Multiple integrals involving a extension of the Hurwitz-Lerch Zeta function, class of pol,ynomials, multivariable I-function, multivariable Aleph-function and product of two Jacobi polynomials

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ABSTRACT
In the present paper we evaluate the multiple integrals involving the product of a multivariable Aleph-function, a extension of the Hurwitz-Lerch Zeta function, the multivariable I-function defined by Prasad [2], the product of two Jacobi polynomials and general class of polynomials of several variables. The importance of the result established in this paper lies in the fact they involve the Aleph-function of several variables which is sufficiently general in nature and capable to yielding a large of results merely by specializating the parameters their in.

Keywords:Multivariable Aleph-function, general class of polynomials, multiple integral, extension of the Hurwitz-Lerch Zeta function, multivariable I-function, multivariable H-function, Jacobi polynomial

2010 Mathematics Subject Classification. 33C99, 33C60, 44A20

1.Introduction and preliminaries.

The function Aleph of several variables is an extension the multivariable I-function recently studied by C.K. Sharma and Ahmad [3] , itself is a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occurring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We define : $\aleph(z_1,\cdots,z_r)=\aleph^{0,N:M_1,N_1,\cdots,M_r,N_r}_{P_i,Q_i,\tau_i;R:P_{i^{(1)}},Q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;P_{i^{(r)}},Q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c}\mathrm{y}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{y}_r\end{array}\right|$

$$[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,N}],[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{N+1,P_i}]:$$
$$\cdots\cdots\cdots,[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{1,Q_i}]:$$

$$\left.\begin{array}{c}[(\mathrm{c}_j^{(1)});\gamma_j^{(1)})_{1,N_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{N_1+1,P_i^{(1)}}];\cdots;[(\mathrm{c}_j^{(r)});\gamma_j^{(r)})_{1,N_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{N_r+1,P_i^{(r)}}]\\ [(\mathrm{d}_j^{(1)});\delta_j^{(1)})_{1,M_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{M_1+1,Q_i^{(1)}}];\cdots;[(\mathrm{d}_j^{(r)});\delta_j^{(r)})_{1,M_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{M_r+1,Q_i^{(r)}}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)y_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{N}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=N+1}^{P_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{Q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{M_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{N_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=M_k+1}^{Q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=N_k+1}^{P_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

Suppose , as usual , that the parameters

$$b_j, j = 1, \cdots, Q; a_j, j = 1, \cdots, P;$$

$$c_{ji^{(k)}}^{(k)}, j = n_k + 1, \cdots, P_{i^{(k)}}; c_j^{(k)}, j = 1, \cdots, N_k;$$

$$d_{ji^{(k)}}^{(k)}, j = M_k + 1, \cdots, Q_{i^{(k)}}; d_j^{(k)}, j = 1, \cdots, M_k;$$

with $k = 1 \cdots, r, i = 1, \cdots, R$, $i^{(k)} = 1, \cdots, R^{(k)}$

are complex numbers , and the $\alpha' s, \beta' s, \gamma' s$ and $\delta' s$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \tau_i \sum_{j=N+1}^{P_i} \alpha_{ji}^{(k)} - \tau_{i^{(k)}} \sum_{j=N_k+1}^{P_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} + \tau_i \sum_{j=1}^{Q_i} \beta_{ji}^{(k)} - \tau_{i^{(k)}} \sum_{j=M_k+1}^{Q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.4}$$

The reals numbers τ_i are positives for $i = 1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)} s_k)$ with $j = 1$ to m_k are separated from those of $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(k)} s_k)$ with $j = 1$ to N and $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)$ with $j = 1$ to N_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2} A_i^{(k)} \pi$, where

$$A_i^{(k)} = \sum_{j=1}^{N} \alpha_j^{(k)} - \tau_i \sum_{j=N+1}^{P_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{Q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{N_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=N_k+1}^{P_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{M_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=M_k+1}^{Q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k = 1 \cdots, r, i = 1, \cdots, R , i^{(k)} = 1, \cdots, R^{(k)} \tag{1.5}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions oconcerning the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$$\aleph(y_1, \cdots, y_r) = 0(|y_1|^{\alpha_1}, \cdots, |y_r|^{\alpha_r}), max(|y_1|, \cdots, |y_r|) \to 0$$

