A general multiple Eulerian integrals involving general classes of polynomials, a multivariable I-function and the generalized incomplete hypergeometric function

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ABSTRACT
Recently, Goyal and Parashar [2] have given cloosed-form expression for a number of general Eulerian integrals involving multivariable H-function. The object of the present papers is to evaluate a general class of multiple Eulerian integral involving the product of a general class of multivariable polynomials, the generalized incomplete hypergeometric function and the multivariable I-function defined by Nambisan et al [4]. with general arguments. This integral formula provides interesting unifications and extensions of a number of (know and new) integrals. Several particular cases will study.

Keywords :Multivariable I-function, multiple Eulerian integral , multivariable H-function, Srivastava-Daoust polynomial, generalized incomplete hypergeometric function

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1.Introduction

In this paper, we evaluate a multiple Eulerian integral involving the multivariable I-function defined by Prathima et al. [4] , the generalized incomplete hypergeometric function and multivariable class of polynomials with general arguments. The multivariable I-function is a extension of the multivariable H-function defined by Srivastava et al [8]. We will give the contracted form.The I-function is defined and represented in the following manner.

$$I(z_1,\cdots,z_r) = I^{0,n:m_1,n_1;\cdots;m_r,n_r}_{p,q:p_1,q_1;\cdots;p_r,q_r}\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_{1,p} : \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_{1,q} : \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,p_r} \\ \\ (\mathrm{d}_j^{(1)},\bar{\delta}_j^{(1)};D_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},\bar{\delta_j}^{(r)};D_j^{(r)})_{1,q_r} \end{array}\right) \tag{1.1}$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{i=1}^r\theta_i(s_i)z_i^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.2}$$

where $\phi(s_1,\cdots,s_r),\theta_i(s_i), i=1,\cdots,r$ are given by :

$$\psi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^n\Gamma^{A_j}\left(1-a_j+\sum_{i=1}^r\alpha_j^{(i)}s_j\right)}{\prod_{j=n+1}^p\Gamma^{A_j}\left(a_j-\sum_{i=1}^r\alpha_j^{(i)}s_j\right)\prod_{j=1}^q\Gamma^{B_j}\left(1-b_j+\sum_{i=1}^r\beta_j^{(i)}s_j\right)} \tag{1.3}$$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{n_i} \Gamma^{C_j^{(i)}}\left(1 - c_j^{(i)} + \gamma_j^{(i)} s_i\right) \prod_{j=1}^{m_i} \Gamma^{D_j^{(i)}}\left(d_j^{(i)} - \bar{\delta}_j^{(i)} s_i\right)}{\prod_{j=n_i+1}^{p_i} \Gamma^{C_j^{(i)}}\left(c_j^{(i)} - \gamma_j^{(i)} s_i\right) \prod_{j=m_i+1}^{q_i} \Gamma^{D_j^{(i)}}\left(1 - d_j^{(i)} - \bar{\delta}_j^{(i)} s_i\right)} \tag{1.4}$$

where $i = 1, \cdots, r$. Also $z_i \neq 0$ for $i = 1, \cdots, r$

The parameters $m_j, n_j, p_j, q_j (j = 1, \cdots, r), n, p, q$ are non negative integers (for more details, see Nambisan [6])

$\alpha_j^{(i)}(j = 1, \cdots, p; i = 1, \cdots, r), \beta_j^{(i)}(j = 1, \cdots, q; i = 1, \cdots, r), \gamma_j^{(i)}(j = 1, \cdots, p_i; i = 1, \cdots, r)$ and $\delta_j^{(i)}$

$(j = 1, \cdots, q_i; i = 1, \cdots, r)$ are assumed to be positive quantities for standardisation purpose.

$a_j(j = 1, \cdots, p), b_j(j = 1, \cdots, q), c_j^{(i)}(j = 1, \cdots, p_i, i = 1, \cdots, r), d_j^{(i)}(j = 1, \cdots, q_i, i = 1, \cdots, r)$ are complex numbers.

The exposants $A_j(j = 1, \cdots, p), B_j(j = 1, \cdots, q), C_j^{(i)}(j = 1, \cdots, p_i; i = 1, \cdots, r), D_j^{(i)}(j = 1, \cdots, q_i; i = 1, \cdots, r)$

of various gamma function involved in (2.2) and (2.3) may take non integer values.

The contour L_i in the complex s_i-plane is of Mellin Barnes type which runs from $c - i\infty$ to $c + i\infty$ (c real) with indentation, if necessary, in such a manner that all singularities of $\Gamma^{D_j^{(i)}}\left(d_j^{(i)} - \delta_j^{(i)} s_i\right), j = 1, \cdots, m_i$lie to the right and $\Gamma^{C_j^{(i)}}\left(1 - c_j^{(i)} - \gamma_j^{(i)} s_i\right), j = 1, \cdots, n_i$ are to the left of L_i.

