Unsteady MHD Flow and Heat Transfer of Third-Grade Fluid with Variable Viscosity Between Two Porous Plates

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Abstract: Magneto-hydro-dynamic unsteady flow and heat transfer of a third grade fluid between two porous plates with variable viscosity has been investigated. The coupled non-linear partial differential equations governing the flow and heat transfer are reduced to a system of nonlinear algebraic equations using implicit finite difference scheme, and then the system is solved by using damped-Newton method. The effects of various physical parameters on the velocity and temperature field have been studied. One of the important observations obtained is that both velocity and temperature fields increase with an increase in the value of the variable viscosity parameter. The uniqueness of the developed method is that, it is valid for all values of elastic parameters whereas the earlier approximation methods like perturbation and power series methods are valid only for small values of elastic parameters. Further, unlike these methods, presented method does not require a repeated derivation of solution of the governing equation for every change in the boundary conditions.

Keywords: *porous plates, third grade fluid, variable viscosity, Damped-Newton method, implicit finite difference scheme.*

I. INTRODUCTION

The flow characteristics of some of the fluids like hydro-carbon oils, polyglycols, synthetic esters, poly-phenyl ethers, oil and greases, clay coating and suspensions, used in industrial and engineering processes could not be adequately explained on the basis of Newtonian model. To characterize such non-Newtonian flow behaviours several constitutive fluid models are proposed. One such model is the class of third order fluids. Several researchers like Hayat et.al.[1], Erdogan[2], Hayal et.al.[3], Fosdick and Rajagopal [4], Hayat et.al.[5], Kaloni and Siddique[6], Ariel[7] and Nayak et.al.[8] investigated the flow and heat transfer of third grade fluids. Okoya[9] studied the thermal transition of a reactive flow of a third grade fluid with viscous heating and chemical reaction between two horizontal flat plates. In that study, the top plate is moving with a uniform speed and the bottom plate is fixed in the presence of an imposed pressure gradient. He obtained an approximation solution for the flow velocity using Homotopy-Perturbation technique. Sahoo and Do[10] have studied the flow and heat transfer of an electrically conducting non-Newtonian fluid due to a stretched surface subject to partial slip. Sajid et. al.[11] have studied the flow of a third grade fluid past a horizontal porous plate with partial slip and solved the non-linear problem using a finite element method. Recently Makinde and Chinyoka[12] have studied unsteady hydro-magnetic generalized couette flow and heat transfer characteristics of a reactive variable viscosity, incompressible, and electrically conducting third grade fluid in a channel with asymmetric convective cooling at the walls in the presence of uniform transverse magnetic field. He has used the semi-implicit finite difference scheme to solve the coupled non-linear partial differential equations.

The objective of our present work is to study an incompressible electrically conducting, variable viscosity third grade fluid placed between two parallel porous plates in presence of an externally applied homogenous magnetic field, where the lower plate suddenly starts moving with a time varying velocity U. The plate surfaces are subjected to unequal convective heat exchange with the ambient. Approximate solution is obtained using implicit finite difference scheme.

Rest of the paper is organized as follows. In section 2, the mathematical formulation of the problem is obtained followed by the derivation of the solution of the equations using finite difference method in section 3. In section 4, the effects of various parameters on velocity and temperature fields in form of graphs are discussed. The conclusion is in section 5.

II. PROBLEM FORMULATION (SIZE 10 & BOLD)

Consider the transient flow of an incompressible electrically conducting, variable viscosity third grade fluid placed between two parallel porous plates in the presence of an externally applied homogeneous magnetic field as shown in Fig. 1.

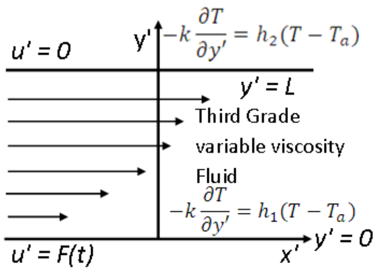

Figure 1: Geometry of the problem.

The x′ axis is taken along the lower wall, y′ axis is taken normal to it and the upper plane is specified by the equation y′ = L, where the number L will be specified later. It is also supposed that the walls extend to infinity in both sides of the x′ axis. The suction or injection is applied at the walls with a constant velocity V. The velocity components u′ and v′ at any point (x′, y′) in the flow field compatible with the equations of continuity can be given as

$$u' = u'(u', t'), \quad v' = V, \tag{1}$$

where u′ and v′ are the velocities of the fluid along x′ and y′ axes respectively and V > 0 indicates the injective velocity.

