Non-Markovian Bulk Service Queue with Unreliable Server and Multiple Vacation Under Restricted Admissibility Policy

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Abstract— *This paper is concerned with the transient and steady state analysis of unreliable server batch arrival general bulk service queueing system with multiple vacation under a restricted admissibility policy of arriving batches. Arrivals occur in batches according to compound Poisson process. Unlike the usual batch arrival queueing system, the restricted admissibility policy differs during a busy period and a vacation period and hence all arriving batches are not allowed to join the system at all times. The service is done in bulk with minimum of 'a' customers and maximum of 'b' customers. The service time follows a general (arbitrary) distribution. In addition, the server subject to active breakdown. As soon as the breakdown occurs the server is sent for repair and the customer who was just being served before server breakdown waits for the remaining service to complete. In the proposed model, the transient and steady state results for queue size distribution by applying the supplementary variable technique are derived. Some performance measures, special and particular cases are also discussed. Numerical illustration is provided to see the effect and validity of the results.*

Keywords— *Bulk service, Unreliable server, Multiple vacation, Restricted admissibility, Mean queue size*

I. INTRODUCTION

Interruption in the customer service is common in many queueing situations. As a result of a sudden breakdown, the service of a customer or a unit undergoing service has to be suspended and the customers have to wait till the server returns to the system or the system becomes operable again. Consequently, such failures have a definite effect on the system; particularly on the queue length and customer's waiting time in the system. Such systems with random breakdown have been studied by many authors, including Gaver [10], Avi-Itzhak and Naor [3] and Madan [16]. Dorda [8] has studied a finite single server queueing system subject to breakdowns where customer's interarrival and service times follow the Erlang distribution defined with certain fixed parameters and the times of failures and repairs are exponentially distributed. Recently, Rajadurai et al. [19] made a study on $M/G/1$ the feedback retrial queue, subject to server breakdown and repair under multiple working vacation policy.

A considerable amount of work has been done in a queueing system with vacation and successfully used in various applied problems such as production/inventory systems, communication systems, computer network, etc. A comprehensive and excellent study on vacation models can be found in Levy and Yechiali [15]. Doshi [9] and Takagi [20] have made a comprehensive survey of queuing systems with vacations. A batch arrival queueing system with multiple vacations was first studied by Baba [5]. Recently, Jeyakumar and Senthilnathan [14] analyzed the steady state behavior of batch arrival and bulk service queueing model with multiple working vacations.

In earlier literature, on different control models of queueing systems, namely control of servers, control of service rates, control of admission of customers and control of queue discipline, one can refer Crabill, Gross and Magazine [7]. Madan and Abu Dayyeah [17] studies some aspect of batch arrivals Bernoulli vacation models with restricted admissibility, where all arriving batches are not allowed into the system at all-time. Haridass and Arumuganathan [11] studied a bulk queueing system with multiple vacations and restricted admissibility policy. Ayyappan and Sathiya [1] analyzed $M^X/G/1$ feedback queue with three stage heterogeneous services, server vacations and restricted admissibility.

The theory of batch service queues originated with the work of Bailey [4]. He considered a queue with Poisson arrival and fixed batch service. He derived the transient solution of the queue size distribution. Neuts [18] proposed the "General Bulk Service Rule" in which service initiates only when a certain number of customers in the queue are available. His general bulk service rule was extended by Borthakur and Medhi. Holman, Chaudhry and Ghosal [12] studied $M/G(a,b)/1$ queues in which the server starts service only if a specified minimum say 'a' of customers have accumulated in the queue and he does not take more than 'b' customers for service in one batch. Ho woo Lee et al. [13] consider the batch service queue with a single vacation. Ayyappan and Devipriya [2] studied a single server fixed batch service queueing system under multiple vacations with gated service.

The rest of the paper is organized as follows. In Section II, the detailed description of the mathematical model and notations are given. In Section III, we consider the queue size distribution at a random epoch. Steady state results and stability condition are derived in Section IV and Section V. Some important performance measures and steady state probabilities are derived in Section VI and Section VII. Special cases and particular cases are obtained in Section VIII and Section IX. In Section X, the effects of various parameters on the system performance are analyzed numerically. Finally, the conclusion of the present work is given in Section XI.

II. The Mathematical Model

We consider a single server queueing system in which arrivals occur according to a compound Poisson process with batches of random size X. Further, it is assumed that not all batches are allowed to join the system at all times. During the busy period of the server, the arrivals are admitted with probability'π',whereas with probability 'θ', they are admitted when the server is on vacation. Such assumptions are quite meaningful in many real life situations. The service follows the "General Bulk Service Rule" in which service done with a minimum of 'a' customers and maximum of 'b' customers. The service time random variable B is assumed to follow a general (arbitrary) distribution with distribution function $B(x)$, Laplace Stieltjes transform $\bar{B}(s)$ and finite moments $E(B^K)(K \geq 1)$. After finishing a service, if the queue length is less than 'a', the server leaves for a vacation of random length. When he returns, if he finds less than 'a' customers, he leaves for another vacation and so on, until he finds 'a' customers in the queue. Next, we assume the vacation time random variable V follows a general (arbitrary) distribution with distribution function $V(x)$, Laplace Stieltjes transform $\bar{V}(s)$ and finite moments $E(V^K)(K \geq 1)$. The busy server may break down at any instant and the service channel will fail for a short interval of time. Now, we assume that the time between breakdowns occur according to a Poisson process with a mean rate of breakdown as'η'. As soon as the breakdown occurs, the server is sent for repair, during that time it stops providing service to the customers till service channel is repaired. The customers, who were just being served before server breakdown, wait for the remaining service to complete. Further, we assume that the repair time random variable R follows a general (arbitrary) distribution with distribution function $R(y)$, Laplace Stieltjes transform $\bar{R}(s)$ and finite moments $E(R^K)(K \geq 1)$.

