Time Delay Stabilizes Chaos Dynamics in Economic System

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Abstract— *This paper discusses the role of time delay feedback which stabilizes the chaos of nonlinear financial model. The interest rate, investment demand and price index are modelled with the help of saving amount, cost per investment, demand elasticity of commercial markets and the strength of feedback. All these parameters are considered to be positive. The stability of financial model is studied by Routh-Hurwitz criterion. The distributed time delay feedback strength stabilized the unstable financial system. Bifurcation of parameter and Lyapunov exponent is simulated for time delay feedback system.*
Using the numerical method, it is observed that the inappropriate combination of saving amount, cost per investment and elasticity of investment demand of commercial market in the financial system is the root cause of chaos. The unstable system is stabilized by introducing the distributed time delay feedback strength. The stability and chaotic behaviour of systems gives the condition and behaviour of economic implications. It is concluded that the system is chaotic and to ensure stability of state, it is controlled by time delay feedback controlled system.

Keywords *—* Nonlinear financial system**,** Distributed time delay, chaos, Routh-Hurwitz criterion, Lypunove exponent, Bifurcation and equilibrium point.
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INTRODUCTION

Nonlinear Financial dynamical system is a complicated system and it may exhibit variety of complex behaviour, like Hopf bifurcation, periodic, quasiperiodic, chaos etc. Delay in financial system is an important factor in economic dynamic system. The time delays emerge in the dynamics of economic variables of the system. It is a period of time between investment decision and implementation of investment that control the chaos in the system. The differential with time delay exhibit more complicated dynamics than the ordinary differential equation. Time delay control the instability of economical dynamic system. Cai and Shen [1] studied the stability of the equilibrium point and Hopf bifurcation of coupled Vander pal and Duffing oscillators. Cai, Hung and Jiangsu [2] studied the nonlinear finance chaotic system, equilibrium point stabilized with the adaptive control method. Chen [3] discussed fixed point, periodic motions and chaotic motion of the fractional order financial system. Dumitree and Opris [4] analysed the Kaldor Kaleki model of business cycle by using Neimatk-Show to show photographs in conalker bifurcation. Kaddar and Alaoui [6] introduced the delay into capital stock and gross product in capital accumulation equation and discussed that local stability Hopf bifurcation exist as the delay cross some critical value. Ma and Chen [8] discussed that the elasticity of the investment demand of commercial market appropriately will help to stable economy as the amount of saving is smaller than the greater fluctuation of the system and it will cause chaos. Ranjan and Bhardwaj [9] concluded that the bifurcation analysis depends on the value of investment demand and also on the elasticity of investment demand of commercial market. Wang and Wu [11] discussed Hopf bifurcation and periodic solution of Kaldor-Kalecki model of business cycle with time delay as the delay crosses some critical values. Wang, Zhai and Wang [12] discussed a continuous time delay complex dynamic behavior of the financial system. Wang, Huang and Shen [13] studied the control of an uncertain fractional order economic system via adaptive sliding mode. Xu & Zhang [15] studied the Lotca Volterra model with time delay and delay dependent parameter. Yu, Cai and Li [16] studied a new hyper chaotic finance system and studied the equilibrium point, stability and also stabilized the hyper chaotic system to unstable equilibrium by linear feedback control and speed feedback control. Zen, Xia and Guodery [17] discussed the complex dynamics of fractional order financial system with time delay and an approximate time delay will enhance or suppress the emergence of chaotic or periodic motion of the system. Zhao, Li and Li [18] studied the synchronization strategies of three different methods -Hybrid feedback control active control and adaptive control method.

In this paper, the stability analysis of economic dynamical systems for financial system with delay feedback is studied. Numerical simulation shows that the system possesses complex dynamic behaviour and the feedback strength suppress the chaotic behaviour. Also, the stability of chaotic economic dynamic system is studied using time delay control method, bifurcation analysis and Lyapunov exponents.

METHODOLOGY

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- **Equilibrium point**

  The point $u \in R^n$ where f(u*)=0 is called equilibrium point of $\dfrac{du}{dt} = f(u)$.

