General class of Eulerian integrals involving the multivariable I-function

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ABSTRACT
In this paper, we derive a general Eulerian integral involving the multivariable I-function defined by Prathima et al. [3], the Aleph-function of one variable and general class of polynomials of several variables . Some of this key formula could provide useful generalizations of some known as well as of some new results concerning the multivariable I-function. We shall study two particular cases.

Keywords :multivariable I-function, Eulerian integral, multivariable H-function, Class of polynomials, Srivastava-Daoust polynomials.

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1.Introduction

In this paper, we evaluate a general class of Eulerian integrals involving the multivariable I-function, the multivariable class of polynomials and the Aleph-function with general arguments.
The multivariable I-function defined by Prathima et al. [3] is an extension of the multivariable H-function defined by Srivastava and Panda [7]. We shall use the contracted form. The I-function of r-variables is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1,\cdots,z_r) = I^{0,n:m_1,n_1;\cdots;m_r,n_r}_{p,q:p_1,q_1;\cdots;p_r,q_r}\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_{1,p} : (\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,p_r} \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_{1,q} : (\mathrm{d}_j^{(1)},\bar{\delta}_j^{(1)};D_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},\bar{\delta}_j^{(r)};D_j^{(r)})_{1,q_r} \end{array}\right) \tag{1.1}$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{i=1}^{r}\theta_i(s_i)z_i^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.2}$$

where $\phi(s_1,\cdots,s_r),\theta_i(s_i), i=1,\cdots,r$ are given by :

$$\psi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{n}\Gamma^{A_j}\left(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_j\right)}{\prod_{j=n+1}^{p}\Gamma^{A_j}\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_j\right)\prod_{j=1}^{q}\Gamma^{B_j}\left(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}s_j\right)} \tag{1.3}$$

and

$$\theta_i(s_i) = \frac{\prod_{j=1}^{n_i}\Gamma^{C_j^{(i)}}\left(1-c_j^{(i)}+\gamma_j^{(i)}s_i\right)\prod_{j=1}^{m_i}\Gamma^{D_j^{(i)}}\left(d_j^{(i)}-\bar{\delta}_j^{(i)}s_i\right)}{\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}\left(c_j^{(i)}-\gamma_j^{(i)}s_i\right)\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}\left(1-d_j^{(i)}-\bar{\delta}_j^{(i)}s_i\right)} \tag{1.4}$$

where $i = 1, \cdots, r$. Also $z_i \neq 0$ for $i = 1, \cdots, r$

The parameters $m_j, n_j, p_j, q_j (j = 1, \cdots, r), n, p, q$ are non negative integers (for more details, see Nambisan [6])

$\alpha_j^{(i)} (j = 1, \cdots, p; i = 1, \cdots, r), \beta_j^{(i)} (j = 1, \cdots, q; i = 1, \cdots, r), \gamma_j^{(i)} (j = 1, \cdots, p_i; i = 1, \cdots, r)$ and $\delta_j^{(i)}$

$(j = 1, \cdots, q_i; i = 1, \cdots, r)$ are assumed to be positive quantities for standardisation purpose.

$a_j (j = 1, \cdots, p), b_j (j = 1, \cdots, q), c_j^{(i)} (j = 1, \cdots, p_i, i = 1, \cdots, r), d_j^{(i)} (j = 1, \cdots, q_i, i = 1, \cdots, r)$ are complex numbers.

The exposants $A_j (j = 1, \cdots, p), B_j (j = 1, \cdots, q), C_j^{(i)} (j = 1, \cdots, p_i; i = 1, \cdots, r), D_j^{(i)} (j = 1, \cdots, q_i; i = 1, \cdots, r)$

of various gamma function involved in (2.2) and (2.3) may take non integer values.

The contour L_i in the complex s_i-plane is of Mellin Barnes type which runs from $c - i\infty$ to $c + i\infty$ (c real) with indentation, if necessary, in such a manner that all singularities of $\Gamma^{D_j^{(i)}} (d_j^{(i)} - \delta_j^{(i)} s_i), j = 1, \cdots, m_i$lie to the right and $\Gamma^{C_j^{(i)}} (1 - c_j^{(i)} - \gamma_j^{(i)} s_i), j = 1, \cdots, n_i$ are to the left of L_i.

