Multiple integrals involving the spheroidal function, a class of polynomials multivariable Aleph-functions and multivariable I-function

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ABSTRACT
In the present paper we evaluate a generalized multiple integral involving the product of a generalized multiple Zeta-function, multivariable Aleph-function, the multivariable I-function and general class of polynomials of several variables. The importance of the result established in this paper lies in the fact they involve the Aleph-function of several variables which is sufficiently general in nature and capable to yielding a large of results merely by specializating the parameters their in.

Keywords:Multivariable Aleph-function, general class of polynomials, multiple integral, generalized multiple Zeta-function, multivariable I-function, multivariable H-function

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1.Introduction and preliminaries.

The function Aleph of several variables generalizes the multivariable I-function recently studied by C.K. Sharma and Ahmad [4] , itself is a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integrals occurring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We define : $$\aleph(z_1,\cdots,z_r)=\aleph^{0,N:M_1,N_1,\cdots,M_r,N_r}_{P_i,Q_i,\tau_i;R:P_{i(1)},Q_{i(1)},\tau_{i(1)};R^{(1)};\cdots;P_{i(r)},Q_{i(r)};\tau_{i(r)};R^{(r)}}\left(\begin{array}{c|}\mathrm{y}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{y}_r\end{array}\right.$$

$$[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,N}]\ ,[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{N+1,P_i}]:$$
$$\cdots\cdots\cdots\cdots\ ,[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{1,Q_i}]:$$

$$\left.\begin{array}{l}[(\mathrm{c}_j^{(1)});\gamma_j^{(1)})_{1,N_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{N_1+1,P_i^{(1)}}];\cdots;[(\mathrm{c}_j^{(r)});\gamma_j^{(r)})_{1,N_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{N_r+1,P_i^{(r)}}]\\ [(\mathrm{d}_j^{(1)});\delta_j^{(1)})_{1,M_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{M_1+1,Q_i^{(1)}}];\cdots;[(\mathrm{d}_j^{(r)});\delta_j^{(r)})_{1,M_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{M_r+1,Q_i^{(r)}}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)y_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{N}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=N+1}^{P_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{Q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{M_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{N_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=M_k+1}^{Q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=N_k+1}^{P_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

Suppose , as usual , that the parameters

$b_j, j = 1, \cdots, Q; a_j, j = 1, \cdots, P;$

$c_{ji^{(k)}}^{(k)}, j = n_k + 1, \cdots, P_{i^{(k)}}; c_j^{(k)}, j = 1, \cdots, N_k;$

$d_{ji^{(k)}}^{(k)}, j = M_k + 1, \cdots, Q_{i^{(k)}}; d_j^{(k)}, j = 1, \cdots, M_k;$

with $k = 1 \cdots, r, i = 1, \cdots, R$, $i^{(k)} = 1, \cdots, R^{(k)}$

are complex numbers , and the $\alpha' s, \beta' s, \gamma' s$ and $\delta' s$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \tau_i \sum_{j=N+1}^{P_i} \alpha_{ji}^{(k)} - \tau_{i^{(k)}} \sum_{j=N_k+1}^{P_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} + \tau_i \sum_{j=1}^{Q_i} \beta_{ji}^{(k)} - \tau_{i^{(k)}} \sum_{j=M_k+1}^{Q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.4}$$

The reals numbers τ_i are positives for $i = 1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)} s_k)$ with $j = 1$ to m_k are separated from those of $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(k)} s_k)$ with $j = 1$ to N and $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)$ with $j = 1$ to N_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2} A_i^{(k)} \pi$, where

$$A_i^{(k)} = \sum_{j=1}^{N} \alpha_j^{(k)} - \tau_i \sum_{j=N+1}^{P_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{Q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{N_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=N_k+1}^{P_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{M_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=M_k+1}^{Q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k = 1 \cdots, r, i = 1, \cdots, R , i^{(k)} = 1, \cdots, R^{(k)} \tag{1.5}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(y_1, \cdots, y_r) = 0(|y_1|^{\alpha_1}, \cdots, |y_r|^{\alpha_r}), max(|y_1|, \cdots, |y_r|) \rightarrow 0$

$\aleph(y_1, \cdots, y_r) = 0(|y_1|^{\beta_1}, \cdots, |y_r|^{\beta_r}), min(|y_1|, \cdots, |y_r|) \rightarrow \infty$

where, with $k = 1, \cdots, r : \alpha_k = min[Re(d_j^{(k)} / \delta_j^{(k)})], j = 1, \cdots, M_k$ and