Following the result of Braaksma ([1], p.278), the I-function of r variables is analytic if :

$$U_i = \sum_{j=1}^{p} A_j \alpha_j^{(i)} - \sum_{j=1}^{q} B_j \beta_j^{(i)} + \sum_{j=1}^{p_i} C_j^{(i)} \gamma_j^{(i)} - \sum_{j=1}^{q_i} D_j^{(i)} \bar{\delta}_j^{(i)}, i = 1, \cdots, r \tag{1.5}$$

The integral (2.1) converges absolutely if $|arg(z_k)| < \frac{1}{2}\Delta_k \pi, k = 1, \cdots, r$ where

$$\Delta_k = -\sum_{j=n+1}^{p} A_j \alpha_j^{(k)} - \sum_{j=1}^{q} B_j \beta_j^{(k)} + \sum_{j=1}^{m_k} D_j^{(k)} \delta_j^{(k)} - \sum_{j=m_k+1}^{q_k} D_j^{(k)} \bar{\delta}_j^{(k)} + \sum_{j=1}^{n_k} C_j^{(k)} \gamma_j^{(k)} - \sum_{j=n_k+1}^{p_k} C_j^{(k)} \gamma_j^{(k)} > 0 \tag{1.6}$$

We shall use these notations for this paper :

$$X = m_1, n_1; \cdots; m_r, n_r; V = p_1, q_1; \cdots; p_r, q_r \tag{1.7}$$

$$A = (a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)}; A_j)_{1,p} \tag{1.8}$$

$$B = (b_j; \beta_j^{(1)}, \cdots, \beta_j^{(r)}; B_j)_{1,q} \tag{1.9}$$

$$\mathrm{C} = (\mathrm{c}_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{1,p_1}; \cdots; (c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)})_{1,p_r} \tag{1.10}$$

$$D = (\mathrm{d}_j^{(1)}, \delta_j^{(1)}; D_j^{(1)})_{1,q_1}; \cdots; (d_j^{(r)}, \delta_j^{(r)}; D_j^{(r)})_{1,q_r} \tag{1.11}$$

the contracted form is

$$I_{p,q;V}^{0,n;X}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A : C} \\ \cdot & \\ \cdot & \cdot\;\cdot\;\cdot \\ \cdot & \\ \mathrm{z}_r & \mathrm{B : D} \end{array}\right) \tag{1.12}$$

Srivastava and Garg [7] introduced and defined a general class of multivariable polynomials as follows

$$S_L^{h_1,\cdots,h_u}[z_1,\cdots,z_u]=\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}(-L)_{h_1R_1+\cdots+h_uR_u}B(R_1,\cdots,R_u)\frac{z_1^{R_1}\cdots z_u^{R_u}}{R_1!\cdots R_u!} \tag{1.13}$$

The coefficients $B(R_1,\cdots,R_u)$ are arbitrary constants, real or complex.

We shall note

$$B_u=\frac{(-L)_{h_1R_1+\cdots+h_uR_u}B(R_1,\cdots,R_u)}{R_1!\cdots R_u!} \tag{1.14}$$

3. Generalized incomplete hypergeometric function

The generalized incomplete hypergeometric function introduced by Srivastava et al [5 page 675, Eq.(4.1) is represented in the following manner.

$${}_p\gamma_q\left[\begin{array}{cc}(\mathrm{e}_1;\sigma),(e_2),\cdots,(e_p) & \\ \cdots & ;z\\ (\mathrm{f}_1),\cdots,(f_q) & \end{array}\right]=\sum_{n=0}^{\infty}\frac{(e_1;\sigma)_n(e_2)_n\cdots(e_p)_n}{(f_1)_n\cdots(f_q)_n}\frac{z^n}{n!} \tag{2.1}$$

where the incomplete Pochhammer symbols are defined as follows :

$$(a;\sigma)_n=\frac{\gamma(a+n;\sigma)}{\Gamma(a)}\quad(a,n\in\mathbb{C};x\geqslant 0) \tag{2.2}$$

$$\gamma(s,x)=\int_0^x t^{s-1}e^{-t}\mathrm{d}t\quad(Re(s)>0,x\geqslant 0) \tag{2.3}$$

We will note $A_{n'}=\dfrac{(e_1;\sigma)_n(e_2)_n\cdots(e_p)_n}{(f_1)_n\cdots(f_q)_n}$ (2.4)

3. Main integral

In this document, we shall establish the following Eulerian multiple integral of multivariable I-function and we shall use the following notations (2.1) and (2.2).