Following the result of Braaksma ([1], p.278) the I-function of r variables is analytic if :

$$U_i = \sum_{j=1}^{p} A_j \alpha_j^{(i)} - \sum_{j=1}^{q} B_j \beta_j^{(i)} + \sum_{j=1}^{p_i} C_j^{(i)} \gamma_j^{(i)} - \sum_{j=1}^{q_i} D_j^{(i)} \bar{\delta}_j^{(i)}, i = 1, \cdots, r \tag{1.5}$$

The integral (2.1) converges absolutely if

$|arg(z_k)| < \frac{1}{2} \Delta_k \pi, k = 1, \cdots, r$ where

$$\Delta_k = - \sum_{j=n+1}^{p} A_j \alpha_j^{(k)} - \sum_{j=1}^{q} B_j \beta_j^{(k)} + \sum_{j=1}^{m_k} D_j^{(k)} \delta_j^{(k)} - \sum_{j=m_k+1}^{q_k} D_j^{(k)} \bar{\delta}_j^{(k)} + \sum_{j=1}^{n_k} C_j^{(k)} \gamma_j^{(k)} - \sum_{j=n_k+1}^{p_k} C_j^{(k)} \gamma_j^{(k)} > 0 \tag{1.6}$$

We will use these notations for this paper :

$$X = m_1, n_1; \cdots; m_r, n_r; V = p_1, q_1; \cdots; p_r, q_r \tag{1.7}$$

$$A = (a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)}; A_j)_{1,p} \tag{1.8}$$

$$B = (b_j; \beta_j^{(1)}, \cdots, \beta_j^{(r)}; B_j)_{1,q} \tag{1.9}$$

$$\mathrm{C} = (\mathrm{c}_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{1,p_1}; \cdots; (c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)})_{1,p_r} \tag{1.10}$$

$$D = (\mathrm{d}_j^{(1)}, \delta_j^{(1)}; D_j^{(1)})_{1,q_1}; \cdots; (d_j^{(r)}, \delta_j^{(r)}; D_j^{(r)})_{1,q_r} \tag{1.11}$$

the contracted form is

$$I_{p,q;V}^{0,n;X} \left(\begin{array}{c|c} z_1 & \mathrm{A} : \mathrm{C} \\ \cdot & \\ \cdot & \cdot \; \cdot \; \cdot \\ \cdot & \\ z_r & \mathrm{B} : \mathrm{D} \end{array} \right) \tag{1.12}$$

Srivastava and Garg [5] introduced and defined a general class of multivariable polynomials as follows

$$S_L^{h_1, \cdots, h_s}[z_1, \cdots, z_s] = \sum_{R_1, \cdots, R_s = 0}^{h_1 R_1 + \cdots h_s R_s \leqslant L} (-L)_{h_1 R_1 + \cdots + h_s R_s} B(R_1, \cdots, R_s) \frac{z_1^{R_1} \cdots z_s^{R_s}}{R_1! \cdots R_s!} \tag{1.13}$$

The coefficients $B(R_1,\cdots,R_s)$ are arbitrary constants, real or complex.

We shall note $:B_s = \frac{(-L)_{h_1R_1+\cdots+h_sR_s}B(E;R_1,\cdots,R_s)}{R_1!\cdots R_s!}$ (1.14)

The Aleph- function introduced by Südland [8] et al. , however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral :

$$\aleph(z)=\aleph^{M,N}_{P_i,Q_i,c_i;r}\left(z \;\middle|\; \begin{matrix}(\mathrm{a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r}\\(\mathrm{b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right)=\frac{1}{2\pi\omega}\int_L \Omega^{M,N}_{P_i,Q_i,c_i;r}(s)z^{-s}\mathrm{d}s \quad (1.15)$$

for all z different to 0 and

$$\Omega^{M,N}_{P_i,Q_i,c_i;r}(s)=\frac{\prod_{j=1}^{M}\Gamma(b_j+B_js)\prod_{j=1}^{N}\Gamma(1-a_j-A_js)}{\sum_{i=1}^{r}c_i\prod_{j=N+1}^{P_i}\Gamma(a_{ji}+A_{ji}s)\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}-B_{ji}s)} \quad (1.16)$$

with $|argz|<\frac{1}{2}\pi\Omega$ where $\Omega=\sum_{j=1}^{M}\beta_j+\sum_{j=1}^{N}\alpha_j-c_i(\sum_{j=M+1}^{Q_i}\beta_{ji}+\sum_{j=N+1}^{P_i}\alpha_{ji})>0\,,i=1,\cdots,r$

For convergence conditions and other details of Aleph-function , see Südland et al. [8]. The serie representation of Aleph-function is given by Chaurasia and Singh [2].

$$\aleph^{M,N}_{P_i,Q_i,c_i;r}(z)=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{B_G g!}z^{-s} \quad (1.17)$$

with $s=\eta_{G,g}=\frac{b_G+g}{B_G}, P_i<Q_i, |z|<1$ and $\Omega^{M,N}_{P_i,Q_i,c_i;r}(s)$ is given in (1.2) (1.18)

2. Required formulae.

We have : $B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},\;\; Re(\alpha)>0, Re(\beta)>0$ (2.1)