The constitutive equation of an incompressible third grade fluid considered here, as given by Coleman, and Noll[13], is

$$P = -pI + \sum_{i=1}^{3} S_i \tag{2}$$

Where

$S_1 = \mu A_1,\ S_2 = \alpha_1 A_2 + \alpha_2 A_1^2, and\ S_3 = \beta_1 A_3 + \beta_2(A_2A_1 + A_1A_2) + \beta_3(trA_2)\,A_1.$

I the identity tensor and A_n represents the kinematical tensors defined by

$$A_1 = \nabla u + (\nabla u)^T, A_{n+1} = \left(\frac{\partial}{\partial t} + u.\nabla\right)A_n + \nabla u.A_n + (\nabla u.A_n)^T,\ n=1,2.$$

Using the stress components obtained from (2) and velocity given by (1), with temperature dependent viscosity, the governing equation of motion is obtained as

$$\rho\left(\frac{\partial u'}{\partial t'} + V\frac{\partial u'}{\partial y'}\right) = \frac{\partial}{\partial y'}\left(\mu'(T)\frac{\partial u'}{\partial y'}\right) + \alpha_1\frac{\partial^3 u'}{\partial y'^2\partial t'} + 6\beta_3\left(\frac{\partial u'}{\partial y'}\right)^2\frac{\partial^2 u'}{\partial y'^2} + \alpha_1 V\frac{\partial^3 u'}{\partial y'^3} - \sigma B_0 u'^2 \tag{3}$$

and heat balance equation is obtained as

$$\rho c_p\left(\frac{\partial T}{\partial t'} + V\frac{\partial T}{\partial y'}\right) = k\frac{\partial^2 T}{\partial y'^2} + \mu'\left(\frac{\partial u'}{\partial y'}\right)^2 + 2\beta_3\left(\frac{\partial u'}{\partial y'}\right)^4 + \sigma B_0 u'^2 \tag{4}$$

The initial and boundary conditions are

$$\begin{array}{ll} t' = 0:\ u' = 0, \quad \forall y' & T(y', 0) = T_0 \ \ for\ t = 0 \\ t' > 0:\ u' = F(t')\ for\ \ y' = 0 & -k\frac{\partial T}{\partial y'}(0, t') = h_1[T(0, t') - T_a] \ \ for\ t' > 0 \\ u' = 0 \quad for\ y' = L & -k\frac{\partial T}{\partial y'}(a, t') = h_2[T(a, t') - T_a] \ \ for\ t' > 0 \end{array} \tag{5}$$

Where $F(t') = U$.

Here T is the temperature, T_a is the ambient temperature, T_0 is the fluid initial temperature, t' is the time, ρ is the density, σ is the fluid electrical conductivity, B_0 is the electro-magnetic induction, k is the thermal conductivity coefficient, C_p is the specific heat at constant pressure, h_1 and h_2 are heat transfer coefficient, and a is the channel width. The temperature dependent viscosity (μ′) can be expressed as $\mu'(T) = \mu_o e^{-b(T-T0)}$, where b is a viscosity variation parameter and μ_o is the initial fluid dynamic viscosity at temperature T_0. We introduce the following dimensionless variables and parameters.

$$y = \frac{y'}{a}\,,\ t = \frac{t'\mu_0}{\rho a^2}\ ,\ u = \frac{u'\rho a}{\mu_0}\ ,\ \theta = \frac{E(T-T_0)}{RT_0^2}\ ,\mu = \frac{\mu'}{\mu_0}$$
$$R_e = \frac{\rho V a}{\mu_0}\,,\ \alpha = \frac{\alpha_1}{\rho a^2}\ ,\ \gamma = \frac{\beta_3 \mu_0}{\rho^2 a^4}$$
$$p_r = \frac{\mu_0 c_p}{k}\ ,\ \epsilon = \frac{RT_0}{E},\ \Omega = \frac{\mu_0^3}{\rho^2 a^2 K T_0}\,,\ m^2 = \frac{\mu_0^3}{\rho^2 a^2 K T_0},\ \delta = \frac{bRT_0^2}{E}$$

Using the above notations, the equation of the motion (3) can be converted into following non-dimensional form

$$\left(\frac{\partial u}{\partial t} + R_e \frac{\partial u}{\partial y}\right) = e^{-\delta\theta}\frac{\partial^2 u}{\partial y^2} - \delta e^{-\delta\theta}\frac{\partial \theta}{\partial y}\frac{\partial u}{\partial y} + \alpha\frac{\partial^3 u}{\partial y^2 \partial t} + 6\gamma\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2} + R_e\alpha\frac{\partial^3 u}{\partial y^3} - m^2 u \qquad (6)$$

and the non-dimensional form of heat balancing equation (4) is

$$\left(p_r\frac{\partial \theta}{\partial t} + p_r R_e\frac{\partial \theta}{\partial y}\right) = \frac{\partial^2 \theta}{\partial y^2} + \frac{\Omega}{\epsilon}\left[m^2 u + \left(\frac{\partial u}{\partial y}\right)^2\left(e^{-\delta\theta} + 2\gamma\left(\frac{\partial u}{\partial y}\right)^2\right)\right] \qquad (7)$$

The initial and boundary conditions (5) reduce to

$$\begin{aligned}
&t = 0 : u = 0\,, \quad \forall y\\
&t > 0 : u = 1,\ for\ y = 0\\
&u = 0, \quad for\ y = 1\\
&t = 0 :\ \theta(y,0) = 0, \quad \forall\ y\\
&t > 0 : \frac{\partial\theta}{\partial y}(0,t) = -B_{i,1}[\theta(0,t) - \theta_a], \quad for\ y = 0\\
&\frac{\partial\theta}{\partial y}(1,t) = -B_{i,2}[\theta(0,t) - \theta_a], \quad for\ y = 1
\end{aligned} \qquad (8)$$

where $R_e,\ P_r,\ Bi_i, \epsilon, \gamma, \delta, \theta_a, \Omega, and\ m$ respectively represent the Reynolds number, Prandtl number, Biot numbers, activation energy parameter, material parameter(second order), non-Newtonian parameter, variable viscosity parameter, viscous heating parameter, the ambient temperature parameter and Hartmann number.