A. Notations

In this section, we first set up the system state equations for the distribution of the queue size (the number of customers in the queue excluding the batch being served, if any) at a random epoch by treating the elapsed service time, elapsed vacation time and elapsed repair time as supplementary variables. Then we solve these equations and derive the probability generating function (PGF) of the queue size.

Now, let us define

λ - batch arrival rate,

X - arrival batch size (a random variable),

c_k = Prob $[X = k], k \geq 1,$

$C(z) = \sum_{k=1}^{\infty} z^k c_k$ is the PGF of X.

Further, it may be noted that since $B(x)$, $V(x)$, $R(y)$ are distribution functions, we have

$B(0) = V(0) = R(0) = 0$ and $B(\infty) = V(\infty) = R(\infty) = 1$

Moreover, since $B(x)$, $V(x)$, $R(y)$ are continuous at $x = 0$ and $y = 0$, so that

$\mu(x)dx = \dfrac{dB(x)}{1 - B(x)}$, $\nu(x)dx = \dfrac{dV(x)}{1 - V(x)}$ *and* $\zeta(y)dy = \dfrac{dR(y)}{1 - R(y)}$ are the first order differential functions (hazard rates) of $B(x)$, $V(x)$, and $R(y)$ respectively.

Let $N_Q(t)$ be the queue size at time 't', $B^0(t)$ be the elapsed service time at time 't', $V^0(t)$ be the elapsed vacation time at time 't' and $R^0(t)$ be the elapsed repair time at time 't'.
Further, we introduce the random variable

$$Y(t) = \begin{cases} 1, \text{ if the server is busy at time `t'}, \\ 2, \text{ if the server is on vacation at time `t'}, \\ 3, \text{ if the server is under repair at time `t'}. \end{cases}$$

Thus, the supplementary variables $B^0(t)$, $V^0(t)$ and $R^0(t)$ are introduced in order to obtain a bivariate Markov process $\{N_Q(t), L(t)\}$, where $L(t) = B^0(t)$ if $Y(t) = 1$, $L(t) = V^0(t)$ if $Y(t) = 2$ and $L(t) = R^0(t)$ if $Y(t) = 3$.

We define the following probabilities:

$P_n(x,t)$ = Probability that at time 't', the server is actively providing service and there are exactly 'n' customers in the queue, excluding the batch under service with an elapsed service time of batch of customers undergoing service is 'x'.

$V_n(x,t)$ = Probability that at time 't', the server is on vacation and there are exactly 'n' customers in the queue with an elapsed vacation time is 'x'.

$R_n(x,y,t)$ = Probability that at time 't', there are exactly 'n 'customers in the queue with an elapsed service time of the batch of customers undergoing service is 'x' and an elapsed repair time of server is 'y'.

For the Process, we define the following limiting probabilities:

$$P_n(x)dx = \lim_{t\to\infty} Pr[Y(t) = 1, N_Q(t) = n, x < B^0(t) \le x + dx]; \quad x > 0, \quad n \ge 0.$$

$$V_n(x)dx = \lim_{t\to\infty} Pr[Y(t) = 2, N_Q(t) = n, x < V^0(t) \le x + dx]; \quad x > 0, \quad n \ge 0.$$

and for fixed value of x and $n \ge 0$

$$R_n(x,y)dy = \lim_{t\to\infty} Pr[Y(t) = 3, N_Q(t) = n, y < R^0(t) \le y + dy / B^0(t) = x]; \quad (x,y) > 0.$$

III. Queue size distribution at a Random Epoch

The supplementary variable technique was introduced by Cox [6]. Using supplementary variables, one can convert non-Markovian models into Markovian models.