(2.1) can be rewritten in the form

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\mathrm{d}t=(b-a)^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, Re(\alpha)>0, Re(\beta)>0, b\neq a \quad (2.2)$$

The binomial expansions for $t\in[a,b]$ yields :

$$(ut+v)^\gamma=(au+v)^\gamma\sum_{m=0}^{\infty}\frac{(-\gamma)_m}{m!}\left\{\frac{-u(a-t)}{au+v}\right\}^m \quad \text{where} \quad \left|\frac{(t-a)u}{au+v}\right|<1 \quad (2.3)$$

With the help of (2.2) we obtain (see Srivastava and Hussain, [6])

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}\mathrm{d}t = (b-a)^{\alpha+\beta-1}B(\alpha,\beta)(at+v)^{\gamma}{}_2F_1\left(\begin{matrix}\alpha,-\gamma\\ \alpha+\beta\end{matrix};-\frac{(b-a)u}{au+v}\right) \quad (2.4)$$

3. General Eulerian integral

Let $g_1(t) = \dfrac{(t-a)^{\delta_1}(b-t)^{\eta_1}(ut+v)^{1-\delta_1-\eta_1}}{B(ut+v)+(A-B)(t-a)}$; $g_2(t) = \dfrac{(t-a)^{\delta_2}(b-t)^{\eta_2}(yt+z)^{1-\delta_2-\eta_2}}{D(yt+z)+(C-D)(t-a)}$

We shall derive the following general Eulerian integral involving the multivariable I-function.

Theorem

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}(yt+z)^{\rho} S_L^{h_1,\cdots,h_s}\left(\begin{matrix} \mathrm{y}_1\,(g_1(t))^{c_1}\,(g_2(t))^{d_1} \\ \cdots \\ \cdots \\ \mathrm{y}_s\,(g_1(t))^{c_s}\,(g_2(t))^{d_s}\end{matrix}\right)$$

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}\left(x\,(g_1(t))^{c}\,(g_2(t))^{d}\right) I_{p,q;V}^{0,n;X}\left(\begin{matrix} \mathrm{z}_1\,(g_1(t))^{u_1}\,(g_2(t))^{v_1} \\ \cdots \\ \cdots \\ \mathrm{z}_r\,(g_1(t))^{u_r}\,(g_2(t))^{v_r}\end{matrix}\right)\mathrm{d}t$$

$$= (b-a)^{\alpha+\beta-1}(au+v)^{\gamma}(by+z)^{\rho}\sum_{l,m,k_1,k_2=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{R_1,\cdots,R_s=0}^{h_1R_1+\cdots h_sR_s\leqslant L}\frac{(B-A/B)^{l}\,(D-C/D)^{m}}{l!m!k_1!k_2!}B_s$$

$$\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!}Y_1^{R_1}\cdots Y_s^{R_s}X^{\eta_{G,g}}\left\{-\frac{(b-a)u}{(au+v)}\right\}^{k_1}\left\{\frac{(b-a)y}{(by+z)}\right\}^{k_2} I_{p+7,q+6;V}^{0,n+7;X}\left(\begin{matrix}\mathrm{Z}_1\\ \cdots \\ \cdots \\ \mathrm{Z}_r\end{matrix}\ \middle|\right.$$

$(1\text{-l+c}\eta_{G,g}-\sum_{i=1}^{s}c_iR_i : u_1,\cdots,u_r)$,$(1\text{-m+d}\eta_{G,g}-\sum_{i=1}^{s}d_iR_i : v_1,\cdots,v_r)$,

$\cdots \qquad \cdots$

$(1\text{+c}\eta_{G,g}-\sum_{i=1}^{s}c_iR_i : u_1,\cdots,u_r)$, $(1\text{+d}\eta_{G,g}-\sum_{i=1}^{s}d_iR_i : v_1,\cdots,v_r)$,

$(1\text{-}\alpha-l-m-k_1+(c\delta_1+d\delta_2)\eta_{G,g}-\sum_{i=1}^{s}(\delta_1c_i+\delta_2d_i)R_i : \delta_1u_1+\delta_2v_1,\cdots,\delta_1u_r+\delta_2v_r)$,

$\cdots$

B_1,

$(1\text{-}\beta-k_2+(c\eta_1+d\eta_2)\eta_{G,g}-\sum_{i=1}^{s}(\eta_1c_i+\eta_2d_i)R_i : \eta_1u_1+\eta_2v_1,\cdots,\eta_1u_r+\eta_2v_r)$,