III. Solution of the Equations

In the following subsection, we discuss the process of reducing the partial differential equation (6) to a system of nonlinear equations using finite difference scheme and to find the solution of this system using Damped-Newton method [14].

A. The Finite Difference Scheme

For solving equation (6) we use an implicit finite difference method of Crank-Nicolson type. For the discretization in space and time, a uniform mesh of step h and time step k are employed so that the grid points are $(ih, j\Delta t)$, $i = 0, 1, \ldots, N+1$, $j = 0, 1, \ldots, M$ where $t_m = \Delta t M$ is the time up to which velocity is computed. The difference scheme is obtained by replacing the derivatives of u by the following difference approximations at the nodes $(ih, j\Delta t)$ using the following notations

$$\begin{aligned}
d_{1i}^j &= u_{i+1}^j - u_{i-1}^j\\
d_{2i}^j &= u_{i+1}^j - 2u_i^j + u_{i-1}^j\\
d_{3i}^j &= -u_{i-2}^j + 2u_{i-1}^j - 2u_{i+1}^j + u_{i-1}^j\\
d_{3i}^{'j} &= -3u_{i-1}^j + 10u_i^j - 12u_{i+1}^j + 6u_{i+2}^j - u_{i+3}^j\\
d_{3i}^{''j} &= u_{i-3}^j - 6u_{i-2}^j + 12u_{i-1}^j - 10u_i^j + 3u_{i+1}^j\\
e_{1i}^j &= \theta_{i+1}^j - \theta_{i-1}^j\\
e_{2i}^j &= \theta_{i+1}^j - 2\theta_i^j + \theta_{i-1}^j
\end{aligned}$$

The difference approximation to the derivatives at nodes $(ih\,, j\Delta t)$, $i = 1,2,\ldots,N+1\ and\ j = 0,1,\ldots,M+1$ are taken as

$$\frac{\partial u}{\partial t} \approx \frac{u_i^{j+1} - u_i^j}{\Delta t}$$
$$\frac{\partial u}{\partial y} \approx \frac{d_{1i}^{j+1} + d_{1i}^j}{4h}$$

$$\frac{\partial^2 u}{\partial y^2} \approx \frac{d_{2i}^{j+1}+d_{2i}^{j}}{2h^2} \tag{9}$$
$$\frac{\partial^3 u}{\partial y^3} \approx \frac{d_{3i}^{j+1}+d_{3i}^{j}}{2h^3},\ i \neq 1, N$$
$$\frac{\partial^3 u}{\partial y^2 \partial t} \approx \frac{d_{2i}^{j+1}-d_{2i}^{j}}{h^2 \Delta t}$$
$$\frac{\partial \theta}{\partial y} \approx \frac{e_{1i}^{j+1}+e_{1i}^{j}}{4h}$$
$$\frac{\partial^2 \theta}{\partial y^2} \approx \frac{e_{2i}^{j+1}+e_{2i}^{j}}{2h^2}$$

At the interior point *(1, jΔt)* and *(N, Δt)* the third grade derivative $\frac{\partial^3 u}{\partial y^3}$ is replaced respectively by $\frac{d_{3i}^{'j+1}+d_{3i}^{'j}}{2h^3}$ and $\frac{d_{3i}^{''j+1}+d_{3i}^{''j}}{2h^3}$. Using the above expressions the equations of velocity and energy with prescribed initial and boundary conditions are written in discretized form. With the implementation of above said notations in (9), the governing equation of velocity is discretized as

$$\frac{u_i^{j+1}-u_i^j}{\Delta t} + R_e \frac{d_{1i}^{j+1}+d_{1i}^{j}}{4h} = \exp\left(-\delta \frac{\theta_i^{j+1}+\theta_i^j}{2}\right)\left(\frac{d_{2i}^{j+1}+d_{2i}^{j}}{2h^2}\right) - \frac{\delta}{16h^2}\exp\left(-\delta \frac{\theta_i^{j+1}+\theta_i^j}{2}\right)(d_{1i}^{j+1}+d_{1i}^{j})(e_{1i}^{J+1}+e_{1i}^{j}) + \alpha \frac{d_{2i}^{j+1}-d_{2i}^{j}}{h^2\Delta t} + R_e \alpha \frac{d_{3i}^{j+1}+d_{3i}^{j}}{2h^3} + \frac{6\gamma}{32h^4}(d_{1i}^{j+1}+d_{1i}^{j})^2(d_{2i}^{j+1}+d_{2i}^{j}) - m^2 \frac{u_i^{j+1}+u_i^j}{2} \tag{10}$$

For *i=2, 3, ..., N-1, j = 0, 1, 2, ..., M.* With the initial and boundary conditions is discretized form as follows:

$$u_i^0 = 0, i = 0,1,\dots,N+1$$
$$u_0^j = 1 \quad \text{and} \tag{11}$$
$$u_{N+1}^j = 0,\ j = 1,\dots,M$$