The Kolmogorov forward equations to govern the system in transient state can be written as follows:

$$\frac{\partial}{\partial x}P_0(x,t) + \frac{\partial}{\partial t}P_0(x,t) + (\lambda + \mu(x) + \eta)P_0(x,t) = \lambda(1-\pi)P_0(x,t) + \int_0^\infty R_0(x,y,t)\zeta(y)dy, \tag{1}$$

$$\frac{\partial}{\partial x}P_n(x,t) + \frac{\partial}{\partial t}P_n(x,t) + (\lambda + \mu(x) + \eta)P_n(x,t) = \lambda(1-\pi)P_n(x,t) + \pi\lambda\sum_{k=1}^{n} c_k P_{n-k}(x,t)$$
$$+ \int_0^\infty R_n(x,y,t)\zeta(y)dy, \quad n \ge 1, \tag{2}$$

$$\frac{\partial}{\partial x}V_0(x,t)+\frac{\partial}{\partial t}V_0(x,t)+(\lambda+\nu(x))V_0(x,t)=\lambda(1-\theta)V_0(x,t), \tag{3}$$

$$\frac{\partial}{\partial x}V_n(x,t)+\frac{\partial}{\partial t}V_n(x,t)+(\lambda+\nu(x))V_n(x,t)=\lambda(1-\theta)V_n(x,t)+\theta\lambda\sum_{k=1}^{n}c_kV_{n-k}(x,t), \quad n\geq 1, \tag{4}$$

$$\frac{\partial}{\partial y}R_0(x,y,t)+\frac{\partial}{\partial t}R_0(x,y,t)+(\lambda+\zeta(y))R_0(x,y,t)=0, \tag{5}$$

$$\frac{\partial}{\partial y}R_n(x,y,t)+\frac{\partial}{\partial t}R_n(x,y,t)+(\lambda+\zeta(y))R_n(x,y,t)=\lambda\sum_{k=1}^{n}c_kR_{n-k}(x,y,t), \ n\geq 1. \tag{6}$$

These set of equations are to be solved under the boundary conditions at $x = 0$:

$$P_0(0,\ t)=\sum_{r=a}^{b}\int_0^\infty P_r(x,t)\mu(x)dx+\sum_{r=a}^{b}\int_0^\infty V_r(x,t)\nu(x)dx, \tag{7}$$

$$P_n(0,\ t)=\int_0^\infty P_{n+b}(x,t)\mu(x)dx+\int_0^\infty V_{n+b}(x,t)\nu(x)dx, \ n\geq 1, \tag{8}$$

$$V_n(0,\ t)=\int_0^\infty P_n(x,t)\mu(x)dx+\int_0^\infty V_n(x,t)\nu(x)dx, \ \ n=0,1,2,..., \quad a-1, \tag{9}$$

$$V_n(0,\ t)=0, \ \ n\geq a, \tag{10}$$

$$R_n(x,0,\ t)=\eta P_n(x,t), \ \ n\geq 0. \tag{11}$$

The initial conditions are $V_n(0) = P_n(0) = R_n(0) = 0$ for $n = 0,1,2,...$ (12)

The normalizing condition is:

$$\sum_{n=0}^{\infty}\int_0^\infty P_n(x)dx+\sum_{n=0}^{\infty}\int_0^\infty V_n(x)dx+\sum_{n=0}^{\infty}\int_0^\infty\int_0^\infty R_n(x,y)dxdy=1$$

To solve the above equations, we define the following Probability generating functions for $|z|\leq 1$:

$$P_q(x,z,t)=\sum_{n=0}^{\infty}z^nP_n(x,t) \text{ and } P_q(0,\ z,t)=\sum_{n=0}^{\infty}z^nP_n(0,\ t),$$

$$V_q(x,z,t)=\sum_{n=0}^{\infty}z^nV_n(x,t) \text{ and } V_q(0,\ z,t)=\sum_{n=0}^{\infty}z^nV_n(0,\ t), \tag{13}$$

$$R_q(x,y,z,t)=\sum_{n=0}^{\infty}z^nR_n(x,y,t) \text{ and } R_q(x,0,\ z,t)=\sum_{n=0}^{\infty}z^nR_n(x,0,\ t).$$

The Laplace transform of a function $f(t)$ is defined as:

$$\bar{f}(s)=\int_0^\infty e^{-st}f(t)dt, \ \Re(s)>0. \tag{14}$$

Taking the Laplace transform of both sides of the equations (1) to (11) and using an equation (12), we get

$$\frac{\partial}{\partial x}\overline{P}_0(x,s)+(s+\lambda\pi+\mu(x)+\eta)\overline{P}_0(x,s)=\int_0^\infty\overline{R}_0(x,y,s)\zeta(y)dy, \tag{15}$$

$$\frac{\partial}{\partial x}\overline{P}_n(x,s)+(s+\lambda\pi+\mu(x)+\eta)\overline{P}_n(x,s)=\pi\lambda\sum_{k=1}^{n}c_k\overline{P}_{n-k}(x,s)+\int_0^\infty\overline{R}_n(x,y,s)\zeta(y)dy,\quad n\geq 1, \tag{16}$$

$$\frac{\partial}{\partial x}\overline{V}_0(x,s)+(s+\lambda\theta+\nu(x))\overline{V}_0(x,s)=0, \tag{17}$$

$$\frac{\partial}{\partial x}\overline{V}_n(x,s)+(s+\lambda\theta+\nu(x))\overline{V}_n(x,s)=\theta\lambda\sum_{k=1}^{n}c_k\overline{V}_{n-k}(x,s),\quad n\geq 1, \tag{18}$$

$$\frac{\partial}{\partial y}\overline{R}_0(x,y,s)+(s+\lambda+\zeta(y))\overline{R}_0(x,y,s)=0, \tag{19}$$