$$\aleph(y_1, \cdots, y_r) = 0(|y_1|^{\beta_1}, \cdots, |y_r|^{\beta_r}), min(|y_1|, \cdots, |y_r|) \to \infty$$

where, with $k = 1, \cdots, r : \alpha_k = min[Re(d_j^{(k)}/\delta_j^{(k)})], j = 1, \cdots, M_k$ and

$$\beta_k = max[Re((c_j^{(k)} - 1)/\gamma_j^{(k)})], j = 1, \cdots, N_k$$

Serie representation of Aleph-function of several variables is given by

$$\aleph(y_1, \cdots, y_r) = \sum_{G_1, \cdots, G_r = 0}^{\infty} \sum_{g_1=0}^{M_1} \cdots \sum_{g_r=0}^{M_r} \frac{(-)^{G_1 + \cdots + G_r}}{\delta_{g_1} G_1! \cdots \delta_{g_r} G_r!} \psi(\eta_{G_1, g_1}, \cdots, \eta_{G_r, g_r})$$

$$\times\ \theta_1(\eta_{G_1,g_1})\cdots\theta_r(\eta_{G_r,g_r})y_1^{-\eta_{G_1,g_1}}\cdots y_r^{-\eta_{G_r,g_r}} \tag{1.6}$$

Where $\psi(.,\cdots,.)$, $\theta_i(.)$,$i=1,\cdots,r$ are given respectively in (1.2), (1.3) and

$$\eta_{G_1,g_1}=\frac{d_{g_1}^{(1)}+G_1}{\delta_{g_1}^{(1)}},\cdots,\ \eta_{G_r,g_r}=\frac{d_{g_r}^{(r)}+G_r}{\delta_{g_r}^{(r)}}$$

which is valid under the conditions $\delta_{g_i}^{(i)}[d_j^i+p_i]\neq\delta_j^{(i)}[d_{g_i}^i+G_i]$ (1.7)

for $j\neq M_i, M_i=1,\cdots\eta_{G_i,g_i};P_i,N_i=0,1,2,\cdots,;y_i\neq 0,i=1,\cdots,r$ (1.8)

In the document , we will note :

$$G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})=\phi(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})\theta_1(\eta_{G_1,g_1})\cdots\theta_r(\eta_{G_r,g_r}) \tag{1.9}$$

where $\phi(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r}),\theta_1(\eta_{G_1,g_1}),\cdots,\theta_r(\eta_{G_r,g_r})$ are given respectively in (1.2) and (1.3)

The multivariable I-function writes :

$$\aleph_{u:w}^{0,N:v}\left(\begin{array}{c}\mathrm{z}_1\\ \cdot\ \cdot\ \cdot\\ \mathrm{z}_r\end{array}\right) \tag{1.10}$$

The multivariable I-function is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1,\cdots,z_s)\ =\ I_{p_2,q_2,p_3,q_3;\cdots;p_s,q_s:p',q';\cdots;p^{(s)},q^{(s)}}^{0,n_2;0,n_3;\cdots;0,n_r:m',n';\cdots;m^{(s)},n^{(s)}}\left(\begin{array}{c|c}\mathrm{z}_1 & (\mathrm{a}_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2};\cdots;\\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_s & (\mathrm{b}_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2};\cdots;\end{array}\right.$$

$$\left.\begin{array}{c}(\mathrm{a}_{sj};\alpha'_{sj},\cdots,\alpha_{sj}^{(s)})_{1,p_s}:(a'_j,\alpha'_j)_{1,p'};\cdots;(a_j^{(s)},\alpha_j^{(s)})_{1,p^{(s)}}\\ \\ (\mathrm{b}_{sj};\beta'_{sj},\cdots,\beta_{sj}^{(s)})_{1,q_s}:(b'_j,\beta'_j)_{1,q'};\cdots;(b_j^{(s)},\beta_j^{(s)})_{1,q^{(s)}}\end{array}\right) \tag{1.11}$$

