Multiple integral involving the extension of the Hurwitz-Lerch Zeta function, class of polynomials and multivariable Aleph-functions

$F.Y.\ AYANT^{1}$

1 Teacher in High School , France

ABSTRACT
In the present paper we evaluate a general multiple integrals involving the product of the extension of the Hurwitz-Lerch Zeta function, the multivariable Aleph-functions and general class of polynomials of several variables. The importance of the result established in this paper lies in the fact they involve the Aleph-function of several variables which is sufficiently general in nature and capable to yielding a large of results merely by specializating the parameters their in.. We will the particular case concerning the multivariable I-function defined by Sharma et al [2].

Keywords:Multivariable Aleph-function, general class of polynomials, multiple integral, multivariable I-function, the extension of the Hurwitz-Lerch Zeta function

2010 Mathematics Subject Classification. 33C99, 33C60, 44A20

1.Introduction and preliminaries.

The Alephfunction of several variables generalizes the multivariable I-function recently studied by C.K. Sharma and Ahmad [2] , itself is a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integrals occurring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We define : $\aleph(z_1,\cdots,z_r)=\aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i(1)},q_{i(1)},\tau_{i(1)};R^{(1)};\cdots;p_{i(r)},q_{i(r)};\tau_{i(r)};R^{(r)}}\left(\begin{array}{c}\mathrm{z}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\end{array}\right|$

$[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}]\ ,[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]:$
$\cdots\cdots\cdots\cdots\ ,[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{1,q_i}]:$

$[(\mathrm{c}_j^{(1)});\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;\ [(\mathrm{c}_j^{(r)});\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}]$
$\left.[(\mathrm{d}_j^{(1)});\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;[(\mathrm{d}_j^{(r)});\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}]\right)$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)y_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \tag{1.2}$$

and $$\theta_k(s_k)=\frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

Suppose , as usual , that the parameters

$a_j, j=1,\cdots,p; b_j, j=1,\cdots,q;$

$c_j^{(k)}, j=1,\cdots,n_k; c_{ji^{(k)}}^{(k)}, j=n_k+1,\cdots,p_{i^{(k)}};$

$d_j^{(k)}, j=1,\cdots,m_k; d_{ji^{(k)}}^{(k)}, j=m_k+1,\cdots,q_{i^{(k)}};$

with $k=1\cdots,r, i=1,\cdots,R\ , i^{(k)}=1,\cdots,R^{(k)}$

are complex numbers , and the $\alpha's, \beta's, \gamma's$ and $\delta's$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = +\tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} + \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.4}$$

The reals numbers τ_i are positives for $i=1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)}=1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma-i\infty$ to $\sigma+i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)$ with $j=1$ to m_k are separated from those of $\Gamma(1-a_j+\sum_{i=1}^{r}\alpha_j^{(k)}s_k)$ with $j=1$ to n and $\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)$ with $j=1$ to n_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k=1\cdots,r, i=1,\cdots,R\ , i^{(k)}=1,\cdots,R^{(k)} \tag{1.5}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1,\cdots,z_r) = 0(\ |z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\ ), max(\ |z_1|,\cdots,|z_r|\ ) \rightarrow 0$

$\aleph(z_1,\cdots,z_r) = 0(\ |z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\ ), min(\ |z_1|,\cdots,|z_r|\ ) \rightarrow \infty$

where, $k=1,\cdots,r : \alpha_k = min[Re(d_j^{(k)}/\delta_j^{(k)})], j=1,\cdots,m_k$ and

$$\beta_k = max[Re((c_j^{(k)}-1)/\gamma_j^{(k)})], j=1,\cdots,n_k$$

Serie representations of Aleph-function of several variables is given by

$$\aleph(y_1,\cdots,y_r) = \sum_{G_1,\cdots,G_r=0}^{\infty} \sum_{g_1=0}^{m_1} \cdots \sum_{g_r=0}^{m_r} \frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!} \psi(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})$$

$$\times\ \theta_1(\eta_{G_1,g_1})\cdots\theta_r(\eta_{G_r,g_r})y_1^{-\eta_{G_1,g_1}}\cdots y_r^{-\eta_{G_r,g_r}} \tag{1.6}$$