$$\frac{\partial}{\partial y}\overline{R}_n(x,y,s)+(s+\lambda+\zeta(y))\overline{R}_n(x,y,s)=\lambda\sum_{k=1}^{n}c_k\overline{R}_{n-k}(x,y,s),\quad n\geq 1, \tag{20}$$

$$\overline{P}_0(0,s)=\sum_{r=a}^{b}\int_0^\infty\overline{P}_r(x,s)\mu(x)dx+\sum_{r=a}^{b}\int_0^\infty\overline{V}_r(x,s)\nu(x)dx, \tag{21}$$

$$\overline{P}_n(0,s)=\int_0^\infty\overline{P}_{n+b}(x,s)\mu(x)dx+\int_0^\infty\overline{V}_{n+b}(x,s)\nu(x)dx,\ n\geq 1, \tag{22}$$

$$\overline{V}_n(0,s)=\int_0^\infty\overline{P}_n(x,s)\mu(x)dx+\int_0^\infty\overline{V}_n(x,s)\nu(x)dx,\quad n=0,1,2,\ldots,\quad a-1, \tag{23}$$

$$\overline{V}_n(0,s)=0,\ n\geq a, \tag{24}$$

$$\overline{R}_n(x,0,s)=\eta\overline{P}_n(x,s),\ n\geq 0. \tag{25}$$

By multiplying equation (16) by z^n, summing over, (n=0,1,2…) and adding to the equation (15) and using (13), we get

$$\frac{\partial}{\partial x}\overline{P}_q(x,z,s)+[s+\lambda\pi(1-C(z))+\mu(x)+\eta]\overline{P}_q(x,z,s)=\int_0^\infty\overline{R}_q(x,y,z,s)\zeta(y)dy. \tag{26}$$

Applying the same process in equation (18) and using (17), gives

$$\frac{\partial}{\partial x}\overline{V}_q(x,z,s)+[s+\lambda\theta(1-C(z))+\nu(x)]\overline{V}_q(x,z,s)=0. \tag{27}$$

Similarly from equations (19) and (20), we get

$$\frac{\partial}{\partial y}\overline{R}_q(x,y,z,s)+[s+\lambda(1-C(z))+\zeta(y)]\overline{R}_q(x,y,z,s)=0. \tag{28}$$

Further integrating equations (26) to (28), we get

$$\overline{P}_q(x,z,s)=\overline{P}_q(0,z,s)(1-\overline{B}(x))e^{-(\psi(z))x}, \tag{29}$$

$$\overline{V}_q(x,z,s)=\overline{V}_q(0,z,s)(1-\overline{V}(x))e^{-(\beta(z))x}, \tag{30}$$

$$\overline{R}_q(x,y,z,s)=\overline{R}_q(x,0,z,s)(1-\overline{R}(y))e^{-(\gamma(z))y}. \tag{31}$$

where $\psi(z)$, $\beta(z)$ and $\gamma(z)$ are given in Appendix – A.

Next, multiply the boundary condition by suitable powers of ‘z’ and taking summation over all possible values of ‘n’ and using generating functions, we get

$$z^b \overline{P}_q(0, z, s) = \int_0^\infty \overline{P}_q(x, z, s)\mu(x)dx + \int_0^\infty \overline{V}_q(x, z, s)\nu(x)dx - \sum_{r=0}^{a-1}\int_0^\infty \overline{P}_r(x, s)\mu(x)z^r dx - \sum_{r=0}^{a-1}\int_0^\infty \overline{V}_r(x, s)\nu(x)z^r dx + \sum_{r=a}^{b-1}\int_0^\infty \overline{P}_r(x, s)\mu(x)(z^b - z^r)dx + \sum_{r=a}^{b-1}\int_0^\infty \overline{V}_r(x, s)\nu(x)(z^b - z^r)dx, \quad (32)$$

$$\overline{V}_q(0, z, s) = \sum_{r=0}^{a-1}\int_0^\infty \overline{P}_r(x, s)\mu(x)z^r dx + \sum_{r=0}^{a-1}\int_0^\infty \overline{V}_r(x, s)\nu(x)z^r dx, \quad (33)$$

$$\overline{R}_q(x, 0, z, s) = \eta \overline{P}_q(x, z, s). \quad (34)$$

Multiplying both sides of the equation (29) by μ(x) and (30) by ν(x) and integrating over ‘x’, we obtain,

$$\int_0^\infty \overline{P}_q(x, z, s)\mu(x)dx = \overline{P}_q(0, z, s)\overline{B}[\psi(z)], \quad (35)$$

$$\int_0^\infty \overline{V}_q(x, z, s)\nu(x)dx = \overline{V}_q(0, z, s)\overline{V}[\beta(z)]. \quad (36)$$

Multiplying both sides of the equation (31) by ζ(y) and integrating over ‘y’ we obtain

$$\int_0^\infty \overline{R}_q(x, y, z, s)\zeta(y)dy = \overline{R}_q(x, 0, z, s)\overline{R}[\gamma(z)]. \quad (37)$$

