Selberg integral involving a extension of the Hurwitz-Lerch Zeta function, class of pol,ynomials, multivariable I-function and multivariable Aleph-function II

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ABSTRACT
In the present paper we evaluate the modified Selberg integral involving the product of a multivariable Aleph-function, a extension of the Hurwitz-Lerch Zeta function, the multivariable I-function defined by Prasad [2] and general class of polynomials of several variables. The importance of the result established in this paper lies in the fact they involve the Aleph-function of several variables which is sufficiently general in nature and capable to yielding a large of results merely by specializating the parameters their in.

Keywords:Multivariable Aleph-function, general class of polynomials, modified Selberg integral, extension of the Hurwitz-Lerch Zeta function, multivariable I-function, multivariable H-function

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1.Introduction and preliminaries.

The function Aleph of several variables is an extension the multivariable I-function recently studied by C.K. Sharma and Ahmad [3] , itself is a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occurring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We define : $\aleph(z_1,\cdots,z_r) = \aleph^{0,N:M_1,N_1,\cdots,M_r,N_r}_{P_i,Q_i,\tau_i;R:P_{i^{(1)}},Q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;P_{i^{(r)}},Q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r \end{array}\right|$

$[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,N}]\ ,[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{N+1,P_i}]:$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ ,[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{1,Q_i}]:$

$[(\mathrm{c}_j^{(1)});\gamma_j^{(1)})_{1,N_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{N_1+1,P_i^{(1)}}];\cdots;[(\mathrm{c}_j^{(r)});\gamma_j^{(r)})_{1,N_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{N_r+1,P_i^{(r)}}]$
$\left.[(\mathrm{d}_j^{(1)});\delta_j^{(1)})_{1,M_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{M_1+1,Q_i^{(1)}}];\cdots;[(\mathrm{d}_j^{(r)});\delta_j^{(r)})_{1,M_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{M_r+1,Q_i^{(r)}}]\right)$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^r\theta_k(s_k)y_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^N\Gamma(1-a_j+\sum_{k=1}^r\alpha_j^{(k)}s_k)}{\sum_{i=1}^R[\tau_i\prod_{j=N+1}^{P_i}\Gamma(a_{ji}-\sum_{k=1}^r\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{Q_i}\Gamma(1-b_{ji}+\sum_{k=1}^r\beta_{ji}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k) = \frac{\prod_{j=1}^{M_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{N_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=M_k+1}^{Q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=N_k+1}^{P_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

Suppose , as usual , that the parameters

$b_j, j = 1, \cdots, Q; a_j, j = 1, \cdots, P;$

$c_{ji^{(k)}}^{(k)}, j = n_k + 1, \cdots, P_{i^{(k)}}; c_j^{(k)}, j = 1, \cdots, N_k;$

$d_{ji^{(k)}}^{(k)}, j = M_k + 1, \cdots, Q_{i^{(k)}}; d_j^{(k)}, j = 1, \cdots, M_k;$

with $k = 1 \cdots, r, i = 1, \cdots, R$, $i^{(k)} = 1, \cdots, R^{(k)}$

are complex numbers , and the $\alpha' s, \beta' s, \gamma' s$ and $\delta' s$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \tau_i \sum_{j=N+1}^{P_i} \alpha_{ji}^{(k)} - \tau_{i^{(k)}} \sum_{j=N_k+1}^{P_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} + \tau_i \sum_{j=1}^{Q_i} \beta_{ji}^{(k)} - \tau_{i^{(k)}} \sum_{j=M_k+1}^{Q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.4}$$

The reals numbers τ_i are positives for $i = 1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)} s_k)$ with $j = 1$ to m_k are separated from those of $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(k)} s_k)$ with $j = 1$ to N and $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)$ with $j = 1$ to N_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|arg z_k| < \frac{1}{2} A_i^{(k)} \pi$, where

$$A_i^{(k)} = \sum_{j=1}^{N} \alpha_j^{(k)} - \tau_i \sum_{j=N+1}^{P_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{Q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{N_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=N_k+1}^{P_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{M_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=M_k+1}^{Q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k = 1 \cdots, r, i = 1, \cdots, R \text{ , } i^{(k)} = 1, \cdots, R^{(k)} \tag{1.5}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \rightarrow 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \rightarrow \infty$ and