Where $\psi(.,\cdots,.), \theta_i(.), i=1,\cdots,r$ are given respectively in (1.2), (1.3) and

$$\eta_{G_1,g_1}=\frac{d_{g_1}^{(1)}+G_1}{\delta_{g_1}^{(1)}},\cdots,\ \eta_{G_r,g_r}=\frac{d_{g_r}^{(r)}+G_r}{\delta_{g_r}^{(r)}}$$

which is valid under the conditions $\delta_{g_i}^{(i)}[d_j^i+p_i]\neq\delta_j^{(i)}[d_{g_i}^i+G_i]$ (1.7)

for $j\neq m_i, m_i=1,\cdots\eta_{G_i,g_i}; p_i,n_i=0,1,2,\cdots,; y_i\neq 0, i=1,\cdots,r$ (1.8)

Consider the Aleph-function of s variables

$$\aleph(z_1,\cdots,z_s)=\aleph_{P_i,Q_i,\iota_i;r':P_{i(1)},Q_{i(1)},\iota_{i(1)};r^{(1)};\cdots;P_{i(s)},Q_{i(s)};\iota_{i(s)};r^{(s)}}^{0,N:M_1,N_1,\cdots,M_s,N_s}\left(\begin{array}{c|c} \mathrm{z}_1 & \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_s & \end{array}\right.$$

$$\begin{array}{ll}[(\mathrm{u}_j;\mu_j^{(1)},\cdots,\mu_j^{(r')})_{1,N}] & ,[\iota_i(u_{ji};\mu_{ji}^{(1)},\cdots,\mu_{ji}^{(r')})_{N+1,P_i}]: \\ \cdots\cdots\cdots\cdots & ,[\iota_i(v_{ji};v_{ji}^{(1)},\cdots,v_{ji}^{(r')})_{1,Q_i}]:\end{array}$$

$$\left.\begin{array}{l}[(\mathrm{a}_j^{(1)});\alpha_j^{(1)})_{1,N_1}],[\iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)};\alpha_{ji^{(1)}}^{(1)})_{N_1+1,P_i^{(1)}}];\cdots;[(\mathrm{a}_j^{(s)});\alpha_j^{(s)})_{1,N_s}],[\iota_{i^{(s)}}(a_{ji^{(s)}}^{(s)};\alpha_{ji^{(s)}}^{(s)})_{N_s+1,P_i^{(s)}}] \\ [(\mathrm{b}_j^{(1)});\beta_j^{(1)})_{1,M_1}],[\iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)};\beta_{ji^{(1)}}^{(1)})_{M_1+1,Q_i^{(1)}}];\cdots;[(\mathrm{b}_j^{(s)});\beta_j^{(s)})_{1,M_s}],[\iota_{i^{(s)}}(b_{ji^{(s)}}^{(s)};\beta_{ji^{(s)}}^{(s)})_{M_s+1,Q_i^{(s)}}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^s}\int_{L_1'}\cdots\int_{L_s'}\zeta(t_1,\cdots,t_s)\prod_{k=1}^s\phi_k(t_k)z_k^{t_k}\,\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{1.9}$$

with $\omega=\sqrt{-1}$

$$\zeta(t_1,\cdots,t_s)=\frac{\prod_{j=1}^N\Gamma(1-u_j+\sum_{k=1}^s\mu_j^{(k)}t_k)}{\sum_{i=1}^{r'}[\iota_i\prod_{j=N+1}^{P_i}\Gamma(u_{ji}-\sum_{k=1}^s\mu_{ji}^{(k)}t_k)\prod_{j=1}^{Q_i}\Gamma(1-v_{ji}+\sum_{k=1}^s v_{ji}^{(k)}t_k)]} \tag{1.10}$$

$$\text{and }\ \phi_k(t_k)=\frac{\prod_{j=1}^{M_k}\Gamma(b_j^{(k)}-\beta_j^{(k)}t_k)\prod_{j=1}^{N_k}\Gamma(1-a_j^{(k)}+\alpha_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{r^{(k)}}[\iota_{i^{(k)}}\prod_{j=M_k+1}^{Q_{i^{(k)}}}\Gamma(1-b_{ji^{(k)}}^{(k)}+\beta_{ji^{(k)}}^{(k)}t_k)\prod_{j=N_k+1}^{P_{i^{(k)}}}\Gamma(a_{ji^{(k)}}^{(k)}-\alpha_{ji^{(k)}}^{(k)}s_k)]} \tag{1.11}$$