$\beta_k = max[Re((c_j^{(k)} - 1) / \gamma_j^{(k)})], j = 1, \cdots, N_k$

Serie representation of Aleph-function of several variables is given by

$$\aleph(y_1, \cdots, y_r) = \sum_{G_1, \cdots, G_r = 0}^{\infty} \sum_{g_1=0}^{M_1} \cdots \sum_{g_r=0}^{M_r} \frac{(-)^{G_1 + \cdots + G_r}}{\delta_{g_1} G_1! \cdots \delta_{g_r} G_r!} \psi(\eta_{G_1, g_1}, \cdots, \eta_{G_r, g_r})$$

$$\times\ \theta_1(\eta_{G_1,g_1})\cdots\theta_r(\eta_{G_r,g_r})y_1^{-\eta_{G_1,g_1}}\cdots y_r^{-\eta_{G_r,g_r}} \tag{1.6}$$

Where $\psi(.,\cdots,.)$, $\theta_i(.)$,$i=1,\cdots,r$ are given respectively in (1.2), (1.3) and

$$\eta_{G_1,g_1}=\frac{d_{g_1}^{(1)}+G_1}{\delta_{g_1}^{(1)}},\cdots,\ \eta_{G_r,g_r}=\frac{d_{g_r}^{(r)}+G_r}{\delta_{g_r}^{(r)}}$$

which is valid under the conditions $\delta_{g_i}^{(i)}[d_j^i+p_i]\neq\delta_j^{(i)}[d_{g_i}^i+G_i]$ (1.7)

for $j\neq M_i, M_i=1,\cdots\eta_{G_i,g_i}; P_i,N_i=0,1,2,\cdots,;y_i\neq 0, i=1,\cdots,r$ (1.8)

In the document , we will note :

$$G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})=\phi(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})\theta_1(\eta_{G_1,g_1})\cdots\theta_r(\eta_{G_r,g_r}) \tag{1.9}$$

where $\phi(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r}),\theta_1(\eta_{G_1,g_1}),\cdots,\theta_r(\eta_{G_r,g_r})$ are given respectively in (1.2) and (1.3)

We will note the Aleph-function of r variables $\aleph_{u:w}^{0,N:v}\left(\begin{array}{c} z_1 \\ \cdot\cdot\cdot \\ z_r \end{array}\right)$ (1.10)

The multivariable I-function is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1,z_2,...z_s)=I_{p_2,q_2,p_3,q_3;\cdots;p_s,q_s:p',q';\cdots;p^{(s)},q^{(s)}}^{0,n_2;0,n_3;\cdots;0,n_r:m',n';\cdots;m^{(s)},n^{(s)}}\left(\begin{array}{c|c} z_1 & (a_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2};\cdots; \\ \cdot & \\ \cdot & \\ \cdot & \\ z_s & (b_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2};\cdots; \end{array}\right.$$

$$\left.\begin{array}{c}(a_{sj};\alpha'_{sj},\cdots,\alpha_{sj}^{(s)})_{1,p_s}:(a'_j,\alpha'_j)_{1,p'};\cdots;(a_j^{(s)},\alpha_j^{(s)})_{1,p^{(s)}} \\ \\ (b_{sj};\beta'_{sj},\cdots,\beta_{sj}^{(s)})_{1,q_s}:(b'_j,\beta'_j)_{1,q'};\cdots;(b_j^{(s)},\beta_j^{(s)})_{1,q^{(s)}}\end{array}\right) \tag{1.11}$$

$$=\frac{1}{(2\pi\omega)^s}\int_{L_1}\cdots\int_{L_s}\xi(t_1,\cdots,t_s)\prod_{i=1}^s\phi_i(t_i)z_i^{t_i}\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{1.12}$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [2]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k|<\frac{1}{2}\Omega_i^{(k)}\pi$, where

$$\Omega_i^{(k)} = \sum_{k=1}^{n^{(i)}} \alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}} \beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta_k^{(i)} + \left(\sum_{k=1}^{n_2} \alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{p_2} \alpha_{2k}^{(i)}\right) + \cdots +$$