$\alpha_k = min[Re(d_j^{(k)} / \delta_j^{(k)})], j = 1, \cdots, M_k$

$\beta_k = max[Re((c_j^{(k)} - 1) / \gamma_j^{(k)})], j = 1, \cdots, N_k$

Serie representations of Aleph-function of several variables is given by

$$\aleph(z_1, \cdots, z_r) = \sum_{G_1, \cdots, G_r = 0}^{\infty} \sum_{g_1=0}^{M_1} \cdots \sum_{g_r=0}^{M_r} \frac{(-)^{G_1 + \cdots + G_r}}{\delta_{g_1} G_1! \cdots \delta_{g_r} G_r!} \psi(\eta_{G_1, g_1}, \cdots, \eta_{G_r, g_r})$$

$$\times \ \theta_1(\eta_{G_1,g_1})\cdots\theta_r(\eta_{G_r,g_r})z_1^{-\eta_{G_1,g_1}}\cdots z_r^{-\eta_{G_r,g_r}} \tag{1.6}$$

Where $\psi(.,\cdots,.), \theta_i(.), i=1,\cdots,r$ are given respectively in (1.2), (1.3) and

$$\eta_{G_1,g_1} = \frac{d_{g_1}^{(1)}+G_1}{\delta_{g_1}^{(1)}},\cdots, \ \eta_{G_r,g_r} = \frac{d_{g_r}^{(r)}+G_r}{\delta_{g_r}^{(r)}}$$

which is valid under the conditions $\delta_{g_i}^{(i)}[d_j^i+p_i] \neq \delta_j^{(i)}[d_{g_i}^i+G_i]$ (1.7)

for $j \neq M_i, M_i = 1,\cdots\eta_{G_i,g_i}; P_i, N_i = 0,1,2,\cdots,; y_i \neq 0, i=1,\cdots,r$ (1.8)

In the document , we will note :

$$G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r}) = \phi(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})\theta_1(\eta_{G_1,g_1})\cdots\theta_r(\eta_{G_r,g_r}) \tag{1.9}$$

where $\phi(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r}), \theta_1(\eta_{G_1,g_1}),\cdots,\theta_r(\eta_{G_r,g_r})$ are given respectively in (1.2) and (1.3)

We shall note the Aleph-function of r variables $\aleph_{u:w}^{0,N:v}\left(\begin{array}{c} \mathrm{z}_1 \\ \cdot\ \cdot\ \cdot \\ \mathrm{z}_r \end{array}\right)$ (1.10)

The multivariable I-function is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1,\cdots,z_s) = I_{p_2,q_2,p_3,q_3;\cdots;p_s,q_s:p',q';\cdots;p^{(s)},q^{(s)}}^{0,n_2;0,n_3;\cdots;0,n_r:m',n';\cdots;m^{(s)},n^{(s)}}\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2};\cdots; \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_s & (\mathrm{b}_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2};\cdots; \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{a}_{sj};\alpha'_{sj},\cdots,\alpha_{sj}^{(s)})_{1,p_s} : (a'_j,\alpha'_j)_{1,p'};\cdots;(a_j^{(s)},\alpha_j^{(s)})_{1,p^{(s)}} \\ \\ (\mathrm{b}_{sj};\beta'_{sj},\cdots,\beta_{sj}^{(s)})_{1,q_s} : (b'_j,\beta'_j)_{1,q'};\cdots;(b_j^{(s)},\beta_j^{(s)})_{1,q^{(s)}} \end{array}\right) \tag{1.11}$$