Let $f(t_j)=(b_j-a_j)+\rho_j(t_j-a_j)+\sigma_j(b_j-t_j)$ (3.1)

$$g(t_j)=\frac{(t_j-a_j)^{\gamma_j}(b_j-t_j)^{\delta_j}\left\{f(t_j)\right\}^{1-\gamma_j-\delta_j}}{\beta_j(b_j-a_j)+(\beta_j\rho_j+\alpha_j-\beta_j)(t_j-a_j)+\beta_j\sigma_j(b_j-t_j)} \tag{3.2}$$

$j=1,\cdots,n$

Lemme ([3] p.287,eq3.198)

$$\int_a^b\frac{(t-a)^{\alpha-1}(b-t)^{\beta-1}}{\{b-a+\lambda(t-a)+\mu(b-t)\}^{\alpha+\beta}}\mathrm{d}t=\frac{(1+\lambda)^{-\alpha}(1+\mu)^{-\beta}\Gamma(\alpha)\Gamma(\beta)}{(b-a)\Gamma(\alpha+\beta)} \tag{3.3}$$

with $t \in [a;b]$ $\quad a \neq b, Re(\alpha) > 0, Re(\beta) > 0, \quad \eta + \lambda(t-a) + \mu(b-t) \neq 0$

We have the general result

Theorem

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \prod_{j=1}^{n} \frac{(t_j - a_j)^{\lambda_j}(b_j - t_j)^{\mu_j}}{[f(t_j)]^{\lambda_j+\mu_j+2}} {}_p\gamma_q \left[\begin{array}{c} (\mathrm{e}_1;\sigma),(e_2),\cdots,(e_p) \\ \cdots \\ (\mathrm{f}_1),\cdots,(f_q) \end{array} ; z\prod_{i=1}^{n}[g(t_j)]^{\zeta_j} \right]$$

$$S_L^{h_1,\cdots,h_u} \left(\begin{array}{c} \mathrm{y}_1 \prod_{j=1}^n \frac{(t_j-a_j)^{s'_j}(b_j-t_j)^{t'_j}}{[f(t_j)]^{s'_j+t'_j}} \\ \cdots \\ \mathrm{y}_u \prod_{j=1}^n \frac{(t_j-a_j)^{s_j^{(u)}}(b_j-t_j)^{t_j^{(u)}}}{[f(t_j)]^{s_j^{(u)}+t_j^{(u)}}} \end{array} \right) I \left(\begin{array}{c} \mathrm{z}_1 \prod_{j=1}^n \left[g(t_j)\right]^{v'_j} \\ \cdots \\ \cdots \\ \mathrm{z}_r \prod_{j=1}^n \left[g(t_j)\right]^{v_j^{(r)}} \end{array} \right) \mathrm{d}t_1 \cdots \mathrm{d}t_n =$$

$$\prod_{j=1}^{n} \left\{ (b_j - a_j)^{-1}(1+\rho_j)^{-\lambda_j - 1}(1+\sigma_j)^{-\mu_j - 1} \sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u \leqslant L} \sum_{r_1,\cdots,r_n=0}^{\infty} \sum_{n'=0}^{\infty} A_{n'} \frac{z^{n'}}{n'!} \right.$$

$$B_u \; y_1^{R_1} \cdots y_u^{R_u} \prod_{j=1}^{n} \left\{ \frac{(\beta_j - \alpha_j)^{r_j}}{r_j!} \frac{(1+\rho_j)^{-(\sum_{k=1}^u R_k s_j^{(k)} + n'\gamma_j\zeta_j + r_j)}(1+\sigma_j)^{-(\sum_{k=1}^u R_k t_j^{(k)} + n'\delta_j\zeta_j)}}{\beta_j^{n'\zeta_j + r_j}} \right\}$$

$$I_{p+3n;q+2n;Y}^{0,\mathrm{n}+3n;X} \left(\begin{array}{c} \mathrm{z}_1 \prod_{j=1}^n \left\{ \beta_j (1+\rho_j)^{\gamma'_j} (1+\sigma_j)^{\delta'_j} \right\}^{-v'_j} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r \prod_{j=1}^n \left\{ \beta_j (1+\rho_j)^{\gamma_j^{(n)}} (1+\sigma_j)^{\delta_j^{(n)}} \right\}^{-v_j^{(r)}} \end{array} \middle| \begin{array}{c} (1\text{-}\mathrm{r}_j - n'\zeta_j; v'_1, \cdots, v_1^{(r)}; 1)_{1,n}, \\ \cdots \\ \cdots \\ (1\text{-n'}\zeta_j; v'_1, \cdots, v_1^{(r)}; 1)_{1,n}, \end{array} \right.$$

$$(\text{-}\lambda_j - \textstyle\sum_{k=1}^u R_k s_j^{(k)} - n'\zeta_j\gamma_j - r_j; \gamma'_j v'_j, \cdots, \gamma_j^{(r)} v_j^{(r)}; 1)_{1,n},$$
$$\cdots$$
$$\cdots$$
$$(\text{-}\lambda_j - \mu_j - \textstyle\sum_{k=1}^u R_k (s_j^{(k)} + t_j^{(k)}) - n'\zeta_j(\gamma_j + \delta_j) - r_j - 1; (\gamma'_j + \delta'_j) v'_j, \cdots, (\gamma_j^{(r)} + \delta_j^{(r)}) v_j^{(r)}; 1)_{1,n},$$

$$\left. \begin{array}{c} (\text{-}\mu_j - \sum_{k=1}^u R_k t_j^{(k)} - n'\zeta_j\delta_j - r_j; \delta'_j v'_j, \cdots, \delta_j^{(r)} v_j^{(r)}; 1)_{1,n}, A : C \\ \cdots \\ \cdots \\ \mathrm{B} : \mathrm{D} \end{array} \right) \tag{3.4}$$