$$=\frac{1}{(2\pi\omega)^s}\int_{L_1}\cdots\int_{L_s}\xi(t_1,\cdots,t_s)\prod_{i=1}^{s}\phi_i(t_i)z_i^{t_i}\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{1.12}$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [2]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k|<\frac{1}{2}\Omega_i^{(k)}\pi$, where

$$\Omega_i^{(k)} = \sum_{k=1}^{n^{(i)}} \alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}} \beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta_k^{(i)} + \left(\sum_{k=1}^{n_2} \alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{p_2} \alpha_{2k}^{(i)}\right) + \cdots +$$

$$\left(\sum_{k=1}^{n_s} \alpha_{sk}^{(i)} - \sum_{k=n_s+1}^{p_s} \alpha_{sk}^{(i)}\right) - \left(\sum_{k=1}^{q_2} \beta_{2k}^{(i)} + \sum_{k=1}^{q_3} \beta_{3k}^{(i)} + \cdots + \sum_{k=1}^{q_s} \beta_{sk}^{(i)}\right) \tag{1.13}$$

where $i = 1, \cdots, s$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1, \cdots, z_s) = 0(\,|z_1|^{\alpha'_1}, \cdots, |z_s|^{\alpha'_s}\,), max(\,|z_1|, \cdots, |z_s|\,) \rightarrow 0$$

$$\aleph(z_1, \cdots, z_s) \; = 0(\,|z_1|^{\beta'_1}, \cdots, |z_s|^{\beta'_s}\,), min(\,|z_1|, \cdots, |z_s|\,) \rightarrow \infty$$

where, with $k = 1, \cdots, z : \alpha'_k = min[Re(b_j^{(k)}/\beta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta'_k = max[Re((a_j^{(k)} - 1)/\alpha_j^{(k)})], j = 1, \cdots, n_k$$

We will use these following notations in this paper :

$$U = p_2, q_2; p_3, q_3; \cdots; p_{s-1}, q_{s-1}; V = 0, n_2; 0, n_3; \cdots; 0, n_{s-1} \tag{1.14}$$

$$W = (p', q'); \cdots; (p^{(s)}, q^{(s)}); X = (m', n'); \cdots; (m^{(s)}, n^{(s)}) \tag{1.15}$$

$$A = (a_{2k}, \alpha'_{2k}, \alpha''_{2k})_{1,p_2}; \cdots; (a_{(s-1)k}), \alpha'_{(s-1)k}, \alpha''_{(s-1)k}, \cdots, \alpha_{(s-1)k}^{(s-1)})_{1,p_{s-1}} \tag{1.16}$$

$$B = (b_{2k}, \beta'_{2k}, \beta''_{2k})_{1,q_2}; \cdots; (b_{(s-1)k}), \beta'_{(s-1)k}, \beta''_{(s-1)k}, \cdots, \beta_{(s-1)k}^{(s-1)})_{1,q_{s-1}} \tag{1.17}$$

$$\mathfrak{A} = (a_{sk}; \alpha'_{sk}, \alpha''_{sk}, \cdots, \alpha_{sk}^{(s)})_{p_s,q_s} : \mathfrak{B} = (b_{sk}; \beta'_{sk}, \beta''_{sk}, \cdots, \beta_{sk}^{(s)})_{1,q_s} \tag{1.18}$$

$$A' = (a'_k, \alpha'_k)_{1,p'}; \cdots; (a_k^{(s)}, \alpha_k^{(s)})_{1,p^{(s)}}; B' = (b'_k, \beta'_k)_{1,q'}; \cdots; (b_k^{(s)}, \beta_k^{(s)})_{1,q^{(s)}} \tag{1.19}$$

The multivariable I-function writes :

$$I(z_1, \cdots, z_s) = I_{U:p_s,q_s;W}^{V;0,n_s;X}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}\;;\mathfrak{A};\mathrm{A'} \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_s & \mathrm{B};\;\mathfrak{B};\;\mathrm{B'} \end{array}\right) \tag{1.20}$$