Suppose , as usual , that the parameters

$u_j, j=1,\cdots,P; v_j, j=1,\cdots,Q;$

$a_j^{(k)}, j=1,\cdots,N_k; a_{ji^{(k)}}^{(k)}, j=n_k+1,\cdots,P_{i^{(k)}};$

$b_{ji^{(k)}}^{(k)}, j=m_k+1,\cdots,Q_{i^{(k)}}; b_j^{(k)}, j=1,\cdots,M_k;$

with $k=1\cdots,s, i=1,\cdots,r', i^{(k)}=1,\cdots,r^{(k)}$

are complex numbers , and the $\alpha's, \beta's, \gamma's$ and $\delta's$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \iota_i \sum_{j=N+1}^{P_i} \mu_{ji}^{(k)} - \iota_{i^{(k)}} \sum_{j=N_k+1}^{P_{i^{(k)}}} \alpha_{ji^{(k)}}^{(k)} + \iota_i \sum_{j=1}^{Q_i} \upsilon_{ji}^{(k)} - \iota_{i^{(k)}} \sum_{j=M_k+1}^{Q_{i^{(k)}}} \beta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.12}$$

The reals numbers τ_i are positives for $i = 1, \cdots, r$, $\iota_{i^{(k)}}$ are positives for $i^{(k)} = 1 \cdots r^{(k)}$

The contour L_k is in the t_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(b_j^{(k)} - \beta_j^{(k)} t_k)$ with $j = 1$ to M_k are separated from those of $\Gamma(1 - u_j + \sum_{i=1}^{s} \mu_j^{(k)} t_k)$with $j = 1$ to N and $\Gamma(1 - a_j^{(k)} + \alpha_j^{(k)} t_k)$ with $j = 1$ to N_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2} B_i^{(k)} \pi$, where

$$B_i^{(k)} = \sum_{j=1}^{N} \mu_j^{(k)} - \iota_i \sum_{j=N+1}^{P_i} \mu_{ji}^{(k)} - \iota_i \sum_{j=1}^{Q_i} \upsilon_{ji}^{(k)} + \sum_{j=1}^{N_k} \alpha_j^{(k)} - \iota_{i^{(k)}} \sum_{j=N_k+1}^{P_{i^{(k)}}} \alpha_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{M_k} \beta_j^{(k)} - \iota_{i^{(k)}} \sum_{j=M_k+1}^{q_{i^{(k)}}} \beta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k = 1 \cdots, s, i = 1, \cdots, r \ , i^{(k)} = 1, \cdots, r^{(k)} \tag{1.13}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_s) = 0(|z_1|^{\alpha_1'}, \cdots, |z_s|^{\alpha_s'}) , max(|z_1|, \cdots, |z_s|) \rightarrow 0$

$\aleph(z_1, \cdots, z_s) = 0(|z_1|^{\beta_1'}, \cdots, |z_s|^{\beta_s'}) , min(|z_1|, \cdots, |z_s|) \rightarrow \infty$

where, $k = 1, \cdots, z : \alpha_k' = min[Re(b_j^{(k)} / \beta_j^{(k)})], j = 1, \cdots, M_k$ and

$$\beta_k' = max[Re((a_j^{(k)} - 1)/\alpha_j^{(k)})], j = 1, \cdots, N_k \tag{1.14}$$

We will use these following notations in this paper

$$U = P_i, Q_i, \iota_i; r' ; V = M_1, N_1; \cdots ; M_s, N_s \tag{1.15}$$

$$\mathrm{W} = P_{i^{(1)}}, Q_{i^{(1)}}, \iota_{i^{(1)}}; r^{(1)}, \cdots, P_{i^{(r)}}, Q_{i^{(r)}}, \iota_{i^{(s)}}; r^{(s)} \tag{1.16}$$

$$A = \{(u_j; \mu_j^{(1)}, \cdots, \mu_j^{(s)})_{1,N}\}, \{\iota_i(u_{ji}; \mu_{ji}^{(1)}, \cdots, \mu_{ji}^{(s)})_{N+1,P_i}\} \tag{1.17}$$

$$B = \{\iota_i(v_{ji}; \upsilon_{ji}^{(1)}, \cdots, \upsilon_{ji}^{(s)})_{M+1,Q_i}\} \tag{1.18}$$

$$C = (a_j^{(1)}; \alpha_j^{(1)})_{1,N_1} , \iota_{i^{(1)}} (a_{ji^{(1)}}^{(1)}; \alpha_{ji^{(1)}}^{(1)})_{N_1+1,P_{i^{(1)}}}, \cdots , (a_j^{(s)}; \alpha_j^{(s)})_{1,N_s} , \iota_{i^{(s)}} (a_{ji^{(s)}}^{(s)}; \alpha_{ji^{(s)}}^{(s)})_{N_s+1,P_{i^{(s)}}} \tag{1.19}$$

$$D = (b_j^{(1)}; \beta_j^{(1)})_{1,M_1}, \iota_{i^{(1)}} (b_{ji^{(1)}}^{(1)}; \beta_{ji^{(1)}}^{(1)})_{M_1+1,Q_{i^{(1)}}}, \cdots ,(b_j^{(s)}; \beta_j^{(s)})_{1,M_s}, \iota_{i^{(s)}} (\beta_{ji^{(s)}}^{(s)}; \beta_{ji^{(s)}}^{(s)})_{M_s+1,Q_{i^{(s)}}} \tag{1.20}$$