Provided that

a) $min\{\lambda_j,\mu_j,s_j^{(k)},t_j^{(k)},\zeta_j,v_j^{(i)}\}>0;j=1,\cdots,n;i=1,\cdots,r;k=1,\cdots,u$

b) $\beta_j\neq 0,b_j-a_j\neq 0,\rho_j\neq -1,\sigma_j\neq -1$

c) $(b_j-a_j)+\rho_j(t_j-a_j)+\sigma_j(b_j-t_j)\neq 0,t_j\in[a_j;b_j]$

d) $|argz_k|<\frac{1}{2}\Delta_k\pi,k=1,\cdots,r$, where Δ_k is given in (1.6)

e) $|(\beta_j-\alpha_j)(t_j-a_j)|<|\beta_j(b_j-a_j)+\rho_j(t_j-a_j)+\sigma_j(b_j-t_j)|$

f) $Re(\lambda_j+n'\zeta_j\gamma_j)+\sum_{i=1}^r\gamma_j^{(i)}v_j^{(i)}\min_{1\leqslant k\leqslant m_i}Re\left[D_k^{(i)}\left(\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)\right]+1>0$;

$$Re(\mu_j+n'\zeta_j\delta_j)+\sum_{i=1}^r\delta_j^{(i)}v_j^{(i)}\min_{1\leqslant k\leqslant m_i}Re\left[D_k^{(i)}\left(\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)\right]+1>0 \text{ with } j=1,\cdots,n$$

g) the multiple serie on the R.H.S of (2.4) converges absolutly

Proof Let $M\{\}=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^r\theta_k(s_k)\{\}$

To establish the formula (3.4), we first use series representation (1.13) and (2.1) for $S_L^{h_1,\cdots,h_u}[.]$ and ${}_p\gamma_q(.)$ respectively and the contour integral representation with the help of (1.1) for the multivariable I-function occuring in its left-hand side. Changing the order of integration ans summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process), we get

$$\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}\sum_{r_1,\cdots,r_n=0}^{\infty}\sum_{n'=0}^{\infty}A_{n'}\frac{z^{n'}}{n'!}\,y_1^{R_1}\cdots y_u^{R_u}\,B_u$$

$$M\left\{\prod_{i=1}^r z_i^{s_i}\prod_{j=1}^n\int_{a_j}^{b_j}(t_j-a_j)^{\lambda_j+\sum_{k=1}^u n'_k s_j^{(k)}+R\zeta_j\gamma_j+\sum_{i=1}^r\gamma_j^{(i)}v_j^{(i)}s_i}\right.$$

$$\frac{(b_j-t_j)^{\mu_j+\sum_{k=1}^u R_k t_j^{(k)}+n'\zeta_j\delta_j+\sum_{i=1}^r\delta_j^{(i)}v_j^{(i)}s_i}}{[f(t_j)]^{\lambda_j+\mu_j+\sum_{k=1}^u R_k(s_j^{(k)}+t_j^{(k)})+\zeta_j(\gamma_j+\delta_j)n'+\sum_{i=1}^r(\gamma_j^{(i)}+\delta_j^{(i)})v_j^{(i)}s_i+2}}\frac{1}{\beta_j^{n'\zeta_j+\sum_{i=1}^r v_j^{(i)}s_j}}$$

$$\{1-\frac{(\beta_j-\alpha_j)(t_j-a_j)}{\beta_jf(t_j)}\}^{-(n'\zeta_j+\sum_{i=1}^r v_j^{(i)}s_i)}dt_j\}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.5}$$

If $|(\beta_j-\alpha_j)(t_j-a_j)|<|\beta_jf(t_j)|$, then we can use binomial expansion and we thus find from (2.5)

$$\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\prod_{j=1}^{n}\frac{(t_j-a_j)^{\lambda_j}(b_j-t_j)^{\mu_j}}{[f(t_j)]^{\lambda_j+\mu_j+2}}\,{}_p\gamma_q\left[\begin{matrix}(\mathrm{e}_1;\sigma),(e_2),\cdots,(e_p)\\ \cdots\\ (\mathrm{f}_1),\cdots,(f_q)\end{matrix};z\prod_{i=1}^{n}[g(t_j)]^{\zeta_j}\right]$$

$$S_L^{h_1,\cdots,h_u}\left(\begin{matrix}\mathrm{y}_1\prod_{j=1}^{n}\frac{(t_j-a_j)^{s'_j}(b_j-t_j)^{t'_j}}{[f(t_j)]^{s'_j+t'_j}}\\ \cdots\\ \mathrm{y}_u\prod_{j=1}^{n}\frac{(t_j-a_j)^{s_j^{(u)}}(b_j-t_j)^{t_j^{(u)}}}{[f(t_j)]^{s_j^{(u)}+t_j^{(u)}}}\end{matrix}\right)I\left(\begin{matrix}\mathrm{z}_1\prod_{j=1}^{n}\left[g(t_j)\right]^{v'_j}\\ \cdots\\ \cdots\\ \mathrm{z}_r\prod_{j=1}^{n}\left[g(t_j)\right]^{v_j^{(r)}}\end{matrix}\right)\mathrm{d}t_1\cdots\mathrm{d}t_n=$$