The energy equation in (7) discretized form is written as

$$P_r \frac{\theta_i^{j+1}-\theta_i^j}{\Delta t} + \frac{R_e P_r}{4h}\left[(e_{1i}^{j+1}+e_{1i}^{j})\right] = \frac{1}{2h^2}\left[(e_{2i}^{j+1}+e_{2i}^{j})\right] + \frac{\Omega}{\epsilon}\left[m^2\left(u_i^j\right)^2 + \frac{1}{4h^2}\left(d_{1i}^{j+1}+d_{1i}^{j}\right)^2\left(e^{-\delta\theta_i^j} + \frac{2\gamma}{4h^2}\left(d_{1i}^{j+1}+d_{1i}^{j}\right)\right)\right] \tag{12}$$

The discretized form of initial conditions is $\theta_i^0 = 0,1,\dots,N+1$ and the discretized form of boundary conditions by taking 2nd order approximation of forward difference for first derivative are

$$\theta_0^j = \frac{2hB_{i1}\theta_a - 4\theta_1^j + \theta_2^j}{2B_{i1}h-3} \ \ and\ \theta_N^j = \frac{2hB_{i2}\theta_a + 4\theta_{N-1}^j - \theta_{N-2}^j}{2B_{i2}h+3}$$

Rearrangement of equation (12) leads to the tri-diagonal form

$$\left(-\frac{R_eP_r\Delta t}{4h} - \frac{\Delta t}{2h^2}\right)\theta_{i-1}^{j+1} + \left(P_r + \frac{\Delta t}{h^2}\right)\theta_i^{j+1} + \left(\frac{R_eP_r\Delta t}{4h} - \frac{\Delta t}{2h^2}\right)\theta_{i+1}^{j+1} = \left(\frac{R_eP_r\Delta t}{4h} + \frac{\Delta t}{2h^2}\right)\theta_{i-1}^{j} + \left(P_r - \frac{\Delta t}{h^2}\right)\theta_i^j + \left(-\frac{R_eP_r\Delta t}{4h} + \frac{\Delta t}{2h^2}\right)\theta_{i+1}^{j} + \frac{\Omega}{\epsilon}\left[m^2\left(u_i^j\right)^2 + \frac{1}{4h^2}\left(d_{1i}^{j+1}+d_{1i}^{j}\right)^2\left(e^{-\delta\theta_i^j} + \frac{2\gamma}{4h^2}\left(d_{1i}^{j+1}+d_{1i}^{j}\right)\right)\right] \tag{13}$$

Where

$$a_i = -\frac{R_eP_r\Delta t}{4h} - \frac{\Delta t}{2h^2}$$
$$b_i = P_r + \frac{\Delta t}{h^2}$$
$$c_i = \frac{R_eP_r\Delta t}{4h} - \frac{\Delta t}{2h^2}$$
$$d_i = A + \left(\frac{R_eP_r\Delta t}{4h} + \frac{\Delta t}{2h^2}\right)\theta_{i-1}^{j} + \left(P_r - \frac{\Delta t}{h^2}\right)\theta_i^j + \left(-\frac{R_eP_r\Delta t}{4h} + \frac{\Delta t}{2h^2}\right)\theta_{i+1}^{j}$$
$$A = \frac{\Omega}{\epsilon}\left[m^2\left(u_i^j\right)^2 + \frac{1}{4h^2}\left(d_{1i}^{j+1}+d_{1i}^{j}\right)^2\left(e^{-\delta\theta_i^j} + \frac{2\gamma}{4h^2}\left(d_{1i}^{j+1}+d_{1i}^{j}\right)\right)\right]$$

B. Solution of the System of non-linear Equations

The system of non-linear equations (9) with initial and boundary conditions (10) is solved by Damped Newton method described in [14]. For the implementation of this method the residuals *(R_i; i = 0, 1, 2, . . ., N)* and non-zero elements of the Jacobian matrix $\left(\frac{\partial R_i}{\partial R_j}\right)$, *i = 1, 2, . . ., N* and *j = 1, 2, . . ., M* of the system are computed as described in appendix A and B.

The following procedures are designed to solve the problem according to the schemes discussed above.

i) The main procedure

Input: $\Delta t, h, \delta, \alpha, \gamma, P_r, R_e, m, \Omega, \theta_a, B_{i1}, B_{i2}, \epsilon$

1. Set t=0.
2. Set the velocities U_i^0 $at\ (t, y_i)\ using\ eq.11$
3. Set the temperature T_i^0 $at\ (t, y_i)\ using\ eq.12.$
4. Choose the initial approximation to the velocities U_i^1 $at\ (t + \Delta t, y_i\ )$ by solving the tridiagonal system given in eq.10 with $\delta = \alpha = \gamma = m = 0.$
5. Compute the temperatures T_i^1 $at\ (t + \Delta t, y_i\ )\ by\ solving\ the\ tridiagonal\ system(given\ in\ eq.13).$
6. Compute the velocities U_i^1 $at\ (t + \Delta t, y_i\ )$ using Damped-Newton method on eq.10.
7. Set $U_i^0 = U_i^1,\ T_i^0 = T_i^1$.
8. Set t= $t + \Delta t$.
9. Repeat step-4 through step-9 until $t \leq M + \Delta t.$