$$= \frac{1}{(2\pi\omega)^s}\int_{L_1}\cdots\int_{L_s}\xi(t_1,\cdots,t_s)\prod_{i=1}^{s}\phi_i(t_i)z_i^{t_i}\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{1.12}$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [2]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type integrals contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2}\Omega_i^{(k)}\pi$, where

$$\Omega_i^{(k)} = \sum_{k=1}^{n^{(i)}} \alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}} \beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta_k^{(i)} + \left(\sum_{k=1}^{n_2} \alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{p_2} \alpha_{2k}^{(i)}\right) + \cdots +$$

$$\left(\sum_{k=1}^{n_s} \alpha_{sk}^{(i)} - \sum_{k=n_s+1}^{p_s} \alpha_{sk}^{(i)}\right) - \left(\sum_{k=1}^{q_2} \beta_{2k}^{(i)} + \sum_{k=1}^{q_3} \beta_{3k}^{(i)} + \cdots + \sum_{k=1}^{q_s} \beta_{sk}^{(i)}\right) \tag{1.13}$$

where $i = 1, \cdots, s$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the the asymptotic expansion in the following convenient form :

$$I(z_1, \cdots, z_s) = 0(\,|z_1|^{\alpha_1'}, \cdots, |z_s|^{\alpha_s'}\,), max(\,|z_1|, \cdots, |z_s|\,) \rightarrow 0$$

$$I(z_1, \cdots, z_s) \;= 0(\,|z_1|^{\beta_1'}, \cdots, |z_s|^{\beta_s'}\,), min(\,|z_1|, \cdots, |z_s|\,) \rightarrow \infty$$

where, with $k = 1, \cdots, z : \alpha_k' = min[Re(b_j^{(k)}/\beta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta_k' = max[Re((a_j^{(k)} - 1)/\alpha_j^{(k)})], j = 1, \cdots, n_k$$

We shall use these following notations in this paper :

$$U = p_2, q_2; p_3, q_3; \cdots; p_{s-1}, q_{s-1}; V = 0, n_2; 0, n_3; \cdots; 0, n_{s-1} \tag{1.14}$$

$$W = (p', q'); \cdots; (p^{(s)}, q^{(s)}); X = (m', n'); \cdots; (m^{(s)}, n^{(s)}) \tag{1.15}$$

$$A = (a_{2k}, \alpha_{2k}', \alpha_{2k}'')_{1,p_2}; \cdots; (a_{(s-1)k}), \alpha'_{(s-1)k}, \alpha''_{(s-1)k}, \cdots, \alpha_{(s-1)k}^{(s-1)})_{1,p_{s-1}} \tag{1.16}$$

$$B = (b_{2k}, \beta_{2k}', \beta_{2k}'')_{1,q_2}; \cdots; (b_{(s-1)k}), \beta'_{(s-1)k}, \beta''_{(s-1)k}, \cdots, \beta_{(s-1)k}^{(s-1)})_{1,q_{s-1}} \tag{1.17}$$

$$\mathfrak{A} = (a_{sk}; \alpha'_{sk}, \alpha''_{sk}, \cdots, \alpha_{sk}^{(s)})_{p_s,q_s} : \mathfrak{B} = (b_{sk}; \beta'_{sk}, \beta''_{sk}, \cdots, \beta_{sk}^{(s)})_{1,q_s} \tag{1.18}$$

$$A' = (a_k', \alpha_k')_{1,p'}; \cdots; (a_k^{(s)}, \alpha_k^{(s)})_{1,p^{(s)}}; B' = (b_k', \beta_k')_{1,q'}; \cdots; (b_k^{(s)}, \beta_k^{(s)})_{1,q^{(s)}} \tag{1.19}$$

The multivariable I-function writes :

$$I(z_1, \cdots, z_s) = I_{U:p_s,q_s;W}^{V;0,n_s;X}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}\ ;\mathfrak{A};\mathrm{A'} \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_s & \mathrm{B};\ \mathfrak{B};\ \mathrm{B'} \end{array}\right) \tag{1.20}$$