$$\sum_{n'=0}^{\infty}\sum_{r_1,\cdots,r_n=0}^{\infty}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}A_{n'}\frac{z^{n'}}{n'!}B_u\;y_1^{R_1}\cdots y_u^{R_u}\frac{\Gamma(r_j+\sum_{i=1}^{n}v_j^{(i)}s_i)}{\Gamma(\sum_{i=1}^{n}v_j^{(i)}s_i)}$$

$$M\left\{\prod_{i=1}^{r}\left[z_i^{s_i}\beta_j^{-(n'\zeta_j+\sum_{i=1}^{r}v_j^{(i)}s_j)}\int_{a_j}^{b_j}(t_j-a_j)^{\lambda_j+\sum_{k=1}^{u}R_ks_j^{(k)}+n'\zeta_j\gamma_j+\sum_{i=1}^{r}\gamma_j^{(i)}v_j^{(i)}s_i}\right.\right.$$

$$\left.\left.\frac{(b_j-t_j)^{\mu_j+\sum_{k=1}^{u}R_kt_j^{(k)}+n'\zeta_j\delta_j+\sum_{i=1}^{r}\delta_j^{(i)}v_j^{(i)}s_i}}{\left[f(t_j)\right]^{\lambda_j+\mu_j+\sum_{k=1}^{u}R_k(s_j^{(k)}+t_j^{(k)})+n'\zeta_j(\gamma_j+\delta_j)+\sum_{i=1}^{r}(\gamma_j^{(i)}+\delta_j^{(i)})v_j^{(i)}s_i+2}}\mathrm{d}t_j\right]\right\}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.6}$$

Use the lemme, we obtain :

$$\prod_{j=1}^{n}\{(b_j-a_j)^{-1}(1+\rho_j)^{-\lambda_j-1}(1+\sigma_j)^{-\mu_j-1}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}\quad\sum_{r_1,\cdots,r_n=0}^{\infty}\sum_{n'=0}^{\infty}A_{n'}\frac{z^{n'}}{n'!}B_u$$

$$\prod_{j=1}^{n}\left\{\frac{(\beta_j-\alpha_j)^{r_j}}{r_j}\frac{(1+\rho_j)^{-(\sum_{k=1}^{u}R_ks_j^{(k)}+n'\gamma_j\zeta_j+r_j)}(1+\sigma_j)^{-(\sum_{k=1}^{u}R_kt_j^{(k)}+n'\delta_j\zeta_j)}}{\beta_j^{\zeta_jR+r_j}}\right\}M\left\{\prod_{j=1}^{n}\frac{\Gamma(r_j+\zeta_jR+\sum_{i=1}^{r}v_j^{(i)}s_j)}{\Gamma(\zeta_jR+\sum_{i=1}^{r}v_j^{(i)}s_j)}\right.$$

$$\frac{\Gamma(\lambda_j+\sum_{k=1}^{u}R_ks_j^{(k)}+n'\gamma_j\zeta_j+r_j+1+\sum_{i=1}^{r}v_j^{(i)}s_j)}{\Gamma(\lambda_j+\mu_j+\sum_{k=1}^{u}R_k(s_j^{(k)}+t_j^{(k)})+n'(\gamma_j+\delta_j)\zeta_j+r_j+1+(\gamma_j+\delta_j)\sum_{i=1}^{r}v_j^{(i)}s_j)}$$

$$\left.\Gamma(\mu_j+\sum_{k=1}^{u}R_kt_j^{(k)}+n'\gamma_j\zeta_j+1+\delta_j+\sum_{i=1}^{r}v_j^{(i)}s_i)\left(\frac{(1+\rho_j)^{-\gamma_j}(1+\sigma_j)^{-\delta_j}}{\beta_j}\right)^{\sum_{i=1}^{r}v_j^{(i)}s_j}\right\}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.7}$$

Finally reinterpreting the multiple Mellin-Barnes contour integral in terms of multivariable I-function, we obtain the desired result (2.4)

4. Particular cases

Corollary 1

Further $n=1$ and for $\rho=\sigma=0$ and $z_i=(b-a)^{(\gamma+\delta-1)v^{(i)}}, i=1,\cdots,r$, we have