Utilizing equations (35) and (36) in equation (32), we obtain

$$[z^b - \overline{B}(\psi(z))]\,\overline{P}_q(0, z, s) = \sum_{r=0}^{a-1}\int_0^\infty [\overline{P}_r(x, s)\mu(x) + \overline{V}_r(x, s)\nu(x)]\, z^r[\overline{V}(\beta(z)) - 1]\, dx + \sum_{r=a}^{b-1}\int_0^\infty [\overline{P}_r(x, s)\mu(x) + \overline{V}_r(x, s)\nu(x)](z^b - z^r)dx, \quad (38)$$

Again integrating equations (29) to (31), we have

$$\int_0^\infty \overline{P}_q(x, z, s)dx = \overline{P}_q(z, s) = \overline{P}_q(0, z, s)\left[\frac{1 - \overline{B}(\psi(z))}{\psi(z)}\right], \quad (39)$$

$$\int_0^\infty \overline{V}_q(x, z, s)dx = \overline{V}_q(z, s) = \overline{V}_q(0, z, s)\left[\frac{1 - \overline{V}(\beta(z))}{\beta(z)}\right], \quad (40)$$

$$\int_0^\infty \overline{R}_q(x, y, z, s)dy = \overline{R}_q(x, z, s) = \overline{R}_q(x, 0, z, s)\left[\frac{1 - \overline{R}(\gamma(z))}{\gamma(z)}\right], \quad (41)$$

then,

$$\overline{R}_q(z, s) = \int_0^\infty \overline{R}_q(x, z, s)dx = \int_0^\infty \overline{R}_q(x, o, z, s)\left[\frac{1 - \overline{R}(\gamma(z))}{\gamma(z)}\right]dx. \quad (42)$$

Inserting equations (34) and (39) in (42), we get

$$\overline{R}_q(z,s) = \eta \overline{P}_q(0, z, s)\left[\frac{1-\overline{B}(\psi(z))}{\psi(z)}\right]\left[\frac{1-\overline{R}(\gamma(z))}{\gamma(z)}\right] \tag{43}$$

Thus, $\overline{P}_q(z,s)$, $\overline{V}_q(z,s)$ and $\overline{R}_q(z,s)$ are completely determined from equations (39), (40) and (43) where $\overline{P}_q(0, z, s)$ and $\overline{V}_q(0, z, s)$ are given in (38) and (33) respectively.

IV. The steady state results

In this section, we shall derive the steady state probability distribution for our queueing model by applying the well-known Tauberian property,

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t). \tag{44}$$

Therefore, in steady state, the PGF of queue size when the server is busy, on vacation and under repair is given below:

$$P_q(z) = \frac{\left(\begin{array}{l}\sum_{r=0}^{a-1}\int_0^\infty [P_r(x)\mu(x)+V_r(x)\nu(x)]z^r[\overline{V}(\beta_1(z))-1]\,dx \\ +\sum_{r=a}^{b-1}\int_0^\infty [P_r(x)\mu(x)+V_r(x)\nu(x)](z^b-z^r)\,dx\end{array}\right)[1-\overline{B}(\psi_1(z))]}{\psi_1(z)(z^b-\overline{B}(\psi_1(z)))}, \tag{45}$$

$$V_q(z) = \frac{\left(\sum_{r=0}^{a-1}\int_0^\infty [P_r(x)\mu(x)+V_r(x)\nu(x)]z^r[1-\overline{V}(\beta_1(z))]\,dx\right)}{\beta_1(z)}, \tag{46}$$

$$R_q(z) = \frac{\left(\begin{array}{l}\left(\begin{array}{l}\sum_{r=0}^{a-1}\int_0^\infty [P_r(x)\mu(x)+V_r(x)\nu(x)]z^r[\overline{V}(\beta_1(z))-1]\,dx \\ +\sum_{r=a}^{b-1}\int_0^\infty [P_r(x)\mu(x)+V_r(x)\nu(x)](z^b-z^r)\,dx\end{array}\right) \\ \times\eta[1-\overline{B}(\psi_1(z))][1-\overline{R}(\gamma_1(z))]\end{array}\right)}{\gamma_1(z)\psi_1(z)[z^b-\overline{B}(\psi_1(z))]}. \tag{47}$$

Finally, the PGF of queue size is

$$P(z) = P_q(z) + V_q(z) + R_q(z) \tag{48}$$

Substituting equations (45) to (47) in equation (48), we get

$$P(z) = \frac{\begin{pmatrix} [\beta_1(z)][\sum_{r=0}^{a-1} q_r z^r (\overline{V}(\beta_1(z)) - 1) + \\ \sum_{r=a}^{b-1} \omega_r (z^b - z^r)][1 - \overline{B}(\psi_1(z))][\gamma_1(z) + \eta(1 - \overline{R}(\gamma_1(z)))] \\ - \gamma_1(z) \sum_{r=0}^{a-1} q_r z^r (\overline{V}(\beta_1(z)) - 1)(z^b - \overline{B}(\psi_1(z)))\psi_1(z) \end{pmatrix}}{\gamma_1(z)\beta_1(z)\psi_1(z)(z^b - \overline{B}(\psi_1(z)))}. \quad (49)$$

where q_r, ω_r, $\psi_1(z)$, $\beta_1(z)$ and $\gamma_1(z)$ are given in Appendix - A.