   Output: U_i^1 and T_i^1 .

ii) Procedure for Damped-Newton method

To solve a system f(x)=0 of n equations in n unknowns, with f a vector valued function having smooth component functions, use the following steps[14]

1. Take a first guess x(0) for a solution x of the system and choose an error tolerance ϵ.
2. Set vector R=Residue(x) (given in appendix (B1-B4)).
3. Set matrix J=Jacobian(x) (given in appendix).
4. Solve the system of linear equations formed by Jh=-R using Gauss-Seidel method.
5. Find $i = \min\left(j : j : 0 \leq j \leq j_{max}, \left| \left\| Residue\left(x + \frac{h}{2^j}\right) \right\|_2 < \| Residue(x) \|_2 \right)$,

   If i not found then report failure and exit.
6. Update x at $(k+1)^{th}$ iteration as $x^{(k+1)} = x^{(k)} + \frac{h}{2^j}$.
7. Repeat step-1 through step-5 till $\left\| \frac{h}{2^j} \right\|_2 > \epsilon$.

IV. Results and Discussions

Fig.2 and Fig.3 show the effect of Reynolds number Re on temperature and velocity respectively. It is observed that when Reynolds number increases both temperature and velocity increase.

Fig.4 and Fig. 5 show the effect of Prandtl number P_r. It is observed from Fig. 4 that large values of the Prandtl number decreases the strength of the source terms in the temperature equation and hence it decreases the overall temperature, whereas from Fig.5, it is observed that the effect of P_r is insignificant.

Fig. 6 and Fig. 7 show the effect of variable viscosity parameter δ. Increase of the values of the parameter δ reduces the fluid viscosity and hence correspondingly diminishes the fluid resistance to the flow and hence fluid velocity increases. This increased velocity in turn increases the viscous heating source terms in the temperature equation and hence it increases the fluid temperature

Fig. 8 and Fig. 9 show the effect of activation energy (∈) on temperature and velocity. It is observed from Fig. 8 that the fluid temperature increases as (∈) decreases. From Fig. 9 it is observed that velocity increases with increase of the value of (∈), but this effect on velocity appears marginal and are not very significant.

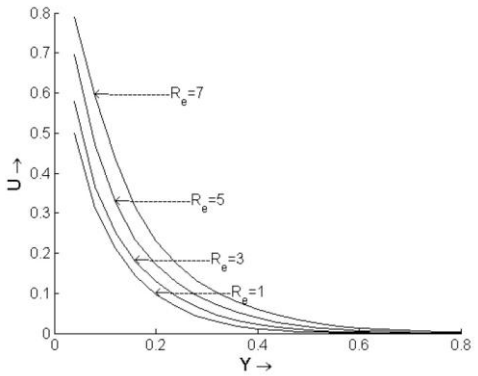

Fig.2 Effect of Reynolds Number (R_e) on Velocity field.($P_r = 8$, $\theta_a = 0.1$, $b_{i1} = 0.1$, $b_{i2} = 1$, $m = 1$, $\varepsilon = 0.1$, $\Omega = 0.1$, $\delta = 0.5$, $\alpha = 0.001$, $\gamma = 0.0001$)

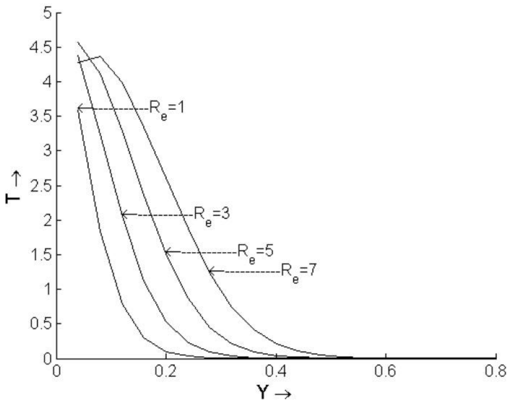

Fig.3 Effect of Reynolds Number (R_e) on Temp. field.($P_r = 8$, $\theta_a = 0.1$, $b_{i1} = 0.1$, $b_{i2} = 1$, $m = 5$, $\varepsilon = 0.1$, $\Omega = 0.1$, $\delta = 0.5$, $\alpha = 0.001$, $\gamma = 0.01$)

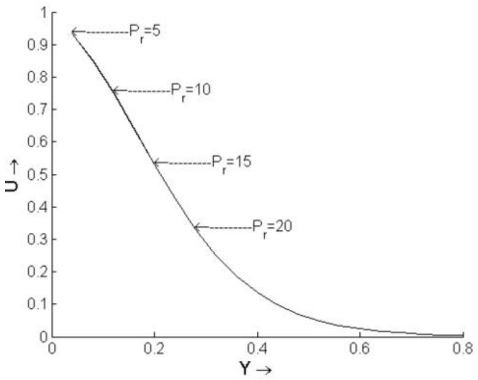

Fig.4 Effect of Prandtl Number (P_r) on Velocity field.($R_e = 5$, $\theta_a = 0.1$, $b_{i1} = 0.1$, $b_{i2} = 1$, $m = 5$, $\varepsilon = 0.1$, $\Omega = 0.1$, $\delta = 0.5$, $\alpha = 0.001$, $\gamma = 0.01$)