The generalized polynomials defined by Srivastava [4], is given in the following manner :

$$S_{N'_1,\cdots,N'_t}^{M'_1,\cdots,M'_t}[y_1,\cdots,y_t] = \sum_{K_1=0}^{[N'_1/M'_1]}\cdots\sum_{K_t=0}^{[N'_t/M'_t]}\frac{(-N'_1)_{M'_1K_1}}{K_1!}\cdots\frac{(-N'_t)_{M'_tK_t}}{K_t!}$$

$$A[N'_1,K_1;\cdots;N'_t,K_t]y_1^{K_1}\cdots y_t^{K_t} \tag{1.21}$$

Where $M'_1,\cdots,M'_s$ are arbitrary positive integers and the coefficients $A[N'_1,K_1;\cdots;N'_t,K_t]$ are arbitrary constants, real or complex. In the present paper, we use the following notation

$$a_1 = \frac{(-N'_1)_{M'_1K_1}}{K_1!}\cdots\frac{(-N'_t)_{M'_tK_t}}{K_t!}A[N'_1,K_1;\cdots;N'_t,K_t] \tag{1.22}$$

2. Extension of the Hurwitz-Lerch Zeta function

The extension of the Hurwitz-Lerch Zeta function $\phi(z,\mathfrak{s},a)$ is introduced by Srivastava et al. ([6],eq.(6.2), page 503) as follows :

$$\phi_{(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_q)}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z;\mathfrak{s},a) = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\lambda_j)_{n\rho_j}}{(a+n)^{\mathfrak{s}}\prod_{j=1}^{q}(\mu_j)_{n\sigma_j}}\times\frac{z^n}{n!} \tag{2.1}$$

with : $p,q\in\mathbb{N}_0,\lambda_j\in\mathbb{C}(j=1,\cdots,p),a,\mu_j\in\mathbb{C}\setminus\mathbb{Z}_0^*\ \ (j=1,\cdots,q),\rho_j,\sigma_k\in\mathbb{R}^+$

$(j=1,\cdots,p;k=1,\cdots,q)$

where $\Delta>-1$ when $\mathfrak{s},z\in\mathbb{C};\Delta=-1$ and $\mathfrak{s}\in\mathbb{C},when|z|<\bigtriangledown^*$, $\Delta=-1$ and $Re(\chi)>\frac{1}{2}$ when $|z|=\bigtriangledown^*$

$$\bigtriangledown^*=\prod_{j=1}^{p}\rho_j^{\rho_j}\prod_{j=1}^{q}\sigma_j^{\sigma_j};\Delta=\sum_{j=1}^{q}\sigma_j-\sum_{j=1}^{p}\rho_j;\chi=\mathfrak{s}+\sum_{j=1}^{q}\mu_j-\sum_{j=1}^{p}\lambda_j+\frac{p-q}{2}$$

We note these conditions the conditions (f).

3. Required integral

We note $S(a,b,c)$, the Selberg integral , see Askey et al ([1], page 402) by :

$$S(a,b,c)=\int_0^1\cdots\int_0^1\prod_{i=1}^{n}x_i^{a-1}(1-x_i)^{b-1}\prod_{1\leqslant j<k\leqslant n}|x_j-x_k|^{2c}\,\mathrm{d}x_1\cdots\mathrm{d}x_n=$$

$$=\prod_{j=0}^{n-1}\frac{\Gamma(a+jc)\Gamma(b+jc)\Gamma(1+(j+1)c)}{\Gamma(a+b+(n-1+j)c)\Gamma(1+c)} \tag{3.1}$$

with $Re(a)>0, Re(b)>0, Re(c)>Max\left\{-\frac{1}{n},-\frac{Re(a)}{n-1},-\frac{Re(b)}{n-1}\right\}$

We consider the new integral , see Askey et al ([1], page 402) defined by :