The generalized polynomials defined by Srivastava [4], is given in the following manner :

$$S_{N_1',\cdots,N_t'}^{M_1',\cdots,M_t'}[y_1,\cdots,y_t] = \sum_{K_1=0}^{[N_1'/M_1']}\cdots\sum_{K_t=0}^{[N_t'/M_t']}\frac{(-N_1')_{M_1'K_1}}{K_1!}\cdots\frac{(-N_t')_{M_t'K_t}}{K_t!}$$

$$A[N_1',K_1;\cdots;N_t',K_t]y_1^{K_1}\cdots y_t^{K_t} \tag{1.21}$$

Where $M_1',\cdots,M_s'$ are arbitrary positive integers and the coefficients $A[N_1',K_1;\cdots;N_t',K_t]$ are arbitrary constants, real or complex. In the present paper, we use the following notation

$$a_1 = \frac{(-N_1')_{M_1'K_1}}{K_1!}\cdots\frac{(-N_t')_{M_t'K_t}}{K_t!}A[N_1',K_1;\cdots;N_t',K_t] \tag{1.22}$$

2. Extension of the Hurwitz-Lerch Zeta function

The extension of the Hurwitz-Lerch Zeta function $\phi(z,\mathfrak{s},a)$ is introduced by Srivastava et al ([6],eq.(6.2), page 503) as follows :

$$\phi_{(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_q)}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z;\mathfrak{s},a) = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\lambda_j)_{n\rho_j}}{(a+n)^{\mathfrak{s}}\prod_{j=1}^{q}(\mu_j)_{n\sigma_j}}\times\frac{z^n}{n!} \tag{2.1}$$

with : $p,q\in\mathbb{N}_0, \lambda_j\in\mathbb{C}(j=1,\cdots,p), a,\mu_j\in\mathbb{C}\setminus\mathbb{Z}_0^*\ \ (j=1,\cdots,q), \rho_j,\sigma_k\in\mathbb{R}^+$

$(j=1,\cdots,p;k=1,\cdots,q)$

where $\Delta>-1$ when $\mathfrak{s},z\in\mathbb{C}$; $\Delta=-1$ and $\mathfrak{s}\in\mathbb{C}, when|z|<\bigtriangledown^*$, $\Delta=-1$ and $Re(\chi)>\frac{1}{2}$ when $|z|=\bigtriangledown^*$

$$\bigtriangledown^* = \prod_{j=1}^{p}\rho_j^{\rho_j}\prod_{j=1}^{q}\sigma_j^{\sigma_j}\,;\Delta=\sum_{j=1}^{q}\sigma_j-\sum_{j=1}^{p}\rho_j;\chi=\mathfrak{s}+\sum_{j=1}^{q}\mu_j-\sum_{j=1}^{p}\lambda_j+\frac{p-q}{2}$$

We note these conditions the conditions (f).

3.Required formula

Let $\bar{bx} = b_1x_1+\cdots+b_nx_n$

We have the following integral, see Marichev et al. ([1], 3.42.3 page 634).

Lemme

$$\int_{x_1\geqslant 0}\cdots\int_{x_n\geqslant 0}\left[1-\left(\frac{x_1^2}{a_1^2}\right)-\cdots-\left(\frac{x_n^2}{a_n^2}\right)\right]^{\lambda-1}P_m^{(\alpha,\beta)}(\bar{bx})P_m^{(\beta,\alpha)}(\bar{bx})\quad \mathrm{d}x_1\cdots\mathrm{d}x_n$$

$$\frac{\sqrt{\pi^n}\Gamma(\lambda)}{\Gamma\left(\lambda+\frac{n}{2}\right)}\binom{m+\alpha}{m}\binom{m+\beta}{m}\prod_{i=1}^{n}a_i\,{}_5F_4\left(\begin{matrix}\text{-m},1,\alpha+\beta+m-1,\frac{\alpha+\beta+1}{2}+1,\frac{\alpha+\beta+3}{2},\lambda+\frac{n-1}{2}\\ \cdots\\ \alpha+1,\beta+1,\alpha+\beta+1,\lambda+\frac{n}{2}\end{matrix};1\right) \tag{3.1}$$

where $\left(\frac{x_1}{a_1}\right)+\cdots+\left(\frac{x_n}{a_n}\right)\leqslant 1, Re(\lambda)>0, Re(\alpha+1)>0, Re(\beta+1)>0, \sum_{l=1}^{n} a_l^2 b_l^2=1$