$$\left(\sum_{k=1}^{n_s} \alpha_{sk}^{(i)} - \sum_{k=n_s+1}^{p_s} \alpha_{sk}^{(i)}\right) - \left(\sum_{k=1}^{q_2} \beta_{2k}^{(i)} + \sum_{k=1}^{q_3} \beta_{3k}^{(i)} + \cdots + \sum_{k=1}^{q_s} \beta_{sk}^{(i)}\right) \tag{1.13}$$

where $i = 1, \cdots, s$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1, \cdots, z_s) = 0(\, |z_1|^{\alpha'_1}, \cdots, |z_s|^{\alpha'_s}\,), max(\, |z_1|, \cdots, |z_s|\,) \rightarrow 0$$

$$\aleph(z_1, \cdots, z_s) = 0(\, |z_1|^{\beta'_1}, \cdots, |z_s|^{\beta'_s}\,), min(\, |z_1|, \cdots, |z_s|\,) \rightarrow \infty$$

where, with $k = 1, \cdots, z : \alpha'_k = min[Re(b_j^{(k)}/\beta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta'_k = max[Re((a_j^{(k)} - 1)/\alpha_j^{(k)})], j = 1, \cdots, n_k$$

We will use these following notations in this paper :

$$U = p_2, q_2; p_3, q_3; \cdots; p_{s-1}, q_{s-1}; V = 0, n_2; 0, n_3; \cdots; 0, n_{s-1} \tag{1.14}$$

$$W = (p', q'); \cdots; (p^{(s)}, q^{(s)}); X = (m', n'); \cdots; (m^{(s)}, n^{(s)}) \tag{1.15}$$

$$A = (a_{2k}, \alpha'_{2k}, \alpha''_{2k})_{1,p_2}; \cdots; (a_{(s-1)k}), \alpha'_{(s-1)k}, \alpha''_{(s-1)k}, \cdots, \alpha_{(s-1)k}^{(s-1)})_{1,p_{s-1}} \tag{1.16}$$

$$B = (b_{2k}, \beta'_{2k}, \beta''_{2k})_{1,q_2}; \cdots; (b_{(s-1)k}), \beta'_{(s-1)k}, \beta''_{(s-1)k}, \cdots, \beta_{(s-1)k}^{(s-1)})_{1,q_{s-1}} \tag{1.17}$$

$$\mathfrak{A} = (a_{sk}; \alpha'_{sk}, \alpha''_{sk}, \cdots, \alpha_{sk}^{(s)})_{p_s,q_s} : \mathfrak{B} = (b_{sk}; \beta'_{sk}, \beta''_{sk}, \cdots, \beta_{sk}^{(s)})_{1,q_s} \tag{1.18}$$

$$A' = (a'_k, \alpha'_k)_{1,p'}; \cdots; (a_k^{(s)}, \alpha_k^{(s)})_{1,p^{(s)}}; B' = (b'_k, \beta'_k)_{1,q'}; \cdots; (b_k^{(s)}, \beta_k^{(s)})_{1,q^{(s)}} \tag{1.19}$$

The multivariable I-function writes :

$$I(z_1, \cdots, z_s) = I_{U:p_s,q_s;W}^{V;0,n_s;X}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}\ ;\mathfrak{A};\mathrm{A'} \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_s & \mathrm{B};\ \mathfrak{B};\ \mathrm{B'} \end{array}\right) \tag{1.20}$$

The generalized polynomials defined by Srivastava [5], is given in the following manner :

$$S_{N'_1,\cdots,N'_t}^{M'_1,\cdots,M'_t}[y_1,\cdots,y_t] = \sum_{K_1=0}^{[N'_1/M'_1]}\cdots\sum_{K_t=0}^{[N'_t/M'_t]}\frac{(-N'_1)_{M'_1K_1}}{K_1!}\cdots\frac{(-N'_t)_{M'_tK_t}}{K_t!}$$

$$A[N'_1,K_1;\cdots;N'_t,K_t]y_1^{K_1}\cdots y_t^{K_t} \tag{1.21}$$

Where $M'_1,\cdots,M'_s$ are arbitrary positive integers and the coefficients $A[N'_1,K_1;\cdots;N'_t,K_t]$ are arbitrary constants, real or complex. In the present paper, we use the following notation

$$a_1 = \frac{(-N'_1)_{M'_1K_1}}{K_1!}\cdots\frac{(-N'_t)_{M'_tK_t}}{K_t!}A[N'_1,K_1;\cdots;N'_t,K_t] \tag{1.22}$$

The spheroidal function $\psi_{\alpha n}(c,\eta)$ of general order $\alpha > -1$ can be expansed as ([3] an [7].