Lemme

$$\int_0^1\cdots\int_0^1\prod_{i=1}^{k}x_i\prod_{i=1}^{n}x_i^{a-1}(1-x_i)^{b-1}\prod_{1\leqslant j<k\leqslant n}|x_j-x_k|^{2c}\,\mathrm{d}x_1\cdots\mathrm{d}x_n=$$

$$=\prod_{i=1}^{k}\frac{(a+(n-i)c)}{(a+b+(2n-i-1)c)}\;S(a,b,c) \tag{3.2}$$

with $Re(a)>0, Re(b)>0, Re(c)>Max\left\{-\frac{1}{n},-\frac{Re(a)}{n-1},-\frac{Re(b)}{n-1}\right\}$ and $k\leqslant n$

where $S(a,b,c)$ is defined by (3.1). In this paper, we shall denote the modified Selberg integral

4. Main integral

Let $X_{u,v,w}=\prod_{i=1}^{n}x_i^u(1-x_i)^v\prod_{1\leqslant j<k\leqslant n}|x_j-x_k|^{2w}$ and $b_m=\frac{\prod_{j=1}^{p}(\lambda_j)_{m\rho_j}}{(a+m)^{\mathfrak{s}}\prod_{j=1}^{q}(\mu_j)_{m\sigma_j}}$

we have the following formula

Theorem

$$\int_0^1\cdots\int_0^1\prod_{i=1}^{k}x_i\prod_{i=1}^{n}x_i^{a-1}(1-x_i)^{b-1}\prod_{1\leqslant j<k\leqslant n}|x_j-x_k|^{2c}\,\phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_q)}\left(zX_{\alpha,\beta,\gamma};\mathfrak{s},a\right)$$

$$S^{M_1,\cdots,M_t}_{N_1,\cdots,N_t}\left(\begin{array}{c}\mathrm{y}_1X_{\alpha_1,\beta_1,\gamma_1}\\ \cdots\\ \mathrm{y}_tX_{\alpha_t,\beta_t,\gamma_t}\end{array}\right)\aleph^{0,N:v}_{u:w}\left(\begin{array}{c}\mathrm{z}_1X_{\delta_1,\psi_1,\phi_1}\\ \cdots\\ \mathrm{z}_rX_{\delta_r,\psi_r,\phi_r}\end{array}\right)I^{V;0,n_s;X}_{U:p_s,q_s;W}\left(\begin{array}{c}\mathrm{Z}_1X_{\epsilon_1,\eta_1,\zeta_1}\\ \cdots\\ \mathrm{Z}_sX_{\epsilon_s,\eta_s,\zeta_s}\end{array}\right)\mathrm{d}x_1\cdots\mathrm{d}x_n=$$

$$\sum_{K_1=0}^{[N_1'/M_1']}\cdots\sum_{K_t=0}^{[N_t'/M_t']}\sum_{n'=0}^{\infty}\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{M_1}\cdots\sum_{g_r=0}^{M_r}\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}G(\eta_{G_1,g_1},\cdots\eta_{G_r,g_r})a_1\frac{b_mz^m}{m!}$$

$$z_1^{\eta_{G_1,g_1}}\cdots z_r^{\eta_{G_r,g_r}}y_1^{K_1}\cdots y_t^{K_t}I^{V;0,n_s+3n+2k;X}_{U:p_s+3n+2k,q_s+2n+2k;W}\left(\begin{array}{c|l}\mathrm{Z}_1 & \mathrm{A}\;;\\ \cdots & \\ \cdots & \\ \mathrm{Z}_s & \mathrm{B};\end{array}\right)$$

$$[1\text{-a-}\alpha' R-\sum_{i=1}^t K_i\alpha_i-\sum_{i=1}^r \eta_{G_i,g_i}\delta_i-j(c+\gamma' R+\sum_{i=1}^t \gamma_i K_i+\sum_{i=1}^r \phi_i\eta_{G_i,g_i});\epsilon_1+j\zeta_1,\cdots,\epsilon_s+j\zeta_s]_{0,n-1}$$