The multivariable Aleph-function writes :

$$\aleph(z_1,\cdots,z_s) = \aleph_{U:W}^{0,N:V}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}:\mathrm{C} \\ \cdot & \cdots \\ \cdot & \\ \mathrm{z}_s & \mathrm{B}:\mathrm{D} \end{array}\right) \tag{1.21}$$

The generalized polynomials defined by Srivastava [5], is given in the following manner :

$$S_{N'_1,\cdots,N'_t}^{M'_1,\cdots,M'_t}[y_1,\cdots,y_t] = \sum_{K_1=0}^{[N'_1/M'_1]}\cdots\sum_{K_t=0}^{[N'_t/M'_t]}\frac{(-N'_1)_{M'_1K_1}}{K_1!}\cdots\frac{(-N'_t)_{M'_tK_t}}{K_t!}$$

$$A[N'_1,K_1;\cdots;N'_t,K_t]y_1^{K_1}\cdots y_t^{K_t} \tag{1.22}$$

Where $M'_1,\cdots,M'_t$ are arbitrary positive integers and the coefficients $A[N'_1,K_1;\cdots;N'_t,K_t]$ are arbitrary constants, real or complex. In the present paper, we use the following notation

$$a_1 = \frac{(-N'_1)_{M'_1K_1}}{K_1!}\cdots\frac{(-N'_t)_{M'_tK_t}}{K_t!}A[N'_1,K_1;\cdots;N'_t,K_t] \tag{1.23}$$

In the document , we note :

$$G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r}) = \phi(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})\theta_1(\eta_{G_1,g_1})\cdots\theta_r(\eta_{G_r,g_r}) \tag{1.24}$$

where $\phi(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r}),\theta_1(\eta_{G_1,g_1}),\cdots,\theta_r(\eta_{G_r,g_r})$ are given respectively in (1.2) and (1.3)

2. Extension of the Hurwitz-Lerch Zeta function

The extension of the Hurwitz-Lerch Zeta function $\phi(z,\mathfrak{s},a)$ is introduced by Srivastava et al. ([6],eq.(6.2), page 503) as follows :

$$\phi_{(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_q)}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z;\mathfrak{s},a) = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\lambda_j)_{n\rho_j}}{(a+n)^{\mathfrak{s}}\prod_{j=1}^{q}(\mu_j)_{n\sigma_j}}\times\frac{z^n}{n!} \tag{2.1}$$

with : $p,q\in\mathbb{N}_0,\lambda_j\in\mathbb{C}(j=1,\cdots,p),a,\mu_j\in\mathbb{C}\setminus\mathbb{Z}_0^*\ \ (j=1,\cdots,q),\rho_j,\sigma_k\in\mathbb{R}^+$

$(j=1,\cdots,p;k=1,\cdots,q)$

where $\Delta>-1$ when $\mathfrak{s},z\in\mathbb{C};\Delta=-1$ and $\mathfrak{s}\in\mathbb{C},when|z|<\bigtriangledown^*$, $\Delta=-1$ and $Re(\chi)>\frac{1}{2}$ when $|z|=\bigtriangledown^*$

$$\bigtriangledown^* = \prod_{j=1}^{p}\rho_j^{\rho_j}\prod_{j=1}^{q}\sigma_j^{\sigma_j};\Delta=\sum_{j=1}^{q}\sigma_j-\sum_{j=1}^{p}\rho_j;\chi=\mathfrak{s}+\sum_{j=1}^{q}\mu_j-\sum_{j=1}^{p}\lambda_j+\frac{p-q}{2}$$

We denote these conditions the conditions (f).

3.Required formula

We have the following multiple integral transformation, see Marichev et al ([1], 33.5 11 page 595).

Lemma.

$$\int_0^1 \cdots \int_0^1 f(x_1 \cdots x_n)\,(1-x_1)^{\upsilon_1 -1}\prod_{i=2}^n x_i^{\upsilon_1+\cdots+\upsilon_i-1}(1-x_i)^{\upsilon_i-1}\mathrm{d}x_1\cdots\mathrm{d}x_n$$

$$=\frac{\Gamma(\upsilon_1)\cdots\Gamma(\upsilon_n)}{\Gamma(\upsilon_1+\cdots+\upsilon_n)}\times\int_0^1 f(x)(1-x)^{\upsilon_1+\cdots+\upsilon_n-1}\,\mathrm{d}x \tag{3.1}$$

where $\upsilon_i > 0, i = 1, \cdots, n$, provided that the integral of the right hand side converges absolutely.