$f(t)=b-a$ and $g(t)=\dfrac{(t-a)^{\gamma}(b-t)^{\delta}}{\alpha(x-a)+\beta(b-x)}$, the integral (3.4) becomes

$$\int_a^b \frac{(t-a)^{\lambda}(b-t)^{\mu}}{\left[(b-a)\right]^{\lambda+\mu+2}} {}_p\gamma_q\left[\begin{matrix}(\mathrm{e}_1;\sigma),(e_2),\cdots,(e_p)\\ \cdots\\ (\mathrm{f}_1),\cdots,(f_q)\end{matrix};z[g(t)]^{\zeta}\right]$$

$$S_L^{h_1,\cdots,h_u}\left(\begin{matrix}\mathrm{y}_1\frac{(t-a)^{s'}(b-t)^{t'}}{(b-a)^{s'+t'}}\\ \cdots\\ \cdots\\ \mathrm{y}_u\frac{(t-a)^{s^{(u)}}(b-t)^{t^{(u)}}}{(b-a)^{s^{(u)}+t^{(u)}}}\end{matrix}\right) I\left(\begin{matrix}\mathrm{z}_1\left[g(t_j)\right]^{v'}\\ \cdots\\ \cdots\\ \mathrm{z}_r\left[g(t_j)\right]^{v^{(r)}}\end{matrix}\right)\mathrm{d}t_1\cdots\mathrm{d}t_n=$$

$$\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}\sum_{r',n'=0}^{\infty} A_{n'}\frac{z^{n'}}{n'!}B_u y_1^{R_1}\cdots y_u^{R_u} I_{p+3;q+2;Y}^{0,\mathfrak{n}+3;X}\left(\begin{matrix}\mathrm{z}_1\frac{b-a)^{(\gamma+\delta-1)v'}}{\beta^{v'}}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\frac{(b-a)^{(\gamma+\delta-1)v^{(r)}}}{\beta^{v^{(r)}}}\end{matrix}\right|\begin{matrix}(1\text{-r'-}\zeta n';v_1',\cdots,v_1^{(r)};1),\\ \cdots\\ \cdots\\ (1\text{-}\zeta n';v_1',\cdots,v_1^{(r)};1),\end{matrix}$$

$$(\text{-}\lambda-\textstyle\sum_{k=1}^u R_k s^{(k)}-\zeta\gamma n'-r';\gamma'v',\cdots,\gamma^{(r)}v^{(r)};1),$$
$$\cdots$$
$$\cdots$$
$$(\text{-}\lambda-\mu-\textstyle\sum_{k=1}^u R_k(s^{(k)}+t^{(k)})-\zeta n'(\gamma+\delta)-r'-1;(\gamma'+\delta')v',\cdots,(\gamma^{(r)}+\delta^{(r)})v^{(r)};1),$$

$$\left.\begin{matrix}(\text{-}\mu-\sum_{k=1}^u R_k t^{(k)}-\zeta-r';\delta'v',\cdots,\delta^{(r)}v^{(r)};1),A:C\\ \cdots\\ \cdots\\ \mathrm{B}:\mathrm{D}\end{matrix}\right) \tag{4.1}$$

If $A_j=B_j=C_j^{(i)}=D_j^{(i)}=1$, the multivariable I-function defined by Nambisan et al [4] reduces to the multivariable H-funcction defined by Srivastava et al [8] and we have the following result.

Corollary 2

$$\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\prod_{j=1}^{n}\frac{(t_j-a_j)^{\lambda_j}(b_j-t_j)^{\mu_j}}{\left[f(t_j)\right]^{\lambda_j+\mu_j+2}} {}_p\gamma_q\left[\begin{matrix}(\mathrm{e}_1;\sigma),(e_2),\cdots,(e_p)\\ \cdots\\ (\mathrm{f}_1),\cdots,(f_q)\end{matrix};z\prod_{i=1}^{n}[g(t_j)]^{\zeta_j}\right]$$

$$S_L^{h_1,\cdots,h_u}\left(\begin{array}{c} \mathrm{y}_1\prod_{j=1}^n \frac{(t_j-a_j)^{s'_j}(b_j-t_j)^{t'_j}}{[f(t_j)]^{s'_j+t'_j}} \\ \cdots \\ \mathrm{y}_u\prod_{j=1}^n \frac{(t_j-a_j)^{s_j^{(u)}}(b_j-t_j)^{t_j^{(u)}}}{[f(t_j)]^{s_j^{(u)}+t_j^{(u)}}} \end{array}\right) H\left(\begin{array}{c} \mathrm{z}_1\prod_{j=1}^n \left[g(t_j)\right]^{v'_j} \\ \cdots \\ \cdots \\ \mathrm{z}_r\prod_{j=1}^n \left[g(t_j)\right]^{v_j^{(r)}} \end{array}\right)\mathrm{d}t_1\cdots\mathrm{d}t_n =$$

$$\prod_{j=1}^n\left\{(b_j-a_j)^{-1}(1+\rho_j)^{-\lambda_j-1}(1+\sigma_j)^{-\mu_j-1}\right.\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}\sum_{r_1,\cdots,r_n=0}^{\infty}\sum_{n'=0}^{\infty}A_{n'}\frac{z^{n'}}{n'!}$$