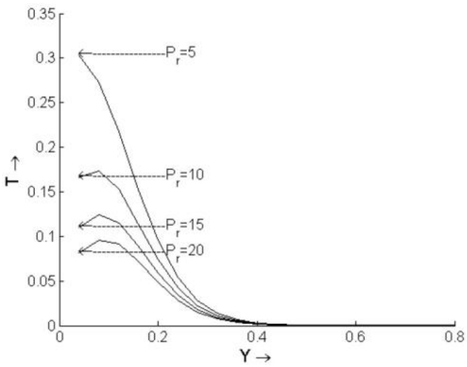

Fig.5 Effect of Prandtl Number (P_r) on Temp. field.($R_e = 5$, $\theta_a = 0.1$, $b_{i1} = 0.1$, $b_{i2} = 1$, $m = 5$, $\varepsilon = 0.1$, $\Omega = 0.1$, $\delta = 0.5$, $\alpha = 0.001$, $\gamma = 0.01$)

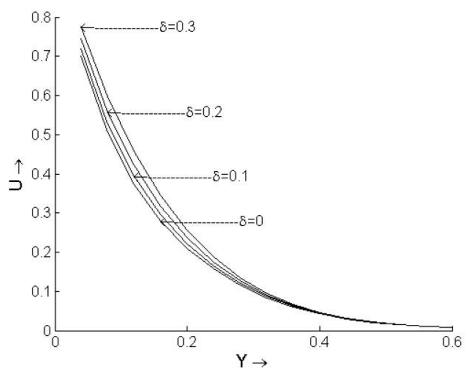

Fig.6 Effect of variable viscosity parameter (δ) on Velocity field. ($R_e = 5$, $\theta_a = 0.1$, $b_{i1} = 0.1$, $b_{i2} = 1$, $m = 5$, $\varepsilon = 0.1$, $\Omega = 0.1$, $\delta = 0.5$, $\alpha = 0.001$, $\gamma = 0.01$)

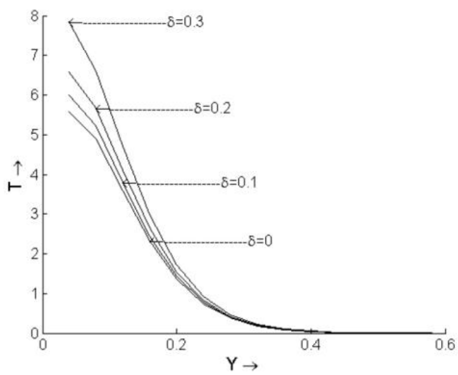

Fig.7 Effect of variable viscosity parameter (δ) on Temp. field. ($R_e = 5$, $\theta_a = 0.1$, $b_{i1} = 0.1$, $b_{i2} = 1$, $m = 5$, $\varepsilon = 0.1$, $\Omega = 0.1$, $\delta = 0.5$, $\alpha = 0.001$, $\gamma = 0.01$)

Fig. 10 and Fig.11 show the effect of viscous heated parameter Ω on the velocity and temperature profile. Increased values of Ω lead to significant increase in viscous heating sources terms and significantly increases the fluid temperature. But velocity decreases when the values of Ω are increasing.

Fig. 12 and Fig. 13 shows the effect of Hartmann number m on velocity and temperature. We observe from Fig. 12 that increase in m leads to the decrease of velocity and increase in damping magnetic properties of the fluid. These forces result in increased resistance to the flow and reduce the fluid velocity. As the Hartmann number m appears as a strong source term in temperature equation, an increase in m increases the fluid temperature.

From Fig. 14 and Fig. 15 we observe that both velocity and temperature decrease with increase of second order elastic parameter α.

Fig. 16 and Fig. 17 show the effect of third order elastic parameter (γ) on velocity and temperature fields. We observed that when γ increases, visco-elasticity of the fluid increases, which reduces both velocity and temperature.

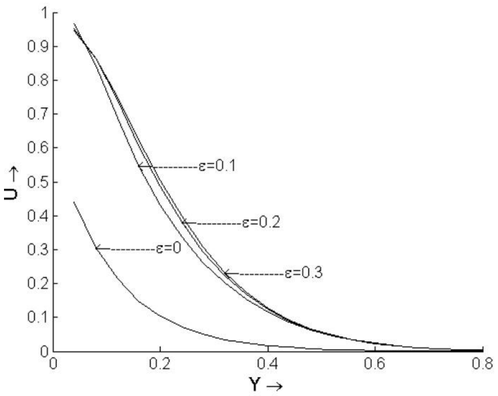

Fig.8 Effect of ε on Velocity field. (R_e = 5, θ_a= 0.1, b_{i1} = 0.1, b_{i2} = 1, m = 5, ε = 0.1, Ω = 0.1, δ = 0.5, α = 0.001, γ = 0.01)