4. Main integral.

Let $X_{\upsilon_1,\cdots,\upsilon_n} = (1-x_1)^{\upsilon_1}\prod_{l=2}^n x_l^{\upsilon_1+\cdots+\upsilon_l}(1-x_l)^{\upsilon_l}$ and $b_n = \dfrac{\prod_{j=1}^p(\lambda_j)_{n\rho_j}}{(a+n)^{\mathfrak{s}}\prod_{j=1}^q(\mu_j)_{n\sigma_j}}$

and we have the following formula :

Theorem .

$$\int_0^1 \cdots \int_0^1 f(x_1\cdots x_n)(1-x_1)^{\upsilon_1}\prod_{l=2}^n x_i^{\upsilon_1+\cdots+\upsilon_l}(1-x_l)^{\upsilon_l-1}\phi_{(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_q)}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(zX_{\xi_1,\cdots,\xi_n};\mathfrak{s},a)$$

$$S_{N_1,\cdots,N_t}^{M'_1,\cdots,M'_t}\begin{pmatrix} \mathrm{y}_1 X_{\gamma_1^{(1)},\cdots\gamma_1^{(n)}} \\ \cdot\ \cdot\ \cdot \\ \mathrm{y}_t X_{\gamma_t^{(1)}\cdots\gamma_t^{(n)}}\end{pmatrix}\aleph_{u:w}^{0,\mathfrak{n}':v}\begin{pmatrix} \mathrm{z}_1 X_{\eta_1^1\cdots\eta_1^n} \\ \cdot\ \cdot\ \cdot \\ \mathrm{z}_r X_{\eta_s^1\cdots\eta_s^n}\end{pmatrix}\aleph_{U:W}^{0,N:V}\begin{pmatrix} \mathrm{Z}_1 X_{\alpha_1^{(1)},\cdots\alpha_1^{(n)}} \\ \cdot\ \cdot\ \cdot \\ \mathrm{Z}_s X_{\alpha_r^{(1)}\cdots\alpha_r^{(n)}}\end{pmatrix}\mathrm{d}x_1\cdots\mathrm{d}x_n$$

$$=\sum_{K_1=0}^{[N'_1/M'_1]}\cdots\sum_{K_t=0}^{[N'_t/M'_t]}\sum_{k=0}^{\infty}\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{M'_1}\cdots\sum_{g_r=0}^{M'_r}\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}G(\eta_{G_1,g_1},\cdots\eta_{G_r,g_r})a_1\frac{b_k z^k}{k!}$$

$$z_1^{\eta_{G_1,g_1}}\cdots z_r^{\eta_{G_r,g_r}}y_1^{K_1}\cdots y_t^{K_t}\int_0^1 (1-x)^{\sum_{i=1}^n(k\xi_i+\sum_{j=1}^t K_j\gamma_j^{(i)}+\sum_{j=1}^r\eta_{G_j,g_j}\alpha_j^{(i)})-1}$$

$$f(x)\quad \aleph_{U_{n,1}:W}^{0,N+n:V}\left(\begin{array}{c|} \mathrm{Z}_1(1-x)^{\eta_1^{(1)}+\cdots+\eta_1^{(n)}} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{Z}_s(1-x)^{\eta_s^{(1)}+\cdots+\eta_s^{(n)}}\end{array}\right.$$

$$\left[1-(\upsilon_i+k\xi_i+\sum_{j=1}^{t}K_i\gamma_j^{(i)}+\sum_{j=1}^{r}\eta_{G_i,g_i}\alpha_j^{(i)});\eta_1^{(i)},\cdots,\eta_s^{(i)}\right]_{1,n},A:C \atop {\cdots \atop \mathrm{B}_1,B:D}\Bigg)\mathrm{d}x \tag{4.1}$$

$$\text{where}:\ B_1=\left\{1-\sum_{i=1}^{n}\left[\upsilon_i+k\xi_i+\sum_{j=1}^{t}K_j\gamma_j^{(i)}+\sum_{j=1}^{r}\eta_{G_j,g_j}\alpha_j^{(i)}\right];\eta_1^{(1)}+\cdots+\eta_1^{(n)},\right.$$