$$\cdots$$

$$(\text{-c-}\gamma' R-\sum_{i=1}^t K_i\gamma_i-\sum_{i=1}^r \phi_i\eta_{G_i,g_i};\zeta_1,\cdots,\zeta_s),\cdots,$$

$$[1\text{-b-}\beta' R-\sum_{i=1}^t K_i\beta_i-\sum_{i=1}^r \eta_{G_i,g_i}\psi_i-j(c+\gamma' R+\sum_{i=1}^t \gamma_i K_i+\sum_{i=1}^r \phi_i\eta_{G_i,g_i});\eta_1+j\zeta_1,\cdots,\eta_s+j\zeta_s]_{0,n-1}$$

$$\cdots$$

$$(\text{-c-}\gamma' R-\sum_{i=1}^t \gamma_i K_i-\sum_{i=1}^r \phi_i\eta_{G_i,g_i};\zeta_1,\cdots,\zeta_s),B_1,B_2,B_3,$$

$$\left.\begin{matrix}\text{-(j+1)(c+}\gamma' R+\sum_{i=1}^t \gamma_i K_i+\sum_{i=1}^r \phi_i\eta_{G_i,g_i});j\zeta_1,\cdots,j\zeta_s)]_{0,n-1},A_2,A_3,\mathfrak{A}:A' \\ \cdots \\ \mathfrak{B}:\text{B'}\end{matrix}\right) \tag{4.1}$$

where $B_1=[1-a-b-(\alpha'+\beta')R-\sum_{i=1}^t K_i(\alpha_i+\beta_i)-\sum_{i=1}^r(\delta_i+\phi_i)\eta_{G_i,g_i}-(n-1+j)\times$

$$(c+\gamma' R+\sum_{i=1}^t K_i\gamma_i+\sum_{i=1}^r \phi_i\eta_{G_i,g_i});\epsilon_1+\eta_1+j\zeta_1,\cdots,\epsilon_s+\eta_s+j\zeta_s]_{0,n-1} \tag{4.2}$$

$$A_2=\big[-a-m\alpha-\sum_{i=1}^t K_i\alpha_i-\sum_{i=1}^r \eta_{G_i,g_i}\delta_i-(n-j)(c+m\gamma+\sum_{i=1}^t \gamma_i K_i+\sum_{i=1}^r \phi_i\eta_{G_i,g_i});$$

$$\epsilon_1+(n-j)\zeta_1,\cdots,\epsilon_s+(n-j)\zeta_s\big]_{1,k} \tag{4.3}$$

$$B_2=\big[1-a-m\alpha-\sum_{i=1}^t K_i\alpha_i-\sum_{i=1}^r \eta_{G_i,g_i}\delta_i-(n-j)(c+m\gamma+\sum_{i=1}^t \gamma_i K_i+\sum_{i=1}^r \phi_i\eta_{G_i,g_i});$$

$$\epsilon_1+(n-j)\zeta_1,\cdots,\epsilon_s+(n-j)\zeta_s\big]_{1,k} \tag{4.4}$$

$$B_3=\big[-a-m\alpha-\sum_{i=1}^t K_i\alpha_i-\sum_{i=1}^r \eta_{G_i,g_i}\delta_i-b-m\beta-\sum_{i=1}^t K_i\beta_i-\sum_{i=1}^r \eta_{G_i,g_i}\psi_i$$

$$-(2n-j-1)(c+m\gamma+\sum_{i=1}^t \gamma_i K_i+\sum_{i=1}^r \phi_i\eta_{G_i,g_i});\epsilon_1+\eta_1+(2n-j-1)\zeta_1,\cdots$$

$$\epsilon_s+\eta_s+(2n-j-1)\zeta_s\big]_{1,k} \tag{4.5}$$

$$A_3=\big[1-a-m\alpha-\sum_{i=1}^t K_i\alpha_i-\sum_{i=1}^r \eta_{G_i,g_i}\delta_i-b-m\beta-\sum_{i=1}^t K_i\beta_i-\sum_{i=1}^r \eta_{G_i,g_i}\psi_i$$