V. Stability condition

The probability generating function has to satisfy $P(1) = 1$. In order to satisfy this condition, apply L'Hopital rule and equating the expression to 1. Consecutively,

$$E(B)(1 + \eta E(R))[\lambda\theta E(X)E(V)\sum_{r=0}^{a-1} q_r + \sum_{r=a}^{b-1} \omega_r (b - r)]$$
$$+ E(V)\sum_{r=0}^{a-1} q_r [b - \lambda E(X)E(B)(\pi + \eta E(R))] = [b - \lambda E(X)E(B)(\pi + \eta E(R))].$$

Since q_r, ω_r are probabilities of 'r' customers being in the queue, if follows that left hand side of the above expression must be positive. Thus $P(1) = 1$ is satisfied if $z^b - \overline{B}(\psi_1(z)) > 0$, if $\rho = \lambda E(X)E(B)(\pi + \eta E(R))/b$ then $\rho < 1$ is the condition to be satisfied for the existence of steady state for the model under consideration. Equation (49) gives the PGF of the number of customers involving only 'b' unknowns. By a Rouche's theorem of complex variables, it can be proved that $[z^b - \overline{B}(\psi_1(z))]$ has b-1 zeros inside and one on the unit circle |z|=1. Since P(z) is analytic within and on the unit circle, the numerator must vanish at these points, which gives 'b' equations in 'b' unknowns. These equations can be solved by any suitable numerical techniques.

VI. Performance Measures

In this section, some useful performance measures of the proposed model like expected number of customers in the queue L_q, expected waiting time in the queue W_q, probability that the server is on vacation P (V), the probability that the server is busy P (B) and probability that the server is under repair P(R) are derived.

A. Expected Queue Length

The expected queue length L_q at an arbitrary epoch is obtained by differentiating $P(Z)$ at $z = 1$ and is given by

$$L_q = \left[\frac{D^{(IV)}(1)\ N^{(V)}(1) - N^{(IV)}(1)\ D^{(V)}(1)}{5(D^{(IV)}(1))^2}\right] \quad (50)$$

where

$$N^{(IV)}(1) = -24\lambda\theta E(X)R_1B_1[\sum_{r=0}^{a-1} q_rV_1 + \sum_{r=a}^{b}\omega_r(b-r)] - 24\lambda E(X)R_2(b-B_1)\sum_{r=0}^{a-1} q_rV_1$$

$$D^{(IV)}(1) = -24\lambda^2\theta E(X)^2R_2(b-B_1)$$

$$N^{(V)}(1) = (-60B_1\lambda\theta E(X)[\lambda E(X^2)S_1 + R_3 + T_1][\sum_{r=0}^{a-1} q_rV_1 + \sum_{r=a}^{b}\omega_r(b-r)]$$

$$-60\lambda^2\theta E(X)^2S_1[B_1(\sum_{r=0}^{a-1} q_rV_2 + 2\sum_{r=0}^{a-1} q_rV_1r + \sum_{r=a}^{b-1}\omega_r(b(b-1)-r(r-1)))$$

$$+(\sum_{r=0}^{a-1} q_rV_1 + \sum_{r=a}^{b-1}\omega_r(b-r))(B_2+T_2)]$$

$$-60\lambda R_2(b-B_1)[E(X^2)\sum_{r=0}^{a-1} q_rV_1 + E(X)(\sum_{r=0}^{a-1} q_rV_2 + 2\sum_{r=0}^{a-1} q_rrV_1)]$$

$$-60\lambda E(X)\sum_{r=0}^{a-1} q_rV_1[R_2(b(b-1)-B_2-T_2)+(b-B_1)(R_4+T_1)]),$$

$$D^{(V)}(1) = -60\lambda^2\theta E(X)[E(X)(b-B_1)(R_4+T_1) + E(X)R_2(b(b-1)-B_2-T_2)$$

$$+2E(X^2)R_2(b-B_1)].$$

where

$\omega_r, q_r, B_1, B_2, V_1, V_2, S_1, S_2, T_1, T_2, R_1, R_2, R_3$, and R_4 are given in Appendix – A

B. Expected waiting time in the queue

The expected waiting time is obtained by using Little's formula as,

$$W_q = \frac{L_q}{\lambda E(X)}, \tag{51}$$

where L_q is given in equation (50).

VII. System state probabilities

A. Probability that the server is on vacation

Let P(V), be the probability that the server is on multiple vacation at time 't'.

From equation (46), we get $P(V) = V_q(1) = E(V)\sum_{r=0}^{a-1} q_r$. (52)

B. Probability that the server is busy

Let P(B), be the probability that the server is in the busy period at time 't'.

From equation (45), we get $P(B) = P_q(1) = \dfrac{E(B)[\sum_{r=0}^{a-1} q_rV_1 + \sum_{r=a}^{b-1}\omega_r(b-r)]}{b-B_1}$. (53)

C. Probability that the server is under repair

Let P(R), be the probability that the server is under repair at time 't'.