$$\left.\cdots,\eta_s^{(1)}+\cdots+\eta_s^{(n)}\right\} \tag{4.2}$$

$U_{n,1}=P_i+n;Q_i+1;\iota_i;r'$

Provided that

a) $min\{\xi_i,\upsilon_i,\gamma_j^{(i)},\alpha_k^{(i)},\eta_l^{(i)},\}>0,i=1,\cdots,n,j=1,\cdots,t,k=1,\cdots,r,l=1,\cdots,s$

b) $Re(\upsilon_i+k\xi_i)+\sum_{j=1}^{r}\alpha_j^{(i)}\min_{1\leqslant k\leqslant m_j}\left(\frac{d_k^{(\mathrm{j})}}{\delta_k^{(\mathrm{j})}}\right)+\sum_{j=1}^{s}\eta_j^{(i)}\min_{1\leqslant k\leqslant M_j}\left(\frac{b_k^{(\mathrm{j})}}{\beta_k^{(\mathrm{j})}}\right)>0\,,i=1,\cdots,n$

c) $|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.5) ; $i=1,\cdots,r$

d) $|argZ_k|<\frac{1}{2}B_i^{(k)}\pi$, where $B_i^{(k)}$ is defined by (1.13) ; $i=1,\cdots,s$

e) $\alpha_i,\beta_i,\gamma_i\in\mathbb{C},i=1,\cdots,m,Re(\alpha_i)>0$

f)The integral of the right hand side converges absolutely.

g) The series occurring on the right-hand side of (4.1) is absolutely and uniformly convergent.

The quantities U,V,W,A,B,C and D are defined by the equations (1.15) to (1;20) respectively.

Proof of (4.1) : Let $M\{\}=\frac{1}{(2\pi\omega)^s}\int_{L_1'}\cdots\int_{L_s'}\zeta(t_1,\cdots,t_s)\prod_{k=1}^{s}\phi_k(t_k)\{\}$. We have :

$$E_{(\alpha_i),(\beta_i)}^{(\gamma_i),m}(zX_{\xi_1,\cdots,\xi_n})S_{N_1,\cdots,N_t}^{M_1',\cdots,M_t'}\begin{pmatrix}\mathrm{y}_1X_{\gamma_1^{(1)},\cdots\gamma_1^{(n)}}\\ \cdots\\ \mathrm{y}_tX_{\gamma_t^{(1)}\cdots\gamma_t^{(n)}}\end{pmatrix}\aleph_{u:w}^{0,\mathrm{n}':v}\begin{pmatrix}\mathrm{z}_1X_{\eta_1^1\cdots\eta_1^n,\epsilon_1}\\ \cdots\\ \mathrm{z}_rX_{\eta_s^1\cdots\eta_s^n,\epsilon_s}\end{pmatrix}$$

$$\aleph_{U:W}^{0,N:V}\begin{pmatrix}\mathrm{Z}_1X_{\alpha_1^{(1)},\cdots\alpha_1^{(n)}}\\ \cdots\\ \mathrm{Z}_sX_{\alpha_r^{(1)}\cdots\alpha_r^{(n)}}\end{pmatrix}=\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{M_1}\cdots\sum_{g_r=0}^{M_r}\sum_{K_1=0}^{[N_1'/M_1']}\cdots\sum_{K_t=0}^{[N_t'/M_t']}\sum_{k=0}^{\infty}\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}$$

$$G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})\, a_1 \frac{b_k z^k}{k!}\, z_1^{\eta_{G_1,g_1}} \cdots z_r^{\eta_{G_r,g_r}}\, y_1^{K_1}\cdots y_t^{K_t} \prod_{j=1}^{t} X^{K_j}_{\gamma_j^{(1)},\cdots,\gamma_j^{(n)}}\, X^{k}_{\xi_{(1)},\cdots,\xi_{(n)}} \prod_{j=1}^{r} X^{\eta_{G_j,g_j}}_{\alpha_j^{(1)},\cdots,\alpha_j^{(n)}}$$

$$M\left[\prod_{j=1}^{s} Z_j^{t_j} X^{t_j}_{\eta_j^{(1)},\cdots,\eta_j^{(n)}}\right] \mathrm{d}t_1\cdots\mathrm{d}t_s \tag{4.3}$$