- **Stability analysis of equilibrium point**

  Consider the system
  $$x'(t) = f(x, y, z), y'(t) = g(x, y, z), z'(t) = h(x, y, z)$$

where f, g and h are differentiable with continuous partial derivatives and they vanish at the equilibrium point (x_0, y_0, z_0). Let J denote the Jacobian matrix at the equilibrium point, then

$$J = \begin{bmatrix} f_x(x_0, y_0, z_0) & f_y(x_0, y_0, z_0) & f_z(x_0, y_0, z_0) \\ g_x(x_0, y_0, z_0) & g_y(x_0, y_0, z_0) & g_z(x_0, y_0, z_0) \\ h_x(x_0, y_0, z_0) & h_y(x_0, y_0, z_0) & h_z(x_0, y_0, z_0) \end{bmatrix}$$

The following conditions are defined:

1. If all eigenvalues of J have negative real part then (x_0, y_0, z_0) is asymptotically stable.

2. If some eigenvalue of J has positive real part, then (x_0, y_0, z_0) is unstable.

- **Routh Hurwiz Theorem (RHT)**

  It states that

1. For the cubic characteristic equation given as
   $$\lambda^3 + \beta_1 \lambda^2 + \beta_2 \lambda + \beta_3 = 0,$$

   where $H_1 = \beta_n > 0, n = 1, 2, 3$ and $H_2 = \begin{vmatrix} \beta_1 & \beta_3 \\ 1 & \beta_2 \end{vmatrix} = \beta_1 \beta_2 - \beta_3$

   case 1: If $H_1 > 0$ and $H_2 > 0$,
      then the system is asymptotic stable;

   case 2: If $H_1 > 0$ and $H_2 = 0$,
      then the system is critically stable;

   case 3: If $H_1 > 0$ and $H_2 < 0$,
      then the system is chaotic.

2. For the fourth-degree characteristic polynomial given as
   $$\lambda^4 + \alpha_1 \lambda^3 + \alpha_2 \lambda^2 + \alpha_3 \lambda + \alpha_4$$

   where
   $$\Delta_1 = \alpha_n > 0, n = 1, 2, 3, 4; \Delta_2 = \alpha_1 \alpha_2 - \alpha_3;$$
   $$\Delta_3 = \alpha_1 \alpha_2 \alpha_3 - \alpha_3^2 - \alpha_1^2 \alpha_4$$

   case 1: If $\Delta_1 > 0, \Delta_2 > 0$ and $\Delta_3 > 0$,
      then the system is asymptotic stable;

   case 2: If $\Delta_1 > 0, \Delta_2 > 0$ and $\Delta_3 = 0$,
      then the system is critically stable;

   case 3: If $\Delta_1 > 0, \Delta_2 > 0$ and $\Delta_3 < 0$,
      then the system is chaotic.

Lyapunov Exponents

Lyapunov exponents for the fixed point u^* in the dynamical system $\dfrac{du}{dt} = f(u)$, be defined as: Let

$\lambda_1, \lambda_2, \ldots\ldots \lambda_n$ be the eigen values of the linearized equation $\dfrac{du}{dt} = A\left(u^*\right)$ such that

$$m_1\left(t\right) = e^{\lambda t} \ and \ \lambda_i' = \lim_{t \to \infty} \frac{1}{t}\ln\left|e^{\lambda_i t}\right| = \mathrm{Re}\left[\lambda_i\right]$$

Lyapunov exponents are equal to the real parts of the eigen values at the critical points. Following table gives the behavior of the chaotic system as per the value of the Lyapunov exponent:

Attractor	Lyapunov Exponent
Equilibrium	$0 > \lambda_1' \geq \lambda_2' \geq \lambda_3' \ldots\ldots \geq \lambda_n'$
Periodic limit cycle	$\lambda_1' = 0,\ 0 > \lambda_2' \geq \lambda_3' \geq \lambda_4' \ldots\ldots \geq \lambda_n'$
2-periodic	$\lambda_1' = \lambda'_2 = 0,$ $0 > \lambda_3' \geq \lambda_4' \geq \ldots\ldots \geq \lambda_n'$
k-periodic	$\lambda_1' = \lambda_2' = \lambda_3' = \ldots\ldots \lambda_k' = 0,\ 0 > \lambda'_{k+1} \geq \lambda'_{k+2} \geq \ldots\ldots \geq \lambda'_{k+n}$
Strange chaotic	$\lambda_1' > 0,\ \sum\limits_{i=1}^{n} \lambda_i' < 0$
Hyper chaotic	$\lambda_1' > 0, \lambda_2' > 0, \sum\limits_{i=1}^{n} \lambda_i' < 0$

- **Bifurcation Analysis**

    Consider the continuous dynamical system described by the ODE $\dfrac{du}{dt} = f\left(u\right)$ A local bifurcation occurs at a point if the Jacobian matrix has an eigenvalue with zero real part. If the eigenvalue is equal to zero, the bifurcation is a steady state bifurcation, but if the eigenvalue is non-zero but purely imaginary, this is a Hopf bifurcation.