$$\psi_{\alpha n}(c,\eta)= \frac{i^n\sqrt{2\pi}}{V_{\alpha n(c)}}\sum_{k=0,or1}^{\infty *} a_k(c|\alpha n)(c\eta)^{-\alpha-\frac{1}{2}}J_{k+\alpha+\frac{1}{2}}(c\eta) \tag{1.25}$$

which represents the function uniformly on (∞,∞), where the coefficients $a_k(c|\alpha n)$ satisfy the recursion formula [14, eq. 67]and the asterisk over the summation sign indicates that the sum is taken over only even or odd values of k according as n is even or odd. As $c\rightarrow 0, a_k(c|\alpha n)\rightarrow 0, k\neq n.$

2. Required integrals

We have the following multiple integrals ,see Marichev et al ([1], 3.3.3 Eq.7, page 589)

Lemme.

$$\int_{x_1\geqslant 0}\cdots\int_{x_n\geqslant 0}\left(\frac{1-x_1^{\alpha_1}-\cdots-x_n^{\alpha_n}}{1+x_1^{\alpha_1}+\cdots+x_n^{\alpha_n}}\right)^{\frac{1}{2}}\prod_{i=1}^{n}x_k^{-\upsilon_k}\,\mathrm{d}x_1\cdots\mathrm{d}x_n$$

$$= \sqrt{\pi}\frac{\Gamma\left(\frac{\upsilon_1}{\alpha_1}\right)\cdots\Gamma\left(\frac{\upsilon_n}{\alpha_n}\right)\Gamma\left(\frac{\beta}{2}\right)}{\Gamma(\beta)\Gamma\left(\frac{\beta+1}{2}\right)} - \frac{\Gamma\left(\frac{\upsilon_1}{\alpha_1}\right)\cdots\Gamma\left(\frac{\upsilon_n}{\alpha_n}\right)\Gamma\left(\frac{\beta+1}{2}\right)}{\Gamma(\beta)\Gamma\left(\frac{\beta+2}{2}\right)}\prod_{k=1}^{n}\alpha_k^{-1} \tag{2.1}$$

where $\beta = \frac{\upsilon_1}{\alpha_1}+\cdots+\frac{\upsilon_n}{\alpha_n}$

and $x_1^{\alpha_1}+\cdots+x_n^{\alpha_n}\leqslant 1, \alpha_i>0, \upsilon_i>0, i=1,\cdots,n$

3. Main integral

We note $X_{\upsilon_1,\cdots,\upsilon_n}= \prod_{i=1}^{n}x_k^{-\upsilon_k}$ and we have the following formula

Theorem

$$\int_{x_1\geqslant 0}\cdots\int_{x_n\geqslant 0}\left(\frac{1-x_1^{\alpha_1}-\cdots-x_n^{\alpha_n}}{1+x_1^{\alpha_1}+\cdots+x_n^{\alpha_n}}\right)^{\frac{1}{2}}\prod_{i=1}^{n}x_k^{-\upsilon_k}\,\psi_{\alpha n}(c^{\sigma},zX_{a_1,\cdots,a_n})$$

$$S_{N_1,\cdots,N_t}^{M'_1,\cdots,M'_t}\begin{pmatrix} \mathrm{y}_1 X_{\gamma_1^1,\cdots\gamma_1^n} \\ \cdot\ \cdot\ \cdot \\ \mathrm{y}_t X_{\gamma_t^1\cdots\gamma_t^n}\end{pmatrix} \aleph_{u:w}^{0,N:v}\begin{pmatrix} \mathrm{z}_1 X_{\alpha_1^1,\cdots\alpha_1^n} \\ \cdot\ \cdot\ \cdot \\ \mathrm{z}_r X_{\alpha_r^1\cdots\alpha_r^n}\end{pmatrix} I_{U:p_s,q_s;W}^{V;0,n_s;X}\begin{pmatrix} \mathrm{Z}_1 X_{\eta_1^1\cdots\eta_1^n} \\ \cdot\ \cdot\ \cdot \\ \mathrm{Z}_s X_{\eta_s^1\cdots\eta_s^n}\end{pmatrix}\mathrm{d}x_1\cdots\mathrm{d}x_n =$$