Multiplying both sides of (4.3) by $f(x_1\cdots x_n)\ (1-x_1)^{\upsilon_1}\prod_{l=2}^{n} x_l^{\upsilon_1+\cdots+\upsilon_l}(1-x_l)^{\upsilon_l-1}$and integrating with respect to $x_1,\cdots,x_s$verifying the conditions e), changing the order of integration ans summations (which is easily seen to be justified due to the absolute convergence of the integrals and the summations involved in the process), we obtain :

$$\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{M_1}\cdots\sum_{g_r=0}^{M_r}\sum_{K_1=0}^{[N_1'/M_1']}\cdots\sum_{K_t=0}^{[N_t'/M_t']}\sum_{k=0}^{\infty}\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})a_1\frac{b_k z^k}{k!}$$

$$z_1^{\eta_{G_1,g_1}}\cdots z_r^{\eta_{G_r,g_r}}\, y_1^{K_1}\cdots y_t^{K_t}\int_0^1\cdots\int_0^1 \prod_{j=1}^{t} X^{K_j}_{\gamma_j^{(1)},\cdots,\gamma_j^{(n)}}\, X^{k}_{\xi_1,\cdots,\xi_n}\prod_{j=1}^{r} X^{\eta_{G_j,g_j}}_{\alpha_j^{(1)},\cdots,\alpha_j^{(n)}}$$

$$\left\{M\left[\prod_{j=1}^{s} Z_j^{t_j} X^{t_j}_{\eta_j^{(1)},\cdots,\eta_j^{(n)}}\right]\mathrm{d}t_1\cdots\mathrm{d}t_s\right\} f(x_1\cdots x_n)\,(1-x_1)^{\upsilon_1}\prod_{l=2}^{n} x_l^{\upsilon_1+\cdots+\upsilon_l}(1-x_l)^{\upsilon_l-1}$$

$$\mathrm{d}x_1\cdots\mathrm{d}x_n \tag{4.4}$$

Change the order of the $(x_1,\cdots,x_n)$-integrals and $(t_1,\cdots,t_s)$-integrals ,we get :

$$\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{M_1}\cdots\sum_{g_r=0}^{M_r}\sum_{K_1=0}^{[N_1'/M_1']}\cdots\sum_{K_t=0}^{[N_t'/M_t']}\sum_{k=0}^{\infty}\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})a_1\frac{b_k z^k}{k!}$$

$$z_1^{\eta_{G_1,g_1}}\cdots z_r^{\eta_{G_r,g_r}}\, y_1^{K_1}\cdots y_t^{K_t} M\left\{\prod_{j=1}^{s} Z_j^{t_j}\int_0^1\cdots\int_0^1 (1-x_1)^{\upsilon_1+\xi_1 k+\sum_{j=1}^{t}K_j\gamma_j^{(1)}+\sum_{j=1}^{r}\eta_{G_j,g_j}\alpha_j^{(1)}+\sum_{j=1}^{s}t_j\eta_j^{(1)}}\right.$$

$$\prod_{l=2}^{n}(1-x_l)^{\upsilon_l+k\xi_l+\sum_{j=1}^{t}K_j\gamma_j^{(l)}+\sum_{j=1}^{r}\eta_{G_j,g_j}\alpha_j^{(l)}+\sum_{j=1}^{s}t_j\eta_j^{(l)})-1}\ f(x_1\cdots x_n)$$

$$\left.\prod_{l=2}^{n} x_l^{\sum_{i=1}^{l}(\xi_i k+\sum_{j=1}^{t}K_j\gamma_j^{(i)}+\sum_{j=1}^{r}\eta_{G_j,g_j}\alpha_j^{(i)}+\sum_{j=1}^{s}t_j\eta_j^{(i)}+\upsilon_i)}\ \mathrm{d}x_1\cdots\mathrm{d}x_n\Big]\mathrm{d}t_1\cdots\mathrm{d}t_s\right\} \tag{4.5}$$

Use the equation (3.1) and interpreting the result thus obtained in the Mellin-barnes contour integral (1.9), we arrive at the desired result.

5. Multivariable I-function

If $\iota_i, \iota_{i^{(1)}}, \cdots, \iota_{i^{(s)}} \to 1$, the Aleph-function of several variables reduces in the I-function of several variables. The multiple integrals transformation have been derived in this section for multivariable I-functions defined by Sharma et Ahmad [2].