4. Main integral

Let $b_k=\frac{\prod_{j=1}^{p}(\lambda_j)_{k\rho_j}}{(a+k)^{\mathfrak{s}}\prod_{j=1}^{q}(\mu_j)_{k\sigma_j}}$ and $X=\left[1-\left(\frac{x_1^2}{a_1^2}\right)-\cdots-\left(\frac{x_n^2}{a_n^2}\right)\right]$

we have the following formula

Theorem

$$\int_{x_1\geqslant 0}\cdots\int_{x_n\geqslant 0}\left[1-\left(\frac{x_1^2}{a_1^2}\right)-\cdots-\left(\frac{x_n^2}{a_n^2}\right)\right]^{\lambda-1} P_m^{(\alpha,\beta)}(\bar{bx})P_m^{(\beta,\alpha)}(\bar{bx})\phi_{(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_q)}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(zX^a;\mathfrak{s},a)$$

$$S_{N_1,\cdots,N_t}^{M_1',\cdots,M_t'}\left(\begin{array}{c}\mathrm{y}_1X^{\gamma_1}\\ \cdot\cdot\cdot\\ \mathrm{y}_tX^{\gamma_t}\end{array}\right)\aleph_{u:w}^{0,N:v}\left(\begin{array}{c}\mathrm{z}_1X^{\alpha_1}\\ \cdot\cdot\cdot\\ \mathrm{z}_rX^{\alpha_r}\end{array}\right)I_{U:p_s,q_s;W}^{V;0,n_s;X}\left(\begin{array}{c}\mathrm{Z}_1X^{\eta_1}\\ \cdot\cdot\cdot\\ \mathrm{Z}_sX^{\eta_s}\end{array}\right)\mathrm{d}x\,\sqrt{\pi^n}\binom{m+\alpha}{m}\binom{m+\beta}{m}$$

$$\sum_{K_1=0}^{[N_1'/M_1']}\cdots\sum_{K_t=0}^{[N_t'/M_t']}\sum_{k=0}^{\infty}\sum_{p=0}^{m+1}\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{M_1}\cdots\sum_{g_r=0}^{M_r}\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}G(\eta_{G_1,g_1},\cdots\eta_{G_r,g_r})a_1$$

$$\frac{(-m)_p(\alpha+\beta+m-1)_p\left(\frac{\alpha+\beta+1}{2}\right)_p\left(\frac{\alpha+\beta+3}{2}\right)_p}{(\alpha+1)_p(\beta+1)_p(\alpha+\beta+1)_p p!}\frac{b_kz^k}{k!}\,z_1^{\eta_{G_1,g_1}}\cdots z_r^{\eta_{G_r,g_r}}y_1^{K_1}\cdots y_t^{K_t}$$

$$I_{U:p_s+2,q_s+2;W}^{V;0,n_s+2;X}\left(\begin{array}{c}\mathrm{Z}_1\\ \cdot\cdot\cdot\\ \ldots\\ \mathrm{Z}_s\end{array}\middle|\begin{array}{c}\mathrm{A}\ ;(\frac{3-n}{2}\text{-}(\lambda+ka+p+\sum_{i=1}^{t}K_i\gamma_i+\sum_{i=1}^{r}\eta_{G_i,g_i}\alpha_i);\eta_1,\cdots,\eta_s),\\ \cdot\cdot\cdot\\ \cdot\cdot\cdot\\ \mathrm{B};\ \ (\frac{3-n}{2}\text{-}(\lambda+ka+\sum_{i=1}^{t}K_i\gamma_i+\sum_{i=1}^{r}\eta_{G_i,g_i}\alpha_i);\eta_1,\cdots,\eta_s),,\end{array}\right.$$