$$-\prod_{i=1}^n \alpha_k^{-1}\frac{\omega^n\sqrt{2\pi}}{V_{\alpha n}(c^\sigma)}\sum_{k=0,or1}^{\infty_*}\sum_{m=0}^{\infty}\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{M_1}\cdots\sum_{g_r=0}^{M_r}\sum_{K_1=0}^{[N_1/M'_1]}\cdots\sum_{K_t=0}^{[N_t/M'_t]} a_1\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}$$

$$G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})\ \frac{(-)^m a_k(c^\sigma|\alpha n)}{m!\Gamma(m+k+\alpha+\frac{3}{2})}\ y_1^{K_1}\cdots y_t^{K_t}\ z_1^{\eta_{G_1,g_1}}\cdots z_r^{\eta_{G_r,g_r}} z^{(2m+k)}$$

$$I_{U:p_s+n+1,q_s+2;W}^{V;0,n_s+n+1;X}\left(\begin{array}{c|c} \mathrm{Z}_1 & \mathrm{A}\ ;\left[1-\frac{\upsilon_i+(2m+k)a_i+\sum_{j=1}^t K_j\gamma_j^i+\sum_{j=1}^r\eta_{G_j,g_j}\alpha_j^i}{\alpha_i};\frac{\eta_1^i}{\alpha_i},\cdots,\frac{\eta_s^i}{\alpha_i}\right]_{1,n},\frac{1-\beta'}{2},\mathfrak{A}:A' \\ \cdot\ \cdot\ \cdot & \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ \mathrm{Z}_s & \mathrm{B}\ ;1\text{-}\beta',-\frac{\beta'}{2},\mathfrak{B}:B' \end{array}\right)$$

$$+\sqrt{\pi}\ \frac{\omega^n\sqrt{2\pi}}{V_{\alpha n}(c^\sigma)}\sum_{k=0,or1}^{\infty_*}\sum_{m=0}^{\infty}\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{M_1}\cdots\sum_{g_r=0}^{M_r}\sum_{K_1=0}^{[N_1/M'_1]}\cdots\sum_{K_t=0}^{[N_t/M'_t]} a_1$$

$$G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})\frac{(-)^m a_k(c^\sigma|\alpha n)}{m!\Gamma(m+k+\alpha+\frac{3}{2})}y_1^{K_1}\cdots y_t^{K_t}z_1^{\eta_{G_1,g_1}}\cdots z_r^{\eta_{G_r,g_r}}I_{U:p_s+n+1,q_s+2;W}^{V;0,n_s+n+1;X}\left(\begin{array}{c|} \mathrm{Z}_1 \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ \mathrm{Z}_s \end{array}\right.$$

$$\left.\begin{array}{c} \mathrm{A}\ ;\left[1-\frac{\upsilon_i+(2m+k)a_i+\sum_{j=1}^t K_j\gamma_j^i+\sum_{j=1}^r\eta_{G_j,g_j}\alpha_j^i}{\alpha_i};\frac{\eta_1^i}{\alpha_i},\cdots,\frac{\eta_s^i}{\alpha_i}\right]_{1,n},1-\frac{\beta'}{2},\mathfrak{A}:B' \\ \cdot\ \cdot\ \cdot \\ \mathrm{B};1\text{-}\beta',-\frac{1-\beta'}{2},\mathfrak{B}:B' \end{array}\right) \tag{3.1}$$

where :

$$1-\beta' = 1-\left[\upsilon_i+(2m+k)a_i+\sum_{j=1}^t K_j\gamma_j^i+\sum_{j=1}^r\eta_{G_j,g_j}\alpha_j^i\right],\eta_1^1,\cdots,\eta_s^1,\cdots,\quad \eta_s^1,\cdots,\eta_s^n) \tag{3.3}$$

$$-\beta' =-\left[\sum_{i=1}^n[\upsilon_i+(2m+k)a_i+\sum_{j=1}^t K_j\gamma_j^i+\sum_{j=1}^r\eta_{G_j,g_j}\alpha_j^i,\eta_1^1,\cdots,\eta_s^1,\cdots,\quad \eta_s^1,\cdots,\eta_s^n)\right. \tag{3.4}$$

$$-\frac{\beta'}{2}=-\left[\sum_{i=1}^n\frac{\upsilon_i+(2m+k)a_i+\sum_{j=1}^t K_j\gamma_j^i+\sum_{j=1}^r\eta_{G_j,g_j}\alpha_j^i}{2}\right],\frac{\eta_1^1}{2},\cdots,\frac{\eta_s^1}{2},\cdots,$$