$\cdots$

$(1\text{+}\gamma-l+(\delta_1+\eta_1)c\eta_{G,g}-\sum_{i=1}^{s}(\delta_1+\eta_1)c_iR_i;(\delta_1+\eta_1)u_1,\cdots,(\delta_1+\eta_1)u_r)$,

$$(1+\rho - m - k_2 + (\eta_2+\delta_2)d\eta_{G,g} - \sum_{i=1}^{s}(\eta_2+\delta_2)d_iR_i : (\eta_2+\delta_2)v_1, \cdots, (\eta_2+\delta_2)v_r),$$
$$\cdots$$
$$(1+\rho - m + (\eta_2+\delta_2)d\eta_{G,g} - \sum_{i=1}^{s}(\eta_2+\delta_2)d_iR_i : (\eta_2+\delta_2)v_1, \cdots, (\eta_2+\delta_2)v_r),$$

$$\left.\begin{matrix}(1+\gamma - l - k_1 + (\delta_1+\eta_1)c\eta_{G,g} - \sum_{i=1}^{s}(\eta_1+\delta_1)c_iR_i : (\eta_1+\delta_1)u_1, \cdots, (\eta_1+\delta_1)u_r), A_1, A : C \\ \cdots \\ \mathrm{B}_2, B : D\end{matrix}\right) \quad (3.1)$$

Where

B_1 =(1-$\alpha - \beta - m - c(\delta_1+\eta_1)\eta_{G,g} + d(\delta_2+\eta_2)\eta_{G,g} - \sum_{i=1}^{s}(\delta_1+\eta_1)c_iR_i - \sum_{i=1}^{s}(\delta_2+\eta_2)d_iR_i;$
$(\delta_1+\eta_1)u_1 + (\delta_2+\eta_2)v_1, \cdots, (\delta_1+\eta_1)u_r + (\delta_2+\eta_2)v_r)$

A_1 =(1-$\alpha - \beta - l - m + c(\delta_1+\eta_1)\eta_{G,g} - d(\delta_2+\eta_2)\eta_{G,g} - \sum_{i=1}^{s}(\delta_1+\eta_1)c_iR_i - \sum_{i=1}^{s}(\delta_2+\eta_2)d_iR_i;$

$(\delta_1+\eta_1)u_1 + (\delta_2+\eta_2)v_1, \cdots, (\delta_1+\eta_1)u_r + (\delta_2+\eta_2)v_r)$

$B_2 =$ (1-$\alpha - \beta - m - l - k_1 - k_2 + c(\delta_1+\eta_1)\eta_{G,g} + d(\delta_2+\eta_2)\eta_{G,g} - \sum_{i=1}^{s}(\delta_1+\eta_1)c_iR_i -$

-$\sum_{i=1}^{s}(\delta_2+\eta_2)d_iR_i;(\delta_1+\eta_1)u_1 + (\delta_2+\eta_2)v_1, \cdots, (\delta_1+\eta_1)u_r + (\delta_2+\eta_2)v_r)$

$$Z_i = \frac{z_i(b-a)^{(\delta_1+\eta_1)u_i+(\delta_2+\eta_2)v_i}}{B^{u_i}D^{v_i}(au+v)^{(\delta_1+\eta_1)u_i}(by+z)^{(\delta_2+\eta_2)v_i}}, i = 1, \cdots, r$$

$$Y_i = \frac{y_i(b-a)^{(\delta_1+\eta_1)c_i+(\delta_2+\eta_2)d_i}}{B^{c_i}D^{d_i}(au+v)^{(\delta_1+\eta_1)c_i}(by+z)^{(\delta_2+\eta_2)d_i}}, i = 1, \cdots, s \text{ and}$$

$$X = \frac{x(b-a)^{(\delta_1+\eta_1)c+(\delta_2+\eta_2)d}}{B^cD^d(au+v)^{(\delta_1+\eta_1)c}(by+z)^{(\delta_2+\eta_2)d}}$$

Provided that

a) $min\{c, d, c_i, d_i, u_j, v_j\} > 0, i = 1, \cdots, s; j = 1, \cdots, r$

b $min\{Re(\alpha), Re(\beta)\} > 0, b \neq a$

c) $max\left\{\left|\frac{u(b-a)}{au+v}\right|, \left|\frac{y(b-a)}{by+z}\right|, \left|\frac{(t-a)(B-A)}{B(ut+v)}\right|, \left|\frac{(t-a)(D-C)}{D(yt+z)}\right|\right\} < 1, t \in [a,b]$

d) $|arg(z_k)| < \frac{1}{2}\Delta_k\pi, k = 1, \cdots, r$ where Δ_k is defined by (1.6)

e) $|argx| < \frac{1}{2}\pi\Omega$ Where $\Omega = \sum_{j=1}^{M}\beta_j + \sum_{j=1}^{N}\alpha_j - c_i(\sum_{j=M+1}^{Q_i}\beta_{ji} + \sum_{j=N+1}^{P_i}\alpha_{ji}) > 0$

Proof of (3.1) Let $M\{\} = \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)\{\}$