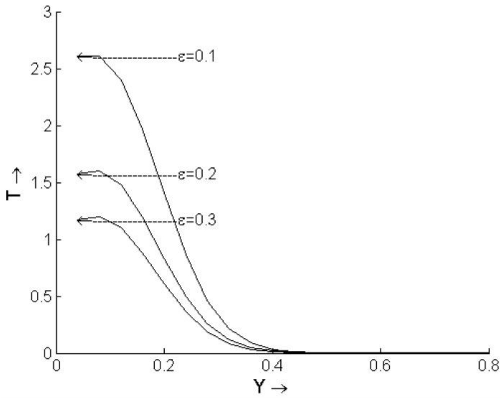

Fig.9 Effect of ε on Temp. field. (R_e = 5, θ_a= 0.1, b_{i1} = 0.1, b_{i2} = 1, m = 5, ε = 0.1, Ω = 0.1, δ = 0.5, α = 0.001, γ = 0.01)

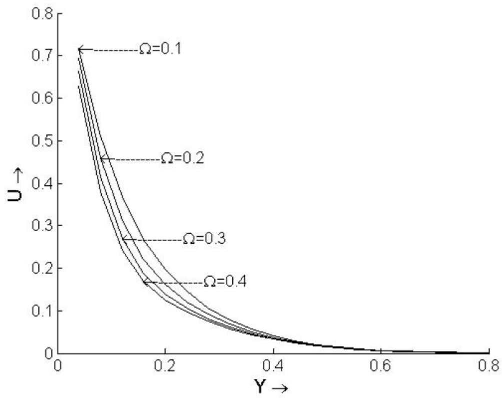

Fig.10 Effect of Ω on Velocity field. (R_e = 5, θ_a= 0.1, b_{i1} = 0.1, b_{i2} = 1, m = 5, ε = 0.1, Ω = 0.1, δ = 0.5, α = 0.001, γ = 0.01)

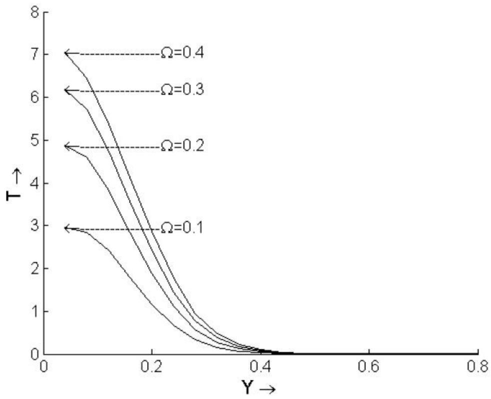

Fig.11 Effect of Ω on Temp. field. (R_e = 5, θ_a= 0.1, b_{i1} = 0.1, b_{i2} = 1, m = 5, ε = 0.1, Ω = 0.1, δ = 0.5, α = 0.001, γ = 0.01)

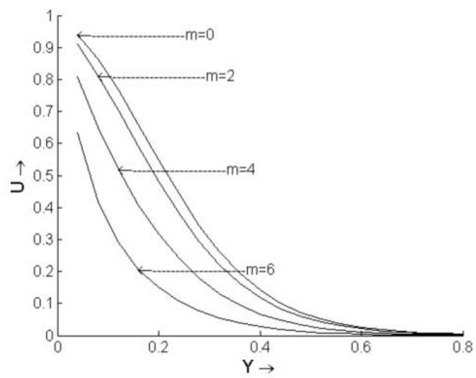

Fig.12 Effect of Hartmann number (m) on Velocity field. (R_e = 5, θ_a= 0.1, b_{i1} = 0.1, b_{i2} = 1, m = 5, ε = 0.1, Ω = 0.1, δ = 0.5, α = 0.001, γ = 0.01)

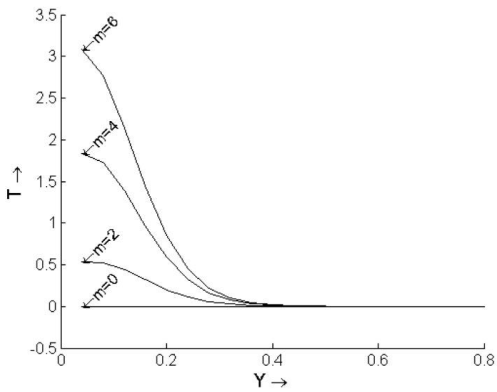

Fig.13 Effect of Hartmann number (m) on Temp. field. (R_e = 5, θ_a= 0.1, b_{i1} = 0.1, b_{i2} = 1, m = 5, ε = 0.1, Ω = 0.1, δ = 0.5, α = 0.001, γ = 0.01)

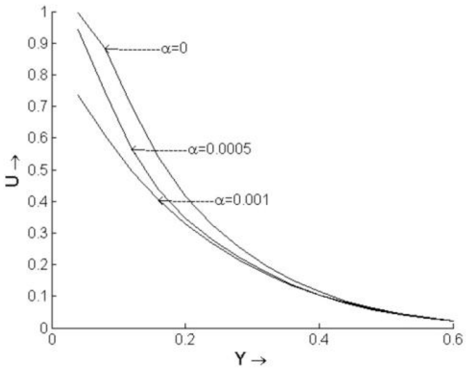

Fig.14 Effect of second order elastic parameter (α) on Velocity field. (R_e = 5, θ_a= 0.1, b_{i1} = 0.1, b_{i2} = 1, m = 5, ε = 0.1, Ω = 0.1, δ = 0.5, α = 0.001, γ = 0.01)