Corollary

$$\int_0^1 \cdots \int_0^1 f(x_1 \cdots x_n)(1-x_1)^{\upsilon_1} \prod_{l=2}^{n} x_i^{\upsilon_1+\cdots+\upsilon_l}(1-x_l)^{\upsilon_l-1} \phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_q)}(zX_{\xi_1,\cdots,\xi_n};\mathfrak{s},a)$$

$$S^{M_1',\cdots,M_t'}_{N_1,\cdots,N_t}\begin{pmatrix} \mathrm{y}_1 X_{\gamma_1^{(1)},\cdots\gamma_1^{(n)}} \\ \cdots \\ \mathrm{y}_t X_{\gamma_t^{(1)}\cdots\gamma_t^{(n)}} \end{pmatrix} I^{0,\mathfrak{n}':v}_{u:w}\begin{pmatrix} \mathrm{z}_1 X_{\eta_1^1\cdots\eta_1^n} \\ \cdots \\ \mathrm{z}_r X_{\eta_s^1\cdots\eta_s^n} \end{pmatrix} I^{0,N:V}_{U:W}\begin{pmatrix} \mathrm{Z}_1 X_{\alpha_1^{(1)},\cdots\alpha_1^{(n)}} \\ \cdots \\ \mathrm{Z}_s X_{\alpha_r^{(1)}\cdots\alpha_r^{(n)}} \end{pmatrix} \mathrm{d}x_1\cdots\mathrm{d}x_n$$

$$= \sum_{K_1=0}^{[N_1'/M_1']} \cdots \sum_{K_t=0}^{[N_t'/M_t']} \sum_{k=0}^{\infty} \sum_{G_1,\cdots,G_r=0}^{\infty} \sum_{g_1=0}^{M_1'} \cdots \sum_{g_r=0}^{M_r'} \frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!} G(\eta_{G_1,g_1},\cdots\eta_{G_r,g_r}) a_1 \frac{b_k z^k}{k!}$$

$$z_1^{\eta_{G_1,g_1}} \cdots z_r^{\eta_{G_r,g_r}} y_1^{K_1} \cdots y_t^{K_t} \int_0^1 (1-x)^{\sum_{i=1}^n (k\xi_i + \sum_{j=1}^t K_j\gamma_j^{(i)} + \sum_{j=1}^r \eta_{G_j,g_j}\alpha_j^{(i)})-1}$$

$$f(x) \quad I^{0,N+n:V}_{U_{n,1}:W}\left(\begin{matrix} \mathrm{Z}_1(1-x)^{\eta_1^{(1)}+\cdots+\eta_1^{(n)}} \\ \cdots \\ \cdots \\ \mathrm{Z}_s(1-x)^{\eta_s^{(1)}+\cdots+\eta_s^{(n)}} \end{matrix} \right|$$

$$\left. \begin{matrix} \left[1-(\upsilon_i + k\xi_i + \sum_{j=1}^t K_i\gamma_j^{(i)} + \sum_{j=1}^r \eta_{G_i,g_i}\alpha_j^{(i)});\eta_1^{(i)},\cdots,\eta_s^{(i)}\right]_{1,n}, A : C \\ \cdots \\ \mathrm{B}_1, B : D \end{matrix} \right) \mathrm{d}x \qquad (4.1)$$

under the same existence conditions and notations that (4.1) with $\iota_i, \iota_{i^{(1)}}, \cdots, \iota_{i^{(s)}} \to 1$

Remarks :

If s= 2 , we obtain the similar relation with the Aleph-function of two variables defined by Sharma [4]

If s= 2 and $\iota_i, \iota_i', \iota_i'' \to 1$, the multivariable Aleph-function reduces to the I-function of two variables defined by Sharma and Mishra [3].

6. Conclusion

In this paper we have evaluated multiple integrals transformation involving the multivariable Aleph-functions, class of polynomials of several variables and the extension of the Hurwitz-Lerch Zeta function $\phi(z,\mathfrak{s},a)$ .This transformation established in this paper is of very general nature as it contains two multivariable Aleph-functions, which are a general function of several variables studied so far. Thus, the result established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

REFERENCES

[1]Marichev O.I. Prudnikov A.P. And Brychkow Y.A. Elementay functions. Integrals and series Vol 1. USSR Academy of sciences . Moscow 1986.

[2] Sharma C.K.and Ahmad S.S.: On the multivariable I-function. Acta ciencia Indica Math , 1994 vol 20,no2, p 113-116.

[3] C.K. Sharma and P.L. mishra : On the I-function of two variables and its properties. Acta Ciencia Indica Math , 1991, Vol 17 page 667-672.

[4] Sharma K. On the integral representation and applications of the generalized function of two variables , International Journal of Mathematical Engineering and Sciences , Vol 3 , issue1 (2014) , page1-13.

[5] Srivastava H.M. A multilinear generating function for the Konhauser set of biorthogonal polynomials suggested by Laguerre polynomial, Pacific. J. Math. 177(1985), page183-191.

[6] H.M. Srivastava, R.K. Saxena, T.K. Pogány, R. Saxena, Integral and computational representations of the extended Hurwitz–Lerch zeta function, Integr.Transf. Spec. Funct. 22 (2011) 487–506