We first replace the multivariable I-function on the L.H.S of (3.1) by its Mellin-barnes contour integrals (1.1), the Aleph-function and general class of polynomials of several variables in series by using respectively (1.17) and (1.13), Now we interchange the order of summations and integrations (which is permissible under the conditions stated) . We get :

$$\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{R_1,\cdots,R_s=0}^{h_1R_1+\cdots h_sR_s\leqslant L} a'\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!}x^{\eta_{G,g}}y_1^{R_1}\cdots y_s^{R_s}\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}$$

$$(yt+z)^{\rho}\Big\{M\Big\{(g_1(t))^{c\eta_{G,g}+\sum_{i=1}^{s}c_iR_i+\sum_{i=1}^{r}u_is_i}(g_2(t))^{d\eta_{G,g}+\sum_{i=1}^{s}d_iR_i+\sum_{i=1}^{r}v_is_i}\ z_1^{s_1}\cdots z_r^{s_r}\Big\}$$

$$\mathrm{d}s_1\cdots\mathrm{d}s_r\Big\}\mathrm{d}t \tag{3.2}$$

We evaluate the inner integrals with the help of (2.1) and (2.3) and applying (1.1)

$$\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{R_1,\cdots,R_s=0}^{h_1R_1+\cdots h_sR_s\leqslant L} B_s\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!}x^{\eta_{G,g}}y_1^{R_1}\cdots y_s^{R_s}$$

$$M\Big\{B^{-(c\eta_{G,g}+\sum_{i=1}^{s}c_iR_i+\sum_{i=1}^{r}u_is_i)}\ D^{-(d\eta_{G,g}+\sum_{i=1}^{s}d_iR_i+\sum_{i=1}^{r}v_is_i)}$$

$$\int_a^b (t-a)^{\alpha+\delta_1(c\eta_{G,g}+\sum_{i=1}^{s}c_iR_i+\sum_{i=1}^{r}u_is_i)+\delta_2(d\eta_{G,g}+\sum_{i=1}^{s}d_iR_i+\sum_{i=1}^{r}v_is_i)-1}$$

$$(b-t)^{\beta+\eta_1(c\eta_{G,g}+\sum_{i=1}^{s}c_iR_i+\sum_{i=1}^{r}u_is_i)+\eta_2(d\eta_{G,g}+\sum_{i=1}^{s}d_iR_i+\sum_{i=1}^{r}v_is_i)-1}z_1^{s_1}\cdots z_r^{s_r}$$

$$(ut+v)^{\gamma-(\delta_1+\eta_1)(c\eta_{G,g}+\sum_{i=1}^{s}c_iR_i+\sum_{i=1}^{r}u_is_i)}(yt+z)^{\rho-(\delta_2+\eta_2)(d\eta_{G,g}+\sum_{i=1}^{s}d_iR_i+\sum_{i=1}^{r}v_is_i)}$$

$$\left(1-\frac{(B-A)(t-a)}{B(ut+v)}\right)^{-(c\eta_{G,g}+\sum_{i=1}^{s}c_iR_i+\sum_{i=1}^{r}u_is_i)}\left(1-\frac{(D-C)(t-a)}{D(yt+z)}\right)^{-(d\eta_{G,g}+\sum_{i=1}^{s}d_iR_i+\sum_{i=1}^{r}v_is_i)}$$

$$\mathrm{d}t z_1^{s_1}\cdots z_r^{s_r}\Big\}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.3}$$

Using binomial expansion (2.3) provided that $max\left\{\left|\frac{(t-a)(B-A)}{B(ut+v)}\right|\left|\frac{(t-a)(D-C)}{D(yt+z)}\right|\right\}<1, t\in[a,b]$

and also that the order of binomial summation and integration can be inversed, we get

$$\sum_{l,m=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{R_1,\cdots,R_s=0}^{h_1R_1+\cdots h_sR_s\leqslant L} B_s\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})(B-A/B)^l(D-C/D)^m}{B_G g!\qquad l!m!}x^{\eta_{G,g}}$$

$$y_1^{R_1} \cdots y_s^{R_s} M\Big\{ B^{-(c\eta_{G,g}+\sum_{i=1}^s c_iR_i+\sum_{i=1}^r u_is_i)} D^{-(d\eta_{G,g}+\sum_{i=1}^s d_iR_i+\sum_{i=1}^r v_is_i)}$$

$$\frac{\Gamma(l+c\eta_{G,g}+\sum_{i=1}^s c_iR_i+\sum_{i=1}^r u_is_i)\Gamma(m+d\eta_{G,g}+\sum_{i=1}^s d_iR_i+\sum_{i=1}^r v_is_i)}{\Gamma(c\eta_{G,g}+\sum_{i=1}^s c_iR_i+\sum_{i=1}^r u_is_i)\Gamma(d\eta_{G,g}+\sum_{i=1}^s d_iR_i+\sum_{i=1}^r v_is_i)}$$