$$B_u\ y_1^{R_1}\cdots y_u^{R_u}\prod_{j=1}^n\left\{\frac{(\beta_j-\alpha_j)^{r_j}}{r_j!}\frac{(1+\rho_j)^{-(\sum_{k=1}^u R_k s_k^{(i)}+n'\gamma_j\zeta_j+r_j)}(1+\sigma_j)^{-(\sum_{k=1}^u R_k t_j^{(k)}+n'\delta_j\zeta_j)}}{\beta_j^{n'\zeta_j+r_j}}\right\}$$

$$H_{p+3n;q+2n;Y}^{0,\mathrm{n}+3n;X}\left(\begin{array}{c} \mathrm{z}_1\prod_{j=1}^n\left\{\beta_j(1+\rho_j)^{\gamma'_j}(1+\sigma_j)^{\delta'_j}\right\}^{-v'_j} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r\prod_{j=1}^n\left\{\beta_j(1+\rho_j)^{\gamma_j^{(n)}}(1+\sigma_j)^{\delta_j^{(n)}}\right\}^{-v_j^{(r)}} \end{array}\right| \begin{array}{c} (1\text{-r}_j-n'\zeta_j;v'_1,\cdots,v_1^{(r)})_{1,n}, \\ \cdots \\ \cdots \\ (1\text{-n'}\zeta_j;v'_1,\cdots,v_1^{(r)})_{1,n}, \end{array}$$

$$\begin{array}{c} (-\lambda_j-\sum_{k=1}^u R_k s_j^{(k)}-n'\zeta_j\gamma_j-r_j;\gamma'_jv'_j,\cdots,\gamma_j^{(r)}v_j^{(r)})_{1,n}, \\ \cdots \\ \cdots \\ (-\lambda_j-\mu_j-\sum_{k=1}^u R_k(s_j^{(k)}+t_j^{(k)})-n'\zeta_j(\gamma_j+\delta_j)-r_j-1;(\gamma'_j+\delta'_j)v'_j,\cdots,(\gamma_j^{(r)}+\delta_j^{(r)})v_j^{(r)})_{1,n}, \end{array}$$

$$\left.\begin{array}{c} (-\mu_j-\sum_{k=1}^u R_k t_j^{(k)}-n'\zeta_j\delta_j-r_j;\delta'_jv'_j,\cdots,\delta_j^{(r)}v_j^{(r)})_{1,n},A:C \\ \cdots \\ \cdots \\ \mathrm{B}:\mathrm{D} \end{array}\right) \tag{3.4}$$

under the same conditions and notations that (3.4) with $A_j = B_j = C_j^{(i)} = D_j^{(i)} = 1$

If $B(R_1,\cdots,R_u) = \dfrac{\prod_{j=1}^{\bar{A}}(a_j)_{R_1\theta'_j+\cdots+R_u\theta_j^{(u)}}\prod_{j=1}^{B'}(b'_j)_{R_1\phi'_j}\cdots\prod_{j=1}^{B^{(u)}}(b_j^{(u)})_{R_u\phi_j^{(u)}}}{\prod_{j=1}^{\bar{C}}(c_j)_{m_1\psi'_j+\cdots+m_u\psi_j^{(u)}}\prod_{j=1}^{D'}(d'_j)_{R_1\delta'_j}\cdots\prod_{j=1}^{D^{(u)}}(d_j^{(u)})_{R_u\delta_j^{(u)}}}$ **(4.3)**

then the general class of multivariable polynomial $S_L^{h_1,\cdots,h_u}[z_1,\cdots,z_u]$ reduces in generalized Srivastava-Daoust polynomial defined by Srivastava and Daoust [6].

$$F_{\bar{C}:D';\cdots;D^{(u)}}^{1+\bar{A}:B';\cdots;B^{(u)}}\left(\begin{array}{c} \mathrm{z}_1 \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{z}_u \end{array}\middle| \begin{array}{c} [(\text{-L});\mathrm{R}_1,\cdots,R_u][(a);\theta',\cdots,\theta^{(u)}]:[(b');\phi'];\cdots;[(b^{(u)});\phi^{(u)}] \\ [(\mathrm{c});\psi',\cdots,\psi^{(u)}]:[(d');\delta'];\cdots;[(d^{(u)});\delta^{(u)}] \end{array}\right) \quad (4.4)$$

and we have the following formula

Corollary 3

$$\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\prod_{j=1}^{n}\frac{(t_j-a_j)^{\lambda_j}(b_j-t_j)^{\mu_j}}{[f(t_j)]^{\lambda_j+\mu_j+2}}{}_p\gamma_q\left[\begin{array}{c}(\mathrm{e}_1;\sigma),(e_2),\cdots,(e_p)\\ \cdot\cdot\cdot \\ (\mathrm{f}_1),\cdots,(f_q)\end{array};z\prod_{i=1}^{n}[g(t_j)]^{\zeta_j}\right]$$