NON-LINEAR DYNAMICAL FINANCIAL SYSTEM

    Let us consider the nonlinear financial model where variable x is used to represent the interest rate, variable y to represent the investment demand and variable z to represent the price exponent. These three variables changing rates about time are thought of as three new state variables. The factors that influence the changes of *x* mainly come from two aspects: firstly, it is the contradiction from the investment market *i.e.* the surplus between investment and savings secondly it is the structure adjustment from goods prices. The changing rate of *y* is in proportion with the rate of investment and in proportion by inversion with the cost of investment and the interest rate. The changes of *z* are controlled by the contradiction between supply and demand of the commercial market and are influenced by the inflation rate. The amount of supplies and demands of commercials is constant in a certain period of time and that the amount of supplies and demands of commercials is in proportion by inversion with the prices. The changes of the inflation rate can be represented by the changes of the real interest rate and the inflation rate equals the nominal interest rate subtracts the real interest rate. Therefore, by choosing the appropriate coordinate system and setting an appropriate dimension to every state variable, the following simplified model is obtained

$$\left.\begin{aligned} \frac{dx}{dt} &= z + \left(y - a\right)x \\ \frac{dy}{dt} &= 1 - by - x^2 \\ \frac{dz}{dt} &= -x - cz \end{aligned}\right\} \quad \ldots\ldots \mathrm{Model}(1)$$

where variable x is used to represent the interest rate, variable y to represent the investment demand and variable z to represent the price exponent. a ($\geq$0) is the saving amount, b ($\geq$0) is the per- investment cost, c ($\geq$0) is the elasticity of demands of commercials market. Ranjan and Bhardwaj [9] studied the role of investment on dynamics of the above mentioned micro-economic model (1) and observed that time bifurcation depends on the investment demand and elasticity of commercial markets parameter.

Stability analysis of Model (1)

For model (1), we get three equilibrium points, which are given as

$$p_1\left(0,\frac{1}{b},0\right), p_2\left(\sqrt{\frac{c-abc-b}{c}},\frac{1}{c},-\sqrt{\frac{c-abc-b}{c}}\right), \text{and } p_3\left(-\sqrt{\frac{c-abc-b}{c}},\frac{1}{c},\sqrt{\frac{c-abc-b}{c}}\right)$$

The stability conditions give:

- If $c-abc-b<0,$ then the equilibrium point p_2 and p_3 are complex, thus, model (1) has only one equilibrium point $p_1\left(0,\frac{1}{b},0\right)$;

- If $c-abc-b\geq 0 \Rightarrow c\geq\frac{b}{1-ab}=c_1$. In this case the system has three equilibrium points

$$p_1\left(0,\frac{1}{b},0\right), p_2\left(\sqrt{\frac{c-abc-b}{c}},\frac{1}{c},-\sqrt{\frac{c-abc-b}{c}}\right), p_3\left(-\sqrt{\frac{c-abc-b}{c}},\frac{1}{c},\sqrt{\frac{c-abc-b}{c}}\right)$$

For $c-abc-b\geq 0$, Jacobian matrix of Model (1) at equilibrium point p_2 and p_3 is given as

$$J=\begin{bmatrix} \dfrac{1}{c} & \pm\sqrt{\dfrac{c-abc-b}{c}} & 1 \\ \mp 2\sqrt{\dfrac{c-abc-b}{c}} & -b & 0 \\ -1 & 0 & -c \end{bmatrix}$$

The characteristic equation of above Jacobian matrix is given as

$$\lambda^3+\lambda^2\left(b+c-\frac{1}{c}\right)+\lambda\left\{b\left(c-\frac{1}{c}\right)+2\left(\frac{c-abc-b}{c}\right)\right\}+2(c-abc-b)=0$$

Comparing with cubic polynomial

$$\lambda^3+\beta_1\lambda^2+\beta_2\lambda+\beta_3=0 \text{ we get}$$

$$\beta_1=b+c-\frac{1}{c}; \beta_2=b\left(c-\frac{1}{c}\right)+2\left(\frac{c-abc-b}{c}\right); \beta_3=2(c-abc-b);$$

$$H_1=\beta_n>0, n=1,2,3;$$

$$H_2=\begin{vmatrix}\beta_1 & \beta_3 \\ 1 & \beta_2\end{vmatrix}=\beta_1\beta_2-\beta_3$$

$$=bc^4+b^2c^3-2ab^2c^2+\left(2ab-2-3b^2\right)c+3b=0$$

Let us consider the bifurcation boundary at which $H_2=0$. By choosing c as bifurcation parameter and using the parameter as given by Gao and Ma, it is observed that model (1) is chaotic when a<9, b=0.1 and c=1. For fixed a=0.9, b=0.1, the critical value of c is obtained as $c_c=2.5620$. It is easy to see, that when $c_c>c$ Model (1) is chaotic (by case2 of RHT). The eigenvalues are computed at the equilibrium point which are given as:

$p_2(0.86, 1.4, -4.3)$; $\lambda_1=0.0472+1.420\,i$, $\lambda_2=0.0472-1.420\,i$, $\lambda_3=-1.6945$

$p_3(-0.86, 1.4, 4.3)$; $\lambda_1=0.0472+1.420i$, $\lambda_2=0.0472-1.420\,i$, $\lambda_3=-1.6945$

For both equilibrium points, real part of eigenvalues is positive. Therefore, trajectory of the model (1) is unstable. Also, the value of β_1 and β_2 are computed as:

$\beta_1=1.6, \beta_2=1.87, \beta_3=3.44,$

$H_1>0, n=1,2,3. H_2=-0.688$

Thus, from Routh Hurwitz criteria Model (1) is chaotic. Lypunov exponent of model (1) at a=0.9, b=0.1 and c=2 are given as: $\lambda_1'=0.1780, \lambda_2'=-1.2257, \lambda_3'=-1.9523$.

There is one positive Lypunov exponent and $\sum_{i=1}^{3}\lambda'<0$ thus the model (1) is chaotic.

NONLINEAR DELAY FINANCIAL DYNAMICAL SYSTEM

To control the chaos of financial model (1), the time delay feedback control is introduced. We introduced the distributive time delay force into the second equation of model (1). The function F(t) satisfies the following delayed system which is obtained as:

$$\left.\begin{array}{l} \dfrac{dx}{dt} = x(t) + \big(y(t) - a\big)x(t) \\[2mm] \dfrac{dy}{dt} = 1 - by(t) - x^2(t) + k\displaystyle\int_{-\infty}^{t} F(t-\tau)y(\tau)d\tau \\[2mm] \dfrac{dz}{dt} = -x(t) - cz(t) \end{array}\right\} - (Model)(2)$$

where $\tau \geq 0$ is time delay and k is the strength of the feedback. To study the effect of strength of the feedback k in model (2), we transform the model (2) into linear form.

Linear transform of model (2)

Let $F(t) = de^{-dt}, d \geq 0,$ and $\psi(t) = \displaystyle\int_{-\infty}^{t} F(t-\tau)y(\tau)d\tau$, the model (2) can be written as

$$\left.\begin{array}{l} \dfrac{dx}{dt} = z(t) + \big(y(t) - a\big)x(t) \\[2mm] \dfrac{dy}{dt} = 1 - by(t) - x^2(t) + k\psi(t) \\[2mm] \dfrac{dz}{dt} = -x(t) - cz(t) \\[2mm] \dfrac{d\psi}{dt} = d\big(y(t) - \psi(t)\big) \end{array}\right\} - (Model)(3)$$

The role of investment on dynamics of the micro-economic of model (2) is discussed and observed that time bifurcation depends on the feedback strength. Model (1) represents nonlinear financial system, model (2) as delayed system and model (3) as feedback system.

Stability analysis of model (3)

There are three equilibrium points of delay control model (3)

$$E_1 = \left(0, \frac{1}{b-k}, 0, \frac{1}{b-k}\right); \quad E_2 = \left\{+\sqrt{\frac{c-(ac+1)(b-k)}{c}}, \left(a+\frac{1}{c}\right), -\sqrt{\frac{c-(ac+1)(b-k)}{c}}, \left(a+\frac{1}{c}\right)\right\};$$

$$E_3 = \left\{-\sqrt{\frac{c-(ac+1)(b-k)}{c}}, \left(a+\frac{1}{c}\right), +\sqrt{\frac{c-(ac+1)(b-k)}{c}}, \left(a+\frac{1}{c}\right)\right\}$$

Case-1: Let $c - (ac+1)(b-k) < 0$ Then there is only one equilibrium point as $\sqrt{c-(ac+1)(b-k)}$ is complex number. $E_1 = \left(0, \frac{1}{b-k}, 0, \frac{1}{b-k}\right)$

Case-2: Let $c - (ac+1)(b-k) \geq 0 \Rightarrow k \geq b - \frac{c}{(ac+1)} = (c_2)$

Then the delay control model (3) has three equilibrium points

$$E_1 = \left(0, \frac{1}{b-k}, 0, \frac{1}{b-k}\right); \quad E_2 = \left\{+\sqrt{\frac{c-(ac+1)(b-k)}{c}}, \left(a+\frac{1}{c}\right), -\sqrt{\frac{c-(ac+1)(b-k)}{c}}, \left(a+\frac{1}{c}\right)\right\}$$

$$E_3 = \left\{-\sqrt{\frac{c-(ac+1)(b-k)}{c}}, \left(a+\frac{1}{c}\right), +\sqrt{\frac{c-(ac+1)(b-k)}{c}}, \left(a+\frac{1}{c}\right)\right\}$$