$$\int_a^b (t-a)^{\alpha+l+m+\delta_1(c\eta_{G,g}+\sum_{i=1}^s c_iR_i+\sum_{i=1}^r u_is_i)+\delta_2(d\eta_{G,g}+\sum_{i=1}^s d_iR_i+\sum_{i=1}^r v_is_i)-1}$$

$$(b-t)^{\beta+\eta_1(c\eta_{G,g}+\sum_{i=1}^s c_iR_i+\sum_{i=1}^r u_is_i)+\eta_2(d\eta_{G,g}+\sum_{i=1}^s d_iR_i+\sum_{i=1}^r v_is_i)-1}$$

$$(ut+v)^{\gamma-l-(\delta_1+\eta_1)(c\eta_{G,g}+\sum_{i=1}^s c_iR_i+\sum_{i=1}^r u_is_i)}(yt+z)^{\rho-m-(\delta_2+\eta_2)(d\eta_{G,g}+\sum_{i=1}^s d_iR_i+\sum_{i=1}^r v_is_i)}$$

$$\mathrm{d}t z_1^{s_1}\cdots z_r^{s_r}\Big\}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.4}$$

The inner integral in (3.4) can be evaluated by using the following extension of Eulerian integral of Beta function given by Hussain and Srivastava [6].

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}(yt+z)^{\rho}\mathrm{d}t = (b-a)^{\alpha+\beta-1}(au+v)^{\gamma}(by+z)^{\delta}B(\alpha,\beta)$$

$$\times F_3\left[\alpha,\beta,-\gamma,-\rho;\alpha+\beta;-\frac{(b-a)u}{au+v},\frac{(b-a)y}{by+z}\right] \tag{3.5}$$

where for convergence $min\{Re(\alpha),Re(\beta)\}>0, b\neq a$ and $max\left\{\left|\frac{u(b-a)}{au+v}\right|,\left|\frac{y(b-a)}{by+t}\right|\right\}<1$

and where F_3 denote the Appell function of two variables, see Appell and Kampe de Feriet [1]. Finally interpreting the resulting Mellin-Barnes contour integrals as a multivariable I-function, we obtain the desired result (3.1).

4. Multivariable H-function

If $A_j=B_j=C_j^{(i)}=D_j^{(i)}=1$, the multivariable I-function defined by Prathima et al. [3] reduces in the multivariable H-funcction defined by Srivastava and Panda [7] and we have the following result.

Corollary1

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}(yt+z)^{\rho}S_L^{h_1,\cdots,h_s}\begin{pmatrix} \mathrm{y}_1\,(g_1(t))^{c_1}\,(g_2(t))^{d_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{y}_s\,(g_1(t))^{c_s}\,(g_2(t))^{d_s} \end{pmatrix}$$

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}\Big(x\,(g_1(t))^{c}\,(g_2(t))^{d}\Big)H_{p,q;V}^{0,n;X}\begin{pmatrix} \mathrm{z}_1\,(g_1(t))^{u_1}\,(g_2(t))^{v_1} \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{z}_r\,(g_1(t))^{u_r}\,(g_2(t))^{v_r} \end{pmatrix}\mathrm{d}t$$

$$= (b-a)^{\alpha+\beta-1}(au+v)^{\gamma}(by+z)^{\rho}\sum_{l,m,k_1,k_2=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{R_1,\cdots,R_s=0}^{h_1R_1+\cdots h_sR_s\leqslant L}\frac{(B-A/B)^l\,(D-C/D)^m}{l!m!k_1!k_2!}B_s$$

$$\frac{(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r'}(\eta_{G,g})}{B_G g!}Y_1^{R_1}\cdots Y_s^{R_s}X^{\eta_{G,g}}\left\{-\frac{(b-a)u}{(au+v)}\right\}^{k_1}\left\{\frac{(b-a)y}{(by+z)}\right\}^{k_2}H^{0,n+7;X}_{p+7,q+6;V}\left(\begin{array}{c}\mathrm{Z}_1\\ \cdot\cdot\cdot\\ \cdot\cdot\cdot\\ \mathrm{Z}_r\end{array}\right|$$

$$\begin{array}{c}(1\text{-l+c}\eta_{G,g}-\sum_{i=1}^s c_iR_i:u_1,\cdots,u_r),(1\text{-m+d}\eta_{G,g}-\sum_{i=1}^s d_iR_i:v_1,\cdots,v_r),\\ \cdot\cdot\cdot \qquad\qquad\qquad \cdot\cdot\cdot\\ (1\text{+c}\eta_{G,g}-\sum_{i=1}^s c_iR_i:u_1,\cdots,u_r),\quad(1\text{+d}\eta_{G,g}-\sum_{i=1}^s d_iR_i:v_1,\cdots,v_r),\end{array}$$