$$-(2n-j-1)(c+m\gamma+\sum_{i=1}^{t}\gamma_iK_i+\sum_{i=1}^{r}\phi_i\eta_{G_i,g_i});\epsilon_1+\eta_1+(2n-j-1)\zeta_1,\cdots$$

$$\epsilon_s+\eta_s+(2n-j-1)\zeta_s\big]_{1,k} \tag{4.6}$$

Provided that

a) $min\{\alpha,\beta,\gamma,\alpha_i,\beta_i,\gamma_i,\delta_j,\psi_j,\phi_j,\epsilon_l,\eta_l,\zeta_l\}>0, i=1,\cdots,t,j=1,\cdots,r,l=1,\cdots,s$,

b)$A=Re(a+m\alpha)+\sum_{i=1}^{r}\delta_i\min_{1\leqslant j\leqslant M_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)+\sum_{i=1}^{s}\epsilon_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)>0$

c) $B=Re(b+m\beta)+\sum_{i=1}^{r}\psi_i\min_{1\leqslant j\leqslant M_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)+\sum_{i=1}^{s}\eta_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)>0$

$$C=Re(c+m\gamma)+\sum_{i=1}^{r}\phi_i\min_{1\leqslant j\leqslant M_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)+\sum_{i=1}^{s}\zeta_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)>$$

$$Max\left\{-\frac{1}{n},-\frac{A}{n-1},-\frac{B}{n-1}\right\}$$

e)$|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.5) ; $i=1,\cdots,r$

f) The conditions (f) are satisfied and $k\leqslant n$

g) $|argZ_k|<\frac{1}{2}\Omega_i^{(k)}\pi$, where $\Omega_i^{(k)}$ is defined by (1.11) ; $i=1,\cdots,s$

h) The multiple serie occurring on the right-hand side of (3.1) is absolutely and uniformly convergent.

Proof

First, expressing the generalized the sequence of functions $\phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_q)}(zX_{\alpha,\beta,\gamma};\mathfrak{s},a)$in serie with the help of equation (2.1), the Aleph-function of r variables in series with the help of equation (1.6), the general class of polynomial of several variables $S_{N_1',\cdots,N_t'}^{M_1',\cdots,M_t'}[y_1,\cdots,y_t]$ with the help of equation (1.21) and the I-function of s variables in defined by Prasad [4] in Mellin-Barnes contour integrals with the help of equation (1.12), changing the order of integrations ans summations (which is easily seen to be justified due to the absolute convergence of the integrals and the summations involved in the process). Now evaluating the resulting modified Selberg integral with the help of equation (3.2). Use the following relations $\Gamma(a)(a)_n=\Gamma(a+n)$ and $a=\frac{\Gamma(a+1)}{\Gamma(a)}$ several times with $Re(a)>0$. Finally interpreting the result thus obtained with the Mellin-barnes contour integrals, we arrive at the desired result.

5. Particular case

If $U=V=A=B=0$, the multivariable I-function defined by Prasad degenere in multivariable H-function defined by Srivastava and Panda [5]. We have the following result.

$$\int_0^1\cdots\int_0^1\prod_{i=1}^k x_i\prod_{i=1}^n x_i^{a-1}(1-x_i)^{b-1}\prod_{1\leqslant j<k\leqslant n}|x_j-x_k|^{2c}\,\phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_q)}\left(zX_{\alpha,\beta,\gamma};\mathfrak{s},a\right)$$

$$S^{M_1,\cdots,M_t}_{N_1,\cdots,N_t}\left(\begin{array}{c}\mathrm{y}_1X_{\alpha_1,\beta_1,\gamma_1}\\ \cdot\;\cdot\;\cdot\\ \mathrm{y}_tX_{\alpha_t,\beta_t,\gamma_t}\end{array}\right)\aleph^{0,N:v}_{u:w}\left(\begin{array}{c}\mathrm{z}_1X_{\delta_1,\psi_1,\phi_1}\\ \cdot\;\cdot\;\cdot\\ \mathrm{z}_rX_{\delta_r,\psi_r,\phi_r}\end{array}\right)H^{0,n_s;X}_{p_s,q_s;W}\left(\begin{array}{c}\mathrm{Z}_1X_{\epsilon_1,\eta_1,\zeta_1}\\ \cdot\;\cdot\;\cdot\\ \mathrm{Z}_sX_{\epsilon_s,\eta_s,\zeta_s}\end{array}\right)\mathrm{d}x_1\cdots\mathrm{d}x_n=$$