Jacobian matrix of delay control system (3) at equilibrium point is given as

$$J = \begin{bmatrix} \dfrac{1}{c} & \pm\sqrt{\dfrac{c-(ac+1)(b-k)}{c}} & 1 & 0 \\ \mp 2\sqrt{\dfrac{c-(ac+1)(b-k)}{c}} & -b & 0 & k \\ -1 & 0 & -c & 0 \\ 0 & d & 0 & -d \end{bmatrix}$$

Characteristic equation of above Jacobian matrix is given as

$$\Rightarrow \lambda^4 + (b+d+c-\tfrac{1}{c})\lambda^3 + \{(b+d)\left(c-\tfrac{1}{c}\right)+2\dfrac{c-(ac+1)(b-k)}{c}+d(b-k)\}\lambda^2 + \{(b-k)d\left(c-\tfrac{1}{c}\right)$$

$$+2\dfrac{c-(ac+1)(b-k)}{c}(c+d)\}\lambda + 2\{c-(ac+1)(b-k)\}d = 0$$

Comparing with fourth degree polynomial

$$\lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + \alpha_3\lambda + \alpha_4 = 0$$

where

$$\alpha_1 = b+d+c-\tfrac{1}{c}; \quad \alpha_2 = (b+d)\left(c-\tfrac{1}{c}\right)+2\dfrac{c-(ac+1)(b-k)}{c}+d(b-k);$$

$$\alpha_3 = (b-k)d\left(c-\tfrac{1}{c}\right)+2\dfrac{c-(ac+1)(b-k)}{c}(c+d); \quad \alpha_4 = 2\{c-(ac+1)(b-k)\}d;$$

According to the Routh-Hurwitz criterion, constraints are imposed as follows:

$$\Delta_1 = \alpha_n > 0, n = 1,2,3,4; \quad \Delta_2 = \alpha_1\alpha_2 - \alpha_3;$$

$$\Delta_3 = \alpha_1\alpha_2\alpha_3 - \alpha_3^2 - \alpha_1^2\alpha_4$$

Let us consider the bifurcation boundary at which $\Delta_3 = 0$, for choosing d as bifurcation parameter. In delay control model (3), fixed the parameter a=.9, b=.1, c=2 and from condition (c_2), $k \geq -0.6142857 --- (c_3)$, thus let us choose k=-0.3.

Let us consider $\Delta_3 > 0$, then the following proposition hold at the above fixed value of parameter and we get

$1.052d^3 + 1.5452d^2 - 0.11296d - 0.19112 ------- (c_4)$,d has at least one positive value. Since $d \geq 0$.

Let us consider the bifurcation boundary at which $\Delta_3 = 0$ .By choosing d as bifurcation parameter, the critical value of $d_c = 0.3517$. It is easy to see, when $d > d_c$,(from case1 of RHT), the delay control model (3) is stable. For fixed value a=0.9, b=0.1, c=2, k=-0.3.and d=0.4 the eigenvalues computed at the equilibrium point

- $E_2\left(0.663325, 1.4, -0.663325, 1.4\right)$ are given as

  $\lambda_1 = -0.0462 + 1.11044i, \lambda_2 = -0.0462 - 1.11044i, \lambda_3 = -1.6120, \lambda_4 = -0.7956$

- $E_3\left(-0.663325, 1.4, 0.663325, 1.4\right)$ are given as

  $\lambda_1 = -0.0462 + 1.11044i, \lambda_2 = -0.0462 - 1.11044i, \lambda_3 = -1.6120, \lambda_4 = -0.7956$

Here the real part of all eigenvalues is negative for both equilibrium point. Therefore, trajectory of the model (3) is stable. Also, the value of α_1, α_2 and α_3 have been computed.

$$\alpha_1 = 2.5, \alpha_2 = 2.74, \alpha_3 = 3.092, \alpha_4 = 1.584$$

$$\Delta_1 > 0, n = 1,2,3,4 \;\&\; \Delta_2 = 3.778 > 0, \Delta_3 = .78576 > 0$$

From case 1 of RHT, model (3) is stable. Lypunov exponent of model (3) are

$$\lambda_1' = -1.4387, \lambda_2' = -0.2406, \lambda_3' = -1.4613, \lambda_4' = -0.2594$$

at a=.9, b=.1, c=2 . All Lypunov exponents of model(3) are negative and $\sum\limits_{i=1}^{4}\lambda' < 0$ thus the model (3) is stable