$$\begin{array}{c}(1\text{-}\alpha-l-m-k_1+(c\delta_1+d\delta_2)\eta_{G,g}-\sum_{i=1}^s(\delta_1c_i+\delta_2d_i)R_i:\delta_1u_1+\delta_2v_1,\cdots,\delta_1u_r+\delta_2v_r),\\ \cdot\cdot\cdot\\ \mathrm{B}_1,\end{array}$$

$$\begin{array}{c}(1\text{-}\beta-k_2+(c\eta_1+d\eta_2)\eta_{G,g}-\sum_{i=1}^s(\eta_1c_i+\eta_2d_i)R_i:\eta_1u_1+\eta_2v_1,\cdots,\eta_1u_r+\eta_2v_r),\\ \cdot\cdot\cdot\\ (1+\gamma-l+(\delta_1+\eta_1)c\eta_{G,g}-\sum_{i=1}^s(\delta_1+\eta_1)c_iR_i;(\delta_1+\eta_1)u_1,\cdots,(\delta_1+\eta_1)u_r),\end{array}$$

$$\begin{array}{c}(1+\rho-m-k_2+(\eta_2+\delta_2)d\eta_{G,g}-\sum_{i=1}^s(\eta_2+\delta_2)d_iR_i:(\eta_2+\delta_2)v_1,\cdots,(\eta_2+\delta_2)v_r),\\ \cdot\cdot\cdot\\ (1+\rho-m+(\eta_2+\delta_2)d\eta_{G,g}-\sum_{i=1}^s(\eta_2+\delta_2)d_iR_i:(\eta_2+\delta_2)v_1,\cdots,(\eta_2+\delta_2)v_r),\end{array}$$

$$\left.\begin{array}{c}(1+\gamma-l-k_1+(\delta_1+\eta_1)c\eta_{G,g}-\sum_{i=1}^s(\eta_1+\delta_1)c_iR_i:(\eta_1+\delta_1)u_1,\cdots,(\eta_1+\delta_1)u_r),A_1,A:C\\ \cdot\cdot\cdot\\ \mathrm{B}_2,B:D\end{array}\right)\tag{3.1}$$

under the same existence conditions and notations that (3.1) with $A_j=B_j=C_j^{(i)}=D_j^{(i)}=1$

4. Srivastava-Daoust polynomial

If $$B(R_1,\cdots,R_s)=\frac{\prod_{j=1}^{\bar{A}}(a_j)_{R_1\theta'_j+\cdots+R_s\theta_j^{(s)}}\prod_{j=1}^{B'}(b'_j)_{R_1\phi'_j}\cdots\prod_{j=1}^{B^{(s)}}(b_j^{(s)})_{R_s\phi_j^{(s)}}}{\prod_{j=1}^{\bar{C}}(c_j)_{m_1\psi'_j+\cdots+m_s\psi_j^{(s)}}\prod_{j=1}^{D'}(d'_j)_{R_1\delta'_j}\cdots\prod_{j=1}^{D^{(s)}}(d_j^{(s)})_{R_s\delta_j^{(s)}}}\tag{4.1}$$

then the general class of multivariable polynomial $S_L^{h_1,\cdots,h_s}[z_1,\cdots,z_s]$ reduces in generalized Srivastava-Daoust polynomial defined by Srivastava and Daoust [4].

$$F_{\bar{C}:D';\cdots;D^{(s)}}^{1+\bar{A}:B';\cdots;B^{(s)}}\left(\begin{array}{c} \mathrm{z}_1 \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{z}_s \end{array}\middle| \begin{array}{c} [(\text{-L});\mathrm{R}_1,\cdots,R_s][(a);\theta',\cdots,\theta^{(s)}]:[(b');\phi'];\cdots;[(b^{(s)});\phi^{(s)}] \\ [(\mathrm{c});\psi',\cdots,\psi^{(s)}]:[(d');\delta'];\cdots;[(d^{(s)});\delta^{(s)}] \end{array}\right) \quad (4.2)$$

and we have the following formula

Corollary 2

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}(yt+z)^{\rho} F_{\bar{C}:D';\cdots;D^{(s)}}^{1+\bar{A}:B';\cdots;B^{(s)}}\left(\begin{array}{c} \mathrm{y}_1\,(g_1(t))^{c_1}\,(g_2(t))^{d_1} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{y}_s\,(g_1(t))^{c_s}\,(g_2(t))^{d_s} \end{array}\right|$$

$$\left.\begin{array}{c} [(\text{-L});\mathrm{R}_1,\cdots,R_s][(a);\theta',\cdots,\theta^{(s)}]:[(b');\phi'];\cdots;[(b^{(s)});\phi^{(s)}] \\ [(\mathrm{c});\psi',\cdots,\psi^{(s)}]:[(d');\delta'];\cdots;[(d^{(s)});\delta^{(s)}] \end{array}\right)$$

