$L\left[\rho M_{3,2}(t,\propto)\right]=\frac{1}{(S-a)^3+1)(S-a)^{\propto}}$.

The inverse Laplace Transform of this general prefunction is

$L^{-1}\left\{\frac{1}{(S-a)^3+1)(S-a)^{\propto}}\right\}=\ e^{at}\rho M_{3,2}$.

Example 3.4.1 Considerthe first orderinitial value problem,

$\dot{x}(t,\propto)-(1+a)x(t,\propto)=e^{at}\left(\frac{t^{\propto}}{\Gamma(\propto+1)}-1\right)$.(3.1.5)

With initial condition,

$x(0,\propto)=1$(3.1.6)

Applying Laplace Transform to both sides of differential equation (3.1.4), we get

$$L\{\dot{x}\,(t,\propto)\}-(1+a)L\{x(t,\propto)\}=L\left\{e^{at}\left(\frac{t^{\propto}}{\Gamma(\propto+1)}-1\right)\right\}.$$

$sL\{x(t,\propto)\} - x(0,\propto) - (1+a)L\{x(t,\propto)\} = \frac{1}{(s-a)^{\propto+1}} - \frac{1}{s-a}.$

Setting $L\{x(t,\propto)\} = X(s,\propto)$ and using the initial condition (3.1.6), we have

$$X(s,\propto) = \frac{1}{[(s-a)-1](s-a)^{\propto+1}} + \frac{1}{s-a}$$

Taking inverse Laplace Transform, we acquire

$$x(t,\propto) = L^{-1}\left\{\frac{1}{[(s-a)-1](s-a)^{\propto+1}} + \frac{1}{s-a}\right\}.$$

$$x(t,\propto) = e^{at}\, p\, exp(t,\propto).$$

Hence the prefunction

$$x(t,\propto) = e^{at}\, p\, exp(t,\propto)$$

is the solution of the first order initial value problem (3.1.5)- (3.1.6).

Example 3.4.2 Consider the second order initial value problem

$\ddot{x}(t,\propto) - 2a\dot{x}(t,\propto) - (1-a^2)x(t,\propto) = e^{at}\left(\frac{t^{\propto}}{\Gamma(\propto+1)} - 1\right).$ (3.1.7)

With initial conditions

$x(0,\propto) = 1, \dot{x}(0,\propto) = 0.$ (3.1.8)

Applying Laplace Transforms to both sides of the differential equation (3.1.7), we have

$L\{\ddot{x}(t,\propto)\} - 2aL\{\dot{x}(t,\propto)\} - (1-a^2)L\{x(t,\propto)\} = L\left\{e^{at}\left(\frac{t^{\propto}}{\Gamma(\propto+1)} - 1\right)\right\}.$

$s^2L\{x(t,\propto)\} - sx(0,\propto) - \dot{x}(0,\propto) - 2asL\{x(t,\propto)\} + 2ax(0,\propto) - (1-a^2)L\{x(t,\propto)\} = \frac{1}{(s-a)^{\propto+1}} - \frac{1}{s-a}.$

Setting, $L\{x(t,\propto)\} = X(s,\propto)$ and using the initial condition (3.1.8), we get

$$X(s,\propto) = \frac{1}{s-a} + \frac{1}{[(s-a)^2-1](s-a)^{1+\propto}}.$$

Taking the inverse Laplace Transform, we have

$X(t,\propto) = L^{-1}\left\{\frac{1}{s-a} + \frac{1}{[(s-a)^2-1](s-a)^{1+\propto}}\right\}.$

$X(t,\propto) = e^{at}\, Cosh(t,\propto).$

Thus, the prefunction

$$X(t,\propto) = e^{at}\, Cosh(t,\propto)$$

is the general solution of second order initial value problem (3.1.7)-(3.1.8).

Conclusion: In this paper we introduced the general prefunctions and their Laplace transforms and it is interesting we have shownthat these prefunctions are the solutions of IVPs of first order as well as second order. We have applied Laplace Transforms technique fruitfully.

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