There is an equilibrium attractor and thus there does not exist poincare map and power spectrum.For fixed value a=0.9, b=0.1, c=2, k=-0.3 and d=0.9 the eigenvalues computed at the equilibrium point:

$E_2\left(0.663325, 1.4, -0.663325, 1.4\right)$ are

$\lambda_1 = -0.0462 + 1.11044i, \lambda_2 = -0.0462 - 1.11044i, \lambda_3 = -1.6120, \lambda_4 = -0.7956$

$E_3(-0.663325, 1.4, 0.663325, 1.4)$ are

$\lambda_1 = -0.0462 + 1.11044i, \lambda_2 = -0.0462 - 1.11044i, \lambda_3 = -1.6120, \lambda_4 = -0.7956$

Here the real part of all eigenvalues is negative for both equilibrium point. Therefore, trajectory of the model (3) is stable. Also, the value of $\alpha_1, \alpha_2 \& \alpha_3$ have been computed.

$\alpha_1 = 2, \alpha_2 = 1.79, \alpha_3 = 2.3556, \alpha_4 = .704$

$\Delta_1 > 0, n = 1, 2, 3, 4 \& \Delta_2 = 3.2244 > 0,$

$\Delta_3 = 4.77939664 > 0$

From case1 of RHT model (3) is stable. Lypunov exponent of model (3) are

$\lambda_1' = -1.4387, \lambda_2' = -0.4767 \lambda_3' = -1.4613 \lambda_4' = -0.5233$ at $a = 0.9, b = 0.1 \& c = 2$

All Lyapunov exponents of model (3) are negative and thus the model (3) is stable. There is an equilibrium attractor and thus there does not exist Poincare map and power spectrum.

Results and Discussion

The critical values of elasticity of commercial market, equilibrium point and eigenvalue for the fixed amount of saving and cost per investment are computed. Table 1 gives the value of equilibrium point and eigenvalues for the parameter a=.9, b=.1, c=2 and phase portrait, time series, Lyapunov exponent and bifurcation diagram are plotted in the figure 1,2, 3,4 respectively for the model (1). From the table 1 and figure 1-3, it is observed that model (1) is chaotic and its trajectory is unstable. From bifurcation figure 4, it is observed that model (1) becomes stable as elasticity of demand crosses critical value c_c (=2.5620). The critical value of d (time delay factor), time delay feedback strength, equilibrium point and eigenvalue for the fixed amount of saving and cost per investment computed. Table-2 and Table-3 gives the value of equilibrium point and eigenvalues for the parameter a=.9, b=.1, c=2, k=-.3, d=.4 and d=.9 respectively. A phase portrait, time series Lyapunov exponent and bifurcation diagram are plotted in the figure 5, figure 6, figure 7 and figure 8 respectively for the model (3). From the Table 2-3 and figure 5-8, it is observed that model (3) is stable and equilibrium attractor exist. In the bifurcation figure 8, the dynamical financial model switch from oscillatory to the stable steady state as delay crosses the critical values. It is observed that the chaotic model (1) is stabilized by controlling time delay feedback method.

Table-1: Nature of trajectory at equilibrium point with eigenvalues for the parameter a=.9, b=.1 and c=2 in model (1)

Equilibrium point	Eigenvalue (λ_1)	Eigenvalue (λ_2)	Eigenvalue (λ_3)	C	Nature of Model (1)
(0.86, 1.4, -.43)	0.0472+1.42i	0.0472-1.42i	-1.6945	$c_c > c$	Model 1 is unstable as shown in figures.1.2.3 &4
(-0.86, 1.4,.43)	0.0472+1.42i	0.0462-1.42i	-1.6945		

* c_c (=2.5620) is critical value of elasticity of demand (c)

Table-2: Nature of trajectory at equilibrium point with eigenvalues for the parameter a=0.9, b=0.1, c=2, k=-0.3 and d=0.9 for model (3)

Equilibrium point	Eigenvalue (λ_1)	Eigenvalue (λ_2)	Eigenvalue (λ_3)	Eigenvalue (λ_4)	d	Nature of trajectory
(0.663325, 1.4, 0.663325, 1.4)	-0.0462+1.11044i	-0.0462-1.11044i	-1.6120	-0.7956	$d > d_c$	Model (3) is stable as shown in figures.5-8
(-0.663325, 1.4, 0.663325,1.4)	-0.0462+1.11044i	-0.0462+1.11044i	-1.6120	-0.7956		