$$\sum_{K_1=0}^{[N_1'/M_1']}\cdots\sum_{K_t=0}^{[N_t'/M_t']}\sum_{n'=0}^{\infty}\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{M_1}\cdots\sum_{g_r=0}^{M_r}\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}G(\eta_{G_1,g_1},\cdots\eta_{G_r,g_r})a_1\frac{b_mz^m}{m!}$$

$$z_1^{\eta_{G_1,g_1}}\cdots z_r^{\eta_{G_r,g_r}}y_1^{K_1}\cdots y_t^{K_t}H^{0,n_s+3n+2k;X}_{p_s+3n+2k,q_s+2n+2k;W}\left(\begin{array}{c}\mathrm{Z}_1\\ \cdot\;\cdot\;\cdot\\ \cdot\;\cdot\;\cdot\\ \mathrm{Z}_s\end{array}\right|$$

$$[\text{1-a-}\alpha'R-\textstyle\sum_{i=1}^t K_i\alpha_i-\sum_{i=1}^r\eta_{G_i,g_i}\delta_i-j(c+\gamma'R+\sum_{i=1}^t\gamma_iK_i+\sum_{i=1}^r\phi_i\eta_{G_i,g_i});\epsilon_1+j\zeta_1,\cdots,\epsilon_s+j\zeta_s]_{0,n-1}$$

$$\cdot\;\cdot\;\cdot$$

$$(\text{-c-}\gamma'R-\textstyle\sum_{i=1}^t K_i\gamma_i-\sum_{i=1}^r\phi_i\eta_{G_i,g_i};\zeta_1,\cdots,\zeta_s),\cdots,$$

$$[\text{1-b-}\beta'R-\textstyle\sum_{i=1}^t K_i\beta_i-\sum_{i=1}^r\eta_{G_i,g_i}\psi_i-j(c+\gamma'R+\sum_{i=1}^t\gamma_iK_i+\sum_{i=1}^r\phi_i\eta_{G_i,g_i});\eta_1+j\zeta_1,\cdots,\eta_s+j\zeta_s]_{0,n-1}$$

$$\cdot\;\cdot\;\cdot$$

$$(\text{-c-}\gamma'R-\textstyle\sum_{i=1}^t\gamma_iK_i-\sum_{i=1}^r\phi_i\eta_{G_i,g_i};\zeta_1,\cdots,\zeta_s),B_1,B_2,B_3,$$

$$\left.\begin{array}{c}\text{-(j+1)(c+}\gamma'R+\sum_{i=1}^t\gamma_iK_i+\sum_{i=1}^r\phi_i\eta_{G_i,g_i});j\zeta_1,\cdots,j\zeta_s)]_{0,n-1},A_2,A_3,\mathfrak{A}:A'\\ \cdot\;\cdot\;\cdot\\ \mathfrak{B}:\mathrm{B'}\end{array}\right)\tag{4.1}$$

where B_1,B_2,A_2,B_3 and $\AA_3$ are defined respectively by (4.2), (4.3), (4.4), (4.5) and (4.6)

under the same notations and conditions that (4.1) with $U=V=A=B=0$.

6. Conclusion

In this paper we have evaluated a modified Selberg integral involving the multivariable Aleph-function, the multivariable I-function defined by Prasad [2], a class of polynomials of several variables and the general of sequence of functions. The integral established in this paper is of very general nature as it contains multivariable Aleph-function, which is a general function of several variables studied so far. Thus, the integral established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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