$$\frac{\eta_s^1}{2}, \cdots, \frac{\eta_s^n}{2} \Bigg) \tag{3.5}$$

$$1-\frac{\beta'}{2} = 1 - \left[\sum_{i=1}^{n} \frac{\upsilon_i + (2m+k)a_i + \sum_{j=1}^{t} K_j\gamma_j^i + \sum_{j=1}^{r} \eta_{G_j,g_j}\alpha_j^i}{2}\right], \frac{\eta_1^1}{2}, \cdots, \frac{\eta_s^1}{2}, \cdots,$$

$$\frac{\eta_s^1}{2}, \cdots, \frac{\eta_s^n}{2} \Bigg) \tag{3.6}$$

$$\frac{1-\beta'}{2} = \frac{1}{2} - \left[\sum_{i=1}^{n} \frac{\upsilon_i + (2m+k)a_i + \sum_{j=1}^{t} K_j\gamma_j^i + \sum_{j=1}^{r} \eta_{G_j,g_j}\alpha_j^i}{2}\right], \frac{\eta_1^1}{2}, \cdots, \frac{\eta_s^1}{2}, \cdots,$$

$$\frac{\eta_s^1}{2}, \cdots, \frac{\eta_s^n}{2} \Bigg) \tag{3.7}$$

Provided that

a) $min\{a_i, b, \gamma_j^i, \alpha_k^j, \eta_l^i, \} > 0, i = 1, \cdots, n; j = 1, \cdots, t; k = 1, \cdots, r; l = 1, \cdots, s$

b) $Re(\upsilon_i + a_i(2m+k)) + \sum_{j=1}^{r} \beta_j^i \min_{1 \leqslant l \leqslant M_j} Re\left(\frac{d_l^{(j)}}{\delta_l^{(j)}}\right) + \sum_{i=1}^{s} \epsilon_j^i \min_{1 \leqslant l \leqslant m^{(j)}} Re\left(\frac{b_l^{(j)}}{\beta_l^{(j)}}\right) > 0, i = 1, \cdots, n$

c) $|argz_k| < \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.5) ; $i = 1, \cdots, r$

g) $|argZ_k| < \frac{1}{2}\Omega_i^{(k)}\pi$, where $\Omega_i^{(k)}$ is defined by (1.11) ; $i = 1, \cdots, s$

e) The double series occuring on the right-hand side of (3.1) is absolutely and uniformly convergent.

f) $x_1^{\alpha_1} + \cdots + x_n^{\alpha_n} \leqslant 1, \alpha_i > 0, \upsilon_i \in \mathbb{R}, i = 1, \cdots, n$

Proof

First, expressing the generalized the spheroidal function $\psi_{\alpha n}(c, \eta)$ in multiple series with the help of equation (2.1), the Aleph-function of r variables in series with the help of equation (1.6), the general class of polynomial of several variables $S_{N_1',\cdots,N_t'}^{M_1',\cdots,M_t'}[y_1, \cdots, y_t]$ with the help of equation (1.21) and the I-function of s variables in defined by Prasad [4] in Mellin-Barnes contour integrals with the help of equation (1.12), changing the order of integration ans summation (which is easily seen to be justified due to the absolute convergence of the integrals and the summations involved in the process) and then evaluating the resulting $(x_1, \cdots, x_n)$-integrals with the help of equation (3.1). Now interpreting the result thus obtained in the Mellin-barnes integrals contour, we arrive at the desired result.

5. Particular case

If $U = V = A = B = 0$, the multivariable I-function defined by Prasad degenere in multivariable H-function defined by Srivastava et al [6]. We have the following result.

Corollary

$$\int_{x_1\geqslant 0}\cdots\int_{x_n\geqslant 0}\left(\frac{1-x_1^{\alpha_1}-\cdots-x_n^{\alpha_n}}{1+x_1^{\alpha_1}+\cdots+x_n^{\alpha_n}}\right)^{\frac{1}{2}}\prod_{i=1}^{n}x_k^{-\upsilon_k}\,\psi_{\alpha n}(c^\sigma,zX_{a_1,\cdots,a_n})$$