From equation (47), we get $P(R) = R_q(1) = \dfrac{\eta E(B)E(R)[\sum_{r=0}^{a-1} q_r V_1 + \sum_{r=a}^{b-1} \omega_r (b-r)]}{b - B_1}$. (54)

VIII. Special cases

In this model, we take service time and vacation time as general (arbitrary). But for practical purposes, service time and vacation time with particular distribution are required. In this section, some special cases of the proposed model by specifying vacation time random variable and bulk service time random variable as exponentialy distributed are discussed.

Case (i):
Single server batch arrival queue with exponential service time and restricted admissibility policy.

The service time distribution is exponential with a parameter μ , then LST of B is given by $\overline{B}(s) = \dfrac{\mu}{(\mu + s)}$

and $E(B) = \dfrac{1}{\mu}$, so that $\overline{B}(\psi_1(z)) = \dfrac{\mu}{\mu + \psi_1(z)}$

Hence, the PGF of queue size distribution can be obtained by,

$$P(z) = \frac{\left(\begin{array}{l} \beta_1(z)[\sum_{r=0}^{a-1} q_r z^r (\overline{V}(\beta_1(z)) - 1) + \sum_{r=a}^{b-1} \omega_r (z^b - z^r)][\gamma_1(z) + \eta(1 - \overline{R}(\gamma_1(z)))]\psi_1(z) \\ - \gamma_1(z) \sum_{r=0}^{a-1} q_r z^r (\overline{V}(\beta_1(z)) - 1)\psi_1(z)][z^b(\mu + \psi_1(z)) - \mu] \end{array} \right)}{\gamma_1(z)\beta_1(z)\psi_1(z)[z^b(\mu + \psi_1(z)) - \mu)]}. \quad (55)$$

Case (ii):
Single server batch arrival queue with exponential vacation time and restricted admissibility policy.

The vacation time distribution is exponential with a parameter ν , then LST of V is given by $\overline{V}(s) = \dfrac{\nu}{(\nu + s)}$

and $E(V) = \dfrac{1}{\nu}$, so that $\overline{V}(\beta_1(z)) = \dfrac{\nu}{\nu + \beta_1(z)}$

Hence, the PGF of queue size distribution can be obtained by,

$$P(z) = \frac{\left(\begin{array}{l} \beta_1(z)[1 - \overline{B}(\psi_1(z))][\gamma_1(z) + \eta(1 - \overline{R}(\gamma_1(z)))][\sum_{r=a}^{b-1} \omega_r (z^b - z^r)(\nu + \beta_1(z)) \\ - \beta_1(z) \sum_{r=0}^{a-1} q_r z^r] + \gamma_1(z)\psi_1(z)\beta_1(z) \sum_{r=0}^{a-1} q_r z^r (z^b - \overline{B}(\psi_1(z))) \end{array} \right)}{\gamma_1(z)\beta_1(z)\psi_1(z)[z^b - \overline{B}(\psi_1(z))](\nu + \beta_1(z))}. \quad (56)$$

IX. Particular cases

Case (i):

When there is no server breakdown, the equation (49) reduces to

$$P(z)=\frac{\begin{pmatrix}\theta\sum_{r=a}^{b-1}\omega_r(z^b-z^r)(\overline{B}(\alpha_1(z))-1)\\ +\sum_{r=0}^{a-1}q_r z^r(\overline{V}(\beta_1(z))-1)[\theta(\overline{B}(\alpha_1(z))-1)+\pi(z^b-\overline{B}(\alpha_1(z)))]\end{pmatrix}}{\theta\pi(\lambda C(z)-\lambda)(z^b-\overline{B}(\alpha_1(z)))} \qquad (57)$$

which exactly coincides with the PGF of Haridass and Arumuganathan [11] , for an $M^X/G(a,b)/1$ queue with multiple vacations and restricted admissibility of arriving batches.

Case (ii):

When there is no server breakdown and all arrivals are allowed to join the system, equation (49) reduces to

$$P(z)=\frac{\left(\sum_{r=a}^{b-1}\omega_r(z^b-z^r)(\overline{B}(\gamma_1(z))-1)+\sum_{r=0}^{a-1}q_r z^r(\overline{V}(\gamma_1(z))-1)(z^b-1)\right)}{(-\lambda+\lambda C(z))[z^b-\overline{B}(\gamma_1(z))]}. \qquad (58)$$

which exactly coincides with the PGF of Jeyakumar and Senthilnathan [15] for an $M^X/G(a,b)/1$ queue with multiple vacation , without server breakdown and closedown time zero.

X. Numerical Illustration

The unknown probabilities of the queue size distribution are computed using numerical techniques. Using MATLAB, the zeroes of the function $[z^b-\overline{B}(\psi_1(z))]$ are obtained and simultaneous equations are solved.