Table- 3: Nature of trajectory at equilibrium point with eigenvalues for the parameter a=0.9, b=0.1, c=2, k=-0.3 and d=0.4 for model (3)

equilibrium point	Eigenvalue (λ_1)	Eigenvalue (λ_2)	Eigenvalue (λ_3)	Eigenvalue (λ_4)	d	Nature of trajectory
(0.663325, 1.4, -0.663325,1.4)	-0.0034+1.08622i	-0.0034-1.08622i	-1.6263	-0.3669	$d > d_c$	Model 3 is stable as shown in figures .5-8
(-0.663325, 1.4, 0.663325,1.4)	-0.0034+1.08622i	-0.0034-1.08622i	-1.6263	-0.3669		

* $d_c (= 0.3517)$ is critical value of delay(d)
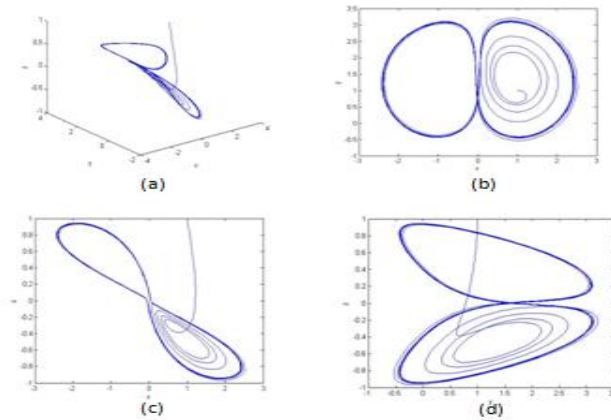
Figure-1: Phase portrait (chaotic Behaviour) of model (1) at a=0.9, b=0.1, c=2 for phase diagram (a) x-y-z (b) x-y (c) x-z (d) y-z
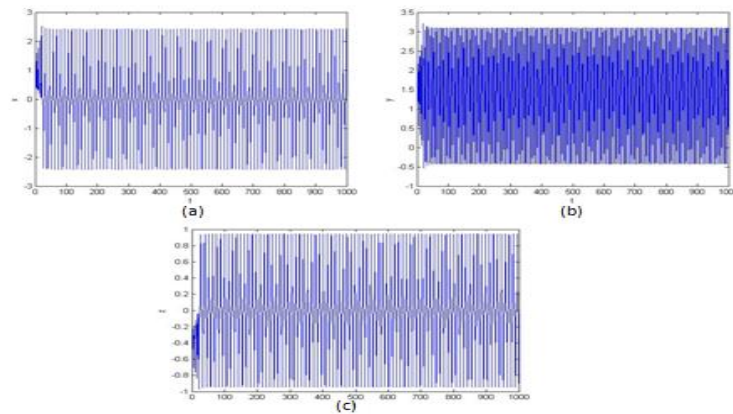
Figure-2: Time series of model (1) at a =0 .9 b=0 .1 c=2 (a) t – x (b) t – y (c) t –z
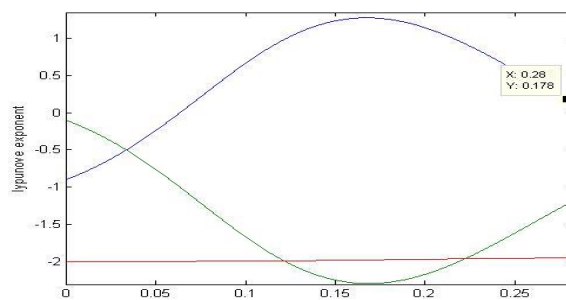
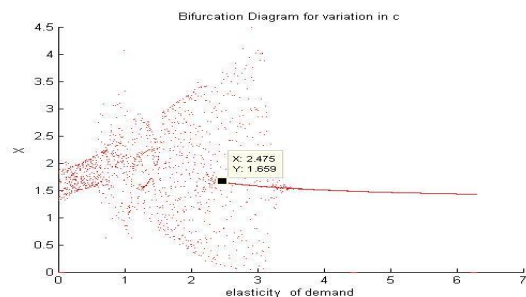**Figure-3**: Lyapunov exponent of model (1) at a=0.9, b=0.1 and c=2

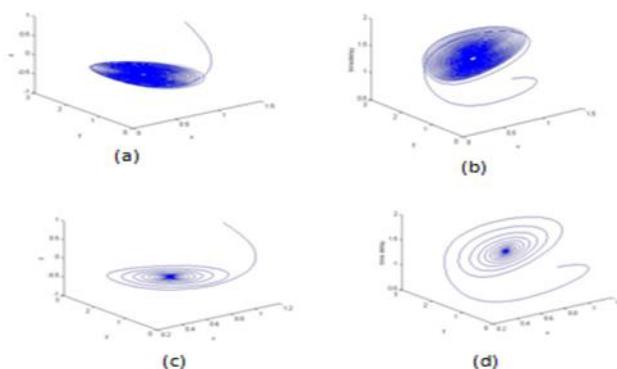**Figure 4**: Hopf Bifurcation of parameter 'c' in financial model (1) at a=0.9,b=0.1