$$\left.\begin{array}{c}(1\text{-}(\lambda+ka+\sum_{i=1}^{t}K_i\gamma_i+\sum_{i=1}^{r}\eta_{G_i,g_i}\alpha_i);\eta_1,\cdots,\eta_s),\mathfrak{A};A'\\ \cdot\cdot\cdot\\ \cdot\cdot\cdot\\ (1\text{-}\frac{n}{2}\text{-}(\lambda+ka+p+\sum_{i=1}^{t}K_i\gamma_i+\sum_{i=1}^{r}\eta_{G_i,g_i}\alpha_i);\eta_1,\cdots,\eta_s),\mathfrak{B};B'\end{array}\right) \tag{4.1}$$

Provided that

a) $min\{\gamma_i,\alpha_j,\eta_k\}>0, i=1,\cdots,t, j=1,\cdots,r, k=1,\cdots,s$

b)$Re(\lambda+ka)+\sum_{i=1}^{r}\alpha_i \min_{1\leqslant j\leqslant M_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)+\sum_{i=1}^{s}\eta_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)>0$

c) $|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.5) ; $i=1,\cdots,r$

d) $|argZ_k|<\frac{1}{2}\Omega_i^{(k)}\pi$, where $\Omega_i^{(k)}$ is defined by (1.11) ; $i=1,\cdots,s$

e) The multiple serie occuring on the right-hand side of (3.1) is absolutely and uniformly convergent.

f) The conditions (f) are satisfied

g) $\left(\frac{x_1}{a_1}\right)+\cdots+\left(\frac{x_n}{a_n}\right)\leqslant 1, Re(\alpha+1)>0, Re(\beta+1)>0, \sum_{l=1}^{n}a_l^2b_l^2=1$

First, expressing the extension of the Hurwitz-Lerch Zeta function $\phi_{(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_q)}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(zx^a;\mathfrak{s},a)$ in serie with the help of equation (2.1), the Aleph-function of r variables in series with the help of equation (1.6), the general class of polynomial of several variables $S_{N_1,\cdots,N_t}^{M_1',\cdots,M_t'}$with the help of equation (1.22) and the Prasad's multivariable I-function of s variables in Mellin-Barnes contour integrals with the help of equation (1.9), changing the order of integrations and summations (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process) and then evaluating the resulting multiple integrals with the help of equation (3.1) and expressing the generalized hypergeometric function ${}_5F_4$ in serie. Use several times the following relations $\Gamma(a)(a)_n=\Gamma(a+n)$ and $a=\frac{\Gamma(a+1)}{\Gamma(a)}$ with $Re(a)>0$. Finally interpreting the result thus obtained in Mellin-barnes contour integrals, we arrive at the desired result.

The quantities $U,V,W,X,A\prime,B,\mathfrak{A},\mathfrak{B},A'$ and B' are defined by the equations (1.14) to (1;19) respectively.

5. Particular case

If $U=V=A=B=0$, the multivariable I-function defined by Prasad [2] degenere in multivariable H-function defined by Srivastava Panda [5]. We have the following result.

Corollary

$$\int_{x_1\geqslant 0}\cdots\int_{x_n\geqslant 0}\left[1-\left(\frac{x_1^2}{a_1^2}\right)-\cdots-\left(\frac{x_n^2}{a_n^2}\right)\right]^{\lambda-1}P_m^{(\alpha,\beta)}(\bar{bx})P_m^{(\beta,\alpha)}(\bar{bx})\phi_{(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_q)}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(zX^a;\mathfrak{s},a)$$

$$S_{N_1,\cdots,N_t}^{M_1',\cdots,M_t'}\begin{pmatrix}\mathrm{y}_1X^{\gamma_1}\\ \cdot\ \cdot\ \cdot\\ \mathrm{y}_tX^{\gamma_t}\end{pmatrix}\aleph_{u:w}^{0,N:v}\begin{pmatrix}\mathrm{z}_1X^{\alpha_1}\\ \cdot\ \cdot\ \cdot\\ \mathrm{z}_rX^{\alpha_r}\end{pmatrix}H_{p_s,q_s;W}^{0,n_s;X}\begin{pmatrix}\mathrm{Z}_1X^{\eta_1}\\ \cdot\ \cdot\ \cdot\\ \mathrm{Z}_sX^{\eta_s}\end{pmatrix}\mathrm{d}x\,\sqrt{\pi^n}\binom{m+\alpha}{m}\binom{m+\beta}{m}$$