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}\left(x\,(g_1(t))^{c}\,(g_2(t))^{d}\right) I_{p,q;V}^{0,n;X}\left(\begin{array}{c} \mathrm{z}_1\,(g_1(t))^{u_1}\,(g_2(t))^{v_1} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{z}_r\,(g_1(t))^{u_r}\,(g_2(t))^{v_r} \end{array}\right)\mathrm{d}t$$

$$= (b-a)^{\alpha+\beta-1}(au+v)^{\gamma}(by+z)^{\rho}\sum_{l,m,k_1,k_2=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{R_1,\cdots,R_s=0}^{h_1R_1+\cdots h_sR_s\leqslant L}\frac{(B-A/B)^l\,(D-C/D)^m}{l!m!k_1!k_2!}B_s'$$

$$\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!}Y_1^{R_1}\cdots Y_s^{R_s}X^{\eta_{G,g}}\left\{-\frac{(b-a)u}{(au+v)}\right\}^{k_1}\left\{\frac{(b-a)y}{(by+z)}\right\}^{k_2} I_{p+7,q+6;V}^{0,n+7;X}\left(\begin{array}{c} \mathrm{Z}_1 \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{Z}_r \end{array}\right|$$

$$\begin{array}{cc} (\text{1-l-c}\eta_{G,g}-\sum_{i=1}^{s}c_iR_i:u_1,\cdots,u_r), & (\text{1-m-d}\eta_{G,g}-\sum_{i=1}^{s}d_iR_i:v_1,\cdots,v_r), \\ \cdot\cdot\cdot & \cdot\cdot\cdot \\ (\text{1-c}\eta_{G,g}-\sum_{i=1}^{s}c_iR_i:u_1,\cdots,u_r), & (\text{1-d}\eta_{G,g}-\sum_{i=1}^{s}d_iR_i:v_1,\cdots,v_r), \end{array}$$

$$\begin{array}{c} (1\text{-}\alpha-l-m-k_1-(c\delta_1+d\delta_2)\eta_{G,g}-\sum_{i=1}^{s}(\delta_1c_i+\delta_2d_i)R_i:\delta_1u_1+\delta_2v_1,\cdots,\delta_1u_r+\delta_2v_r), \\ \cdot\cdot\cdot \\ \mathrm{B}_1, \end{array}$$

$$\begin{array}{c} (1\text{-}\beta-k_2-(c\eta_1+d\eta_2)\eta_{G,g}-\sum_{i=1}^{s}(\eta_1c_i+\eta_2d_i)R_i:\eta_1u_1+\eta_2v_1,\cdots,\eta_1u_r+\eta_2v_r), \\ \cdot\cdot\cdot \\ (1+\gamma-l-(\delta_1+\eta_1)c\eta_{G,g}-\sum_{i=1}^{s}(\delta_1+\eta_1)c_iR_i;(\delta_1+\eta_1)u_1,\cdots,(\delta_1+\eta_1)u_r), \end{array}$$

$$(1+\rho-m-k_2-(\eta_2+\delta_2)d\eta_{G,g}-\textstyle\sum_{i=1}^{s}(\eta_2+\delta_2)d_iR_i:(\eta_2+\delta_2)v_1,\cdots,(\eta_2+\delta_2)v_r),$$

$$\cdots$$

$$(1+\rho-m-(\eta_2+\delta_2)d\eta_{G,g}-\textstyle\sum_{i=1}^{s}(\eta_2+\delta_2)d_iR_i:(\eta_2+\delta_2)v_1,\cdots,(\eta_2+\delta_2)v_r),$$

$$\left.\begin{matrix}(1+\gamma-l-k_1-(\delta_1+\eta_1)c\eta_{G,g}-\textstyle\sum_{i=1}^{s}(\eta_1+\delta_1)c_iR_i:(\eta_1+\delta_1)u_1,\cdots,(\eta_1+\delta_1)u_r),A_1,A:C\\ \cdots\\ \mathrm{B}_2,B:D\end{matrix}\right)\quad(4.3)$$

under the same notations and existence conditions that (3.1) with : $B'_s=\dfrac{(-L)_{h_1R_1+\cdots+h_sR_s}B(R_1,\cdots,R_s)}{R_1!\cdots R_s!}$ and$B(R_1,\cdots,R_s)$ is defined by (4.1).

5. Conclusion

Due to general character of the multivariable I-function defined by Prathima et al. [3], the Aleph-function and the Eulerian integral involving here, our formulae are capable to be reduced into many known and new integrals involving the special functions of one and several variables and polynomials of one and several variables.

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