A numerical example is analyzed with the following assumptions and notations:

1. Service time distribution is Erlang-2.
2. Batch size distribution of the arrival is geometric with mean 2.
3. Vacation time and Repair time are exponential with parameter $\nu = 10$ and $\zeta = 2$.
4. Minimum service capacity is $a = 3$ and maximum service capacity $b = 6$.
5. Probability of arriving batch will be allowed to join the system during the busy period is $\pi = 0.2$.
6. Probability of arriving batch will be allowed to join the system during the vacation period is $\theta = 0.2$.

The expected queue length L_q , expected waiting time in the queue W_q , is calculated for various service rates, breakdown rate, repair rate and the results are tabulated.

From Table (1) to Table (3) the following observations are made.
1. As service rate μ increases, the expected queue length and expected waiting time decreases.
2. As arrival rate λ increases, the expected queue length and expected waiting time increases.
3. For a fixed value of breakdown rate, if we increase the repair rate, the expected queue length and expected waiting time decrease.

Table I

Service rate *(Vs)* performance measure

$a = 3, b = 6, \lambda = 4, \zeta = 2, \nu = 10, \eta = 1, \theta = 0.2$ and $\pi = 0.2$

μ	ρ	L_q	W_q
5	**0.1867**	**1.4039**	**0.1755**
6	**0.1556**	**1.2484**	**0.1561**
7	**0.1333**	**1.1551**	**0.1444**
8	**0.1167**	**1.0938**	**0.1367**
9	**0.1037**	**1.0509**	**0.1314**

Table II

Arrival rate *(Vs)* performance measure

$a = 3, b = 6, \mu = 7, \zeta = 2, \nu = 10, \eta = 1, \theta = 0.2$ and $\pi = 0.2$

λ	ρ	L_q	W_q
6	**0.2000**	**1.6722**	**0.1392**
7	**0.2333**	**2.0387**	**0.1456**
8	**0.2667**	**2.4970**	**0.1561**
9	**0.3000**	**3.0658**	**0.1703**
10	**0.3333**	**3.7678**	**0.1884**

Table III

Breakdown and Repair rate *(Vs)* performance measure

$a = 3, b = 6, \mu = 7, \lambda = 4, \nu = 10, \theta = 0.2$ and $\pi = 0.2$

η	ζ	L_q	W_q
	2	**1.1551**	**0.1444**
1	**3**	**1.0227**	**0.1278**
	4	**0.9746**	**0.1218**
	2	**1.6020**	**0.2003**
2	**3**	**1.2184**	**0.1523**
	4	**1.0918**	**0.1365**
	2	**2.3368**	**0.2921**
3	**3**	**1.4948**	**0.1868**
	4	**1.2448**	**0.1556**

XI. Conclusion

In this paper, an $M^X/G(a,b)/1$ queue with a restricted admissibility policy of arriving batches and multiple vacation for an unreliable server is analyzed. The probability generating function of the number of customers in the queue when the server is busy, on vacation and under repair are found using the supplementary variable technique. Some important performance measures like mean number of customers in the queue and average waiting time in the queue are obtained. Particular and special cases of the model are also presented. From the numerical results, it is observed that the arrival rate increases, the expected queue length and waiting time of the customers increases, and if the service rate increase, the expected queue length and expected waiting time decrease.

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Appendix-A

The following expressions are used throughout this paper

$$\psi(z) = \alpha(z) + \eta(1 - \overline{R}(\gamma(z))),$$
$$\alpha(z) = s + \lambda\pi(1 - C(z)),$$
$$\beta(z) = s + \lambda\theta(1 - C(z)),$$
$$\gamma(z) = s + \lambda(1 - C(z)),$$
$$\psi_1(z) = \alpha_1(z) + \eta(1 - \overline{R}(\gamma_1(z))),$$
$$\alpha_1(z) = \lambda\pi(1 - C(z)),$$
$$\beta_1(z) = \lambda\theta(1 - C(z)),$$
$$\gamma_1(z) = \lambda(1 - C(z)),$$
$$q_r = \int_0^\infty [P_r(x)\mu(x) + V_r(x)\nu(x)]\,dx, 0 \le r \le a-1$$
$$\omega_r = \int_0^\infty [P_r(x)\mu(x) + V_r(x)\nu(x)]\,dx, a \le r \le b-1$$
$$\lambda\theta E(X)E(V) = V_1,$$
$$\lambda\theta E(X^2)E(V) + \lambda^2\theta^2 E(V^2)E(X^2) = V_2,$$
$$\lambda E(X)E(B)S_2 = B_1,$$
$$\lambda E(X^2)E(B)S_2 + \lambda^2 E(X^2)E(B^2)S_2^2 = B_2,$$
$$\lambda E(X)S_1 = R_1,$$
$$\lambda E(X)S_2 = R_2$$
$$\lambda E(X^2)S_1 = R_3,$$
$$\lambda E(X^2)S_2 = R_4,$$
$$\eta\lambda^2 E(X)^2 E(R^2) = T_1,$$
$$\eta\lambda^2 E(X)^2 E(R^2)E(B) = T_2,$$
$$1 + \eta E(R) = S_1,$$
$$\pi + \eta E(R) = S_2,$$
$$C'(1) = E(X) \text{ and } C''(1) = E(X^2).$$