$$S_{N_1,\cdots,N_t}^{M_1',\cdots,M_t'}\left(\begin{array}{c}\mathrm{y}_1X_{\gamma_1^1,\cdots\gamma_1^n}\\ \cdots\\ \mathrm{y}_tX_{\gamma_t^1\cdots\gamma_t^n}\end{array}\right)\aleph_{u:w}^{0,N:v}\left(\begin{array}{c}\mathrm{z}_1X_{\alpha_1^1,\cdots\alpha_1^n}\\ \cdots\\ \mathrm{z}_rX_{\alpha_r^1\cdots\alpha_r^n}\end{array}\right)H_{p_s,q_s;W}^{0,n_s;X}\left(\begin{array}{c}\mathrm{Z}_1X_{\eta_1^1\cdots\eta_1^n}\\ \cdots\\ \mathrm{Z}_sX_{\eta_s^1\cdots\eta_s^n}\end{array}\right)\mathrm{d}x_1\cdots\mathrm{d}x_n=$$

$$-\prod_{i=1}^{n}\alpha_k^{-1}\,\frac{\omega^n\sqrt{2\pi}}{V_{\alpha n}(c^\sigma)}\sum_{k=0,or1}^{\infty_*}\sum_{m=0}^{\infty}\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{M_1}\cdots\sum_{g_r=0}^{M_r}\sum_{K_1=0}^{[N_1/M_1']}\cdots\sum_{K_t=0}^{[N_t/M_t']}a_1\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}$$

$$G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})\,\frac{(-)^m a_k(c^\sigma|\alpha n)}{m!\Gamma(m+k+\alpha+\frac{3}{2})}\;y_1^{K_1}\cdots y_t^{K_t}\;z_1^{\eta_{G_1,g_1}}\cdots z_r^{\eta_{G_r,g_r}}z^{(2m+k)}$$

$$H_{p_s+n+1,q_s+2;W}^{0,n_s+n+1;X}\left(\begin{array}{c}\mathrm{Z}_1\\ \cdots\\ \cdots\\ \mathrm{Z}_s\end{array}\left|\begin{array}{c}\left[1-\frac{\upsilon_i+(2m+k)a_i+\sum_{j=1}^{t}K_j\gamma_j^i+\sum_{j=1}^{r}\eta_{G_j,g_j}\alpha_j^i}{\alpha_i};\frac{\eta_1^i}{\alpha_i},\cdots,\frac{\eta_s^i}{\alpha_i}\right]_{1,n},\frac{1-\beta'}{2},\mathfrak{A}:A'\\ \cdots\\ 1\text{-}\beta',-\frac{\beta'}{2},\mathfrak{B}:B'\end{array}\right.\right)$$

$$+\sqrt{\pi}\;\frac{\omega^n\sqrt{2\pi}}{V_{\alpha n}(c^\sigma)}\sum_{k=0,or1}^{\infty_*}\sum_{m=0}^{\infty}\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{M_1}\cdots\sum_{g_r=0}^{M_r}\sum_{K_1=0}^{[N_1/M_1']}\cdots\sum_{K_t=0}^{[N_t/M_t']}a_1$$

$$G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})\frac{(-)^m a_k(c^\sigma|\alpha n)}{m!\Gamma(m+k+\alpha+\frac{3}{2})}y_1^{K_1}\cdots y_t^{K_t}z_1^{\eta_{G_1,g_1}}\cdots z_r^{\eta_{G_r,g_r}}H_{p_s+n+1,q_s+2;W}^{0,n_s+n+1;X}\left(\begin{array}{c}\mathrm{Z}_1\\ \cdots\\ \cdots\\ \mathrm{Z}_s\end{array}\right|$$

$$\left.\begin{array}{c}\left[1-\frac{\upsilon_i+(2m+k)a_i+\sum_{j=1}^{t}K_j\gamma_j^i+\sum_{j=1}^{r}\eta_{G_j,g_j}\alpha_j^i}{\alpha_i};\frac{\eta_1^i}{\alpha_i},\cdots,\frac{\eta_s^i}{\alpha_i}\right]_{1,n},1-\frac{\beta'}{2},\mathfrak{A}:B'\\ \cdots\\ 1\text{-}\beta',-\frac{1-\beta'}{2},\mathfrak{B}:B'\end{array}\right)\qquad(4.1)$$

under the same notations and conditions that (4.1) with $U=V=A=B=0$

6. Conclusion

In this paper we have evaluated a generalized multiple integrals involving the multivariable Aleph-function, a class of polynomials of several variables the spheroidal function and the multivariable I-function defined by Prasad. The integral established in this paper is of very general nature as it contains multivariable Aleph-function, which is a general function of several variables studied so far. Thus, the integral established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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