$$F_{\bar{C}:D';\cdots;D^{(u)}}^{1+\bar{A}:B';\cdots;B^{(u)}}\left(\begin{array}{c} \mathrm{y}_1\prod_{j=1}^{n}\frac{(t_j-a_j)^{s'_j}(b_j-t_j)^{t'_j}}{[f(t_j)]^{s'_j+t'_j}} \\ \cdot\cdot\cdot \\ \mathrm{y}_u\prod_{j=1}^{n}\frac{(t_j-a_j)^{s_j^{(u)}}(b_j-t_j)^{t_j^{(u)}}}{[f(t_j)]^{s_j^{(u)}+t_j^{(u)}}} \end{array}\right|$$

$$\left.\begin{array}{c}[(\text{-L});\mathrm{R}_1,\cdots,R_u][(a);\theta',\cdots,\theta^{(u)}]:[(b');\phi'];\cdots;[(b^{(u)});\phi^{(u)}] \\ [(\mathrm{c});\psi',\cdots,\psi^{(u)}]:[(d');\delta'];\cdots;[(d^{(u)});\delta^{(u)}]\end{array}\right)$$

$$I\left(\begin{array}{c}\mathrm{z}_1\prod_{j=1}^{n}\left[g(t_j)\right]^{v'_j} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{z}_r\prod_{j=1}^{n}\left[g(t_j)\right]^{v_j^{(r)}}\end{array}\right)\mathrm{d}t_1\cdots\mathrm{d}t_n=$$

$$\prod_{j=1}^{n}\{(b_j-a_j)^{-1}(1+\rho_j)^{-\lambda_j-1}(1+\sigma_j)^{-\mu_j-1}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}\sum_{r_1,\cdots,r_n=0}^{\infty}\sum_{n'=0}^{\infty}A_{n'}\frac{z^{n'}}{n'!}$$

$$B'_u\;y_1^{R_1}\cdots y_u^{R_u}\prod_{j=1}^{n}\left\{\frac{(\beta_j-\alpha_j)^{r_j}}{r_j!}\frac{(1+\rho_j)^{-(\sum_{k=1}^{u}R_ks_k^{(i)}+n'\gamma_j\zeta_j+r_j)}(1+\sigma_j)^{-(\sum_{k=1}^{u}R_kt_j^{(k)}+n'\delta_j\zeta_j)}}{\beta_j^{n'\zeta_j+r_j}}\right\}$$

$$I_{p+3n;q+2n;Y}^{0,\mathrm{n}+3n;X}\left(\begin{array}{c}\mathrm{z}_1\prod_{j=1}^{n}\left\{\beta_j(1+\rho_j)^{\gamma'_j}(1+\sigma_j)^{\delta'_j}\right\}^{-v'_j} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r\prod_{j=1}^{n}\left\{\beta_j(1+\rho_j)^{\gamma_j^{(n)}}(1+\sigma_j)^{\delta_j^{(n)}}\right\}^{-v_j^{(r)}}\end{array}\middle| \begin{array}{c}(1\text{-r}_j-n'\zeta_j;v'_1,\cdots,v_1^{(r)};1)_{1,n}, \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ (1\text{-n}'\zeta_j;v'_1,\cdots,v_1^{(r)};1)_{1,n},\end{array}\right.$$

$$(\text{-}\lambda_j - \sum_{k=1}^{u} R_k s_j^{(k)} - n'\zeta_j\gamma_j - r_j; \gamma_j' v_j', \cdots, \gamma_j^{(r)} v_j^{(r)}; 1)_{1,n},$$

$$\cdots$$

$$\cdots$$

$$(\text{-}\lambda_j - \mu_j - \sum_{k=1}^{u} R_k (s_j^{(k)} + t_j^{(k)}) - n'\zeta_j(\gamma_j + \delta_j) - r_j - 1; (\gamma_j' + \delta_j')v_j', \cdots, (\gamma_j^{(r)} + \delta_j^{(r)})v_j^{(r)}; 1)_{1,n},$$

$$\left.\begin{matrix}(\text{-}\mu_j - \sum_{k=1}^{u} R_k t_j^{(k)} - n'\zeta_j\delta_j - r_j; \delta_j' v_j', \cdots, \delta_j^{(r)} v_j^{(r)}; 1)_{1,n}, A : C \\ \cdots \\ \cdots \\ \mathrm{B : D}\end{matrix}\right) \quad \textbf{(4.5)}$$

under the same notations and exisrence conditions that (3.4)

and $B_u' = \frac{(-L)_{h_1 R_1 + \cdots + h_u R_u} B(E; R_1, \cdots, R_u)}{R_1! \cdots R_u!}$; $B(L; R_1, \cdots, R_u)$ **is defined by (3.4)**

5. Conclusion

Our main integral formula (3.4) is unified in nature. The multivariable I-function defined by Prathima et al. [4] occurring in this integral can be suitably specialized to a remarkably wide variety of special functions which are expressible in terms of E, G and H-function of one and more variables. Again by suitably specializing various parameters and coefficients, the general class of polynomials can be reduced to a large number of orthogonal polynomials and generalized Lauricella polynomials.

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