Figure-5: Phase portrait of model (3) at a=0.9, b=0.1, c=2 and k=-.3 (a) x-y-z phase d=.4 (b) x-y- time delay d=.4 (c) x-y-z phase d=.9 (d) x-y-time delay phase d=.9
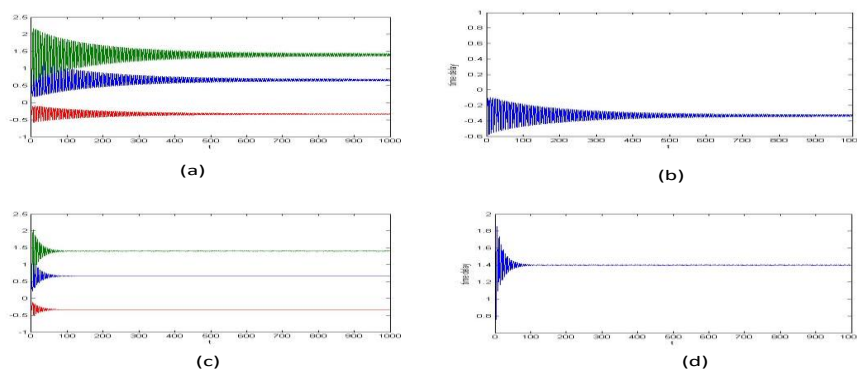

Figure-6: Time series of system (3) at a=0.9, b=0.1, c=2, k= -0.3 (a) t-x, y, z, d=0.4 (b) t-time delay d =0.4 (c) t-x, y, z d=0.9 (d) t-time delay d=0.9
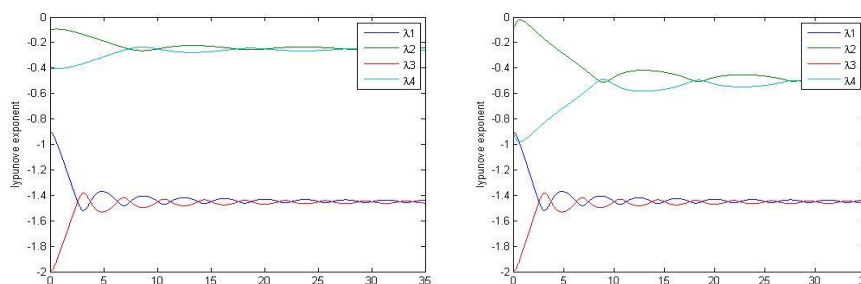

Figure-7: Lyapunov exponent of model (3) (a) a=.9, b=.1, c=2, k=-0.3 and d=0.4 (b) a=0.9, b=0.1, c=2, k=-0.3 and d=0.9.
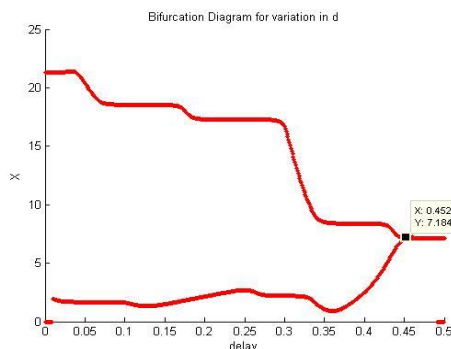
Figure8: Subcritical Hopf Bifurcation of parameter 'd' in delay financial model (3) at a=0.9,b=0.1,c=2 and k=-0.3

CONCLUSIONS

The nonlinear financial system and financial system with distributed time delay have been studied. It is observed that inappropriate combination of parameters in the system is the main cause of chaos and the system tends to out of control in economic dynamics system. The complex behavior in an economic system has been controlled by time delay feedback strength with fixed chaotic value of saving amount, cost per investment and elasticity of demand of commercial market. The critical value of time delay at fixed value of saving amount, cost per investment and elasticity of investment demand of market, which is applied to control the chaotic market and to obtain a regular and stable financial market is observed. The time delay feedback strength and the time delay 'd' must be kept at a proper level through adjusting the interest rate properly, it is also the foundation to stabilized price and keep the society balanced. It indicates that chaotic behavior in such an economic system can be controlled under the appropriate feedback strength and time delay.

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