$$\sum_{K_1=0}^{[N_1'/M_1']}\cdots\sum_{K_t=0}^{[N_t'/M_t']}\sum_{k=0}^{\infty}\sum_{p=0}^{m+1}\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{M_1}\cdots\sum_{g_r=0}^{M_r}\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}G(\eta_{G_1,g_1},\cdots\eta_{G_r,g_r})a_1$$

$$\frac{(-m)_p(\alpha+\beta+m-1)_p\left(\frac{\alpha+\beta+1}{2}\right)_p\left(\frac{\alpha+\beta+3}{2}\right)_p}{(\alpha+1)_p(\beta+1)_p(\alpha+\beta+1)_p p!}\frac{b_k z^k}{k!}\ z_1^{\eta_{G_1,g_1}}\cdots z_r^{\eta_{G_r,g_r}}y_1^{K_1}\cdots y_t^{K_t}$$

$$H^{0,n_s+2;X}_{p_s+2,q_s+2;W}\left(\begin{array}{c|c} Z_1 & (\frac{3-n}{2}\text{-}(\lambda+ka+p+\sum_{i=1}^t K_i\gamma_i+\sum_{i=1}^r \eta_{G_i,g_i}\alpha_i);\eta_1,\cdots,\eta_s), \\ \cdot\cdot\cdot & \cdot\cdot\cdot \\ \cdots & \cdot\cdot\cdot \\ Z_s & (\frac{3-n}{2}\text{-}(\lambda+ka+\sum_{i=1}^t K_i\gamma_i+\sum_{i=1}^r \eta_{G_i,g_i}\alpha_i);\eta_1,\cdots,\eta_s),, \end{array}\right.$$

$$\left.\begin{array}{c} (1\text{-}(\lambda+ka+\sum_{i=1}^t K_i\gamma_i+\sum_{i=1}^r \eta_{G_i,g_i}\alpha_i);\eta_1,\cdots,\eta_s),\mathfrak{A};A' \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ (1\text{-}\frac{n}{2}\text{-}(\lambda+ka+p+\sum_{i=1}^t K_i\gamma_i+\sum_{i=1}^r \eta_{G_i,g_i}\alpha_i);\eta_1,\cdots,\eta_s),\mathfrak{B};B' \end{array}\right) \quad (5.1)$$

6. Conclusion

In this paper we have evaluated a multiple integrals involving the multivariable Aleph-function, multivariable I-function defined by Prasad [2], class of polynomials of several variables, product of two Jacobi polynomials and the extension of the Hurwitz-Lerch Zeta function. The multiple integrals established in this paper is of very general character as it contain multivariable Aleph-function, which is a general function of several variables studied so far. Thus, the multiple integrals established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

REFERENCES

[1]Marichev O.I. Prudnikov A.P. And Brychkow Y.A. Elementay functions. Integrals and series Vol 2. USSR Academy of sciences . Moscow 1986.

[2] Y.N. Prasad , Multivariable I-function , Vijnana Parishad Anusandhan Patrika 29 (1986) , page 231-237.

[3] Sharma C.K.and Ahmad S.S.: On the multivariable I-function. Acta ciencia Indica Math , 1994 vol 20,no2, p 113-116.

[4] Srivastava H.M. A multilinear generating function for the Konhauser set of biorthogonal polynomials suggested by Laguerre polynomial, Pacific. J. Math. 177(1985), page183-191.

[5] H.M. Srivastava And R.Panda. Some expansion theorems and generating relations for the H-function of several complex variables. Comment. Math. Univ. St. Paul. 24(1975), p.119-137.

[6] H.M. Srivastava, R.K. Saxena, T.K. Pogány, R. Saxena, Integral and computational representations of the extended Hurwitz–Lerch zeta function, Integr.Transf. Spec. Funct. 22 (2011) 487–506