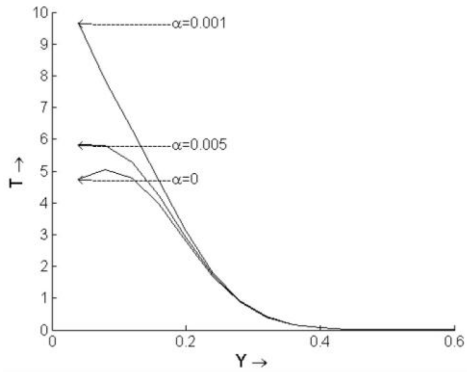

Fig.15 Effect of second order elastic parameter (α) on Temp. field. (R_e = 5, θ_a= 0.1, b_{i1} = 0.1, b_{i2} = 1, m = 5, ε = 0.1, Ω = 0.1, δ = 0.5, α = 0.001, γ = 0.01)

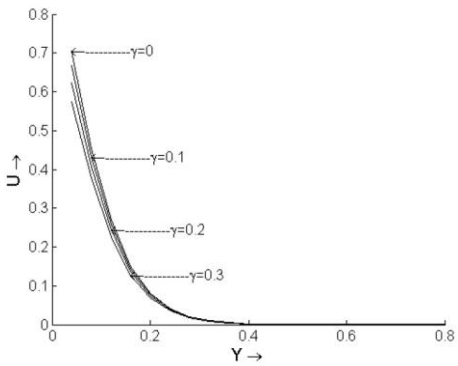

Fig.16 Effect of third order elastic parameter (γ) on Velocity field. (R_e = 5, θ_a= 0.1, b_{i1} = 0.1, b_{i2} = 1, m = 5, ε = 0.1, Ω = 0.1, δ = 0.5, α = 0.001, γ = 0.01)

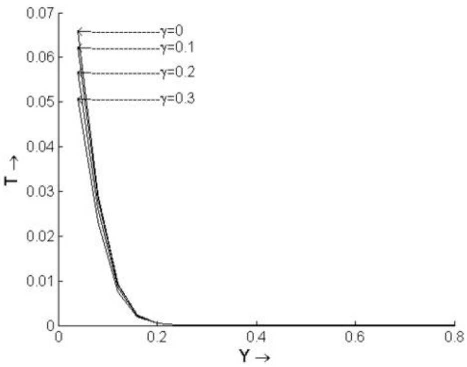

Fig.17 Effect of third order elastic parameter (γ) on Temp. field. (R_e = 5, θ_a= 0.1, b_{i1} = 0.1, b_{i2} = 1, m = 5, ε = 0.1, Ω = 0.1, δ = 0.5, α = 0.001, γ = 0.01)

V. CONCLUSION

In the present work we have considered the unsteady MHD flow and heat Transfer of a third-grade fluid with variable viscosity between two porous plates using an implicit finite difference method. A damped Newton method is used to solve the system of nonlinear equations obtained by discretization which is then implemented using MATLAB.

The discussed method is valid for all values of elastic parameters unlike perturbation method and power series method that are valid only for small values of elastic parameters.

One of the major observations in the present investigation is that the velocity decreases and temperature field increases with an increase in magnetic parameter value m irrespective of the presence of both elastic parameters. Both the velocity and temperature field decreases with increasing values of both second and third order elastic parameter values α and γ respectively, whereas both velocity and temperature field increase when the value of variable viscosity parameter δ increases.

C. Apendix

***Residuals*:**

$$R_1 \equiv \left(u_1^{j+1} - u_1^j\right) + \frac{R_e \Delta t}{4h}\left(d_{11}^{j+1} + d_{11}^j\right) - \frac{\Delta t}{2h^2} exp\left(-\delta \frac{\theta_1^{j+1} + \theta_1^j}{2}\right)\left(d_{21}^{j+1} - d_{21}^j\right)$$
$$+ \frac{\delta \Delta t}{16h^2} exp\left(-\delta \frac{\theta_1^{j+1} + \theta_1^j}{2}\right)\left(d_{11}^{j+1} + d_{11}^j\right)\left(e_{11}^{j+1} + e_{11}^j\right) - \frac{\alpha}{h^2}\left(d_{21}^{j+1} - d_{21}^j\right)$$
$$- \frac{6\gamma \Delta t}{32h^4}\left(d_{11}^{j+1} + d_{11}^j\right)^2\left(d_{21}^{j+1} + d_{21}^j\right) - \frac{\alpha R_e \Delta t}{2h^3}\left(d'^{\,j+1}_{31} + d'^{\,j}_{31}\right) + m^2\left(\frac{u_1^{j+1} + u_1^j}{2}\right)\Delta t$$

$$R_2 \equiv (u_2^{j+1} - u_2^j) + \frac{R_e \Delta t}{4h}(d_{12}^{j+1} + d_{12}^j) - \frac{\Delta t}{2h^2} exp\left(-\delta \frac{\theta_2^{j+1} + \theta_2^j}{2}\right)(d_{22}^{j+1} - d_{22}^j)$$
$$+ \frac{\delta \Delta t}{16h^2} exp\left(-\delta \frac{\theta_2^{j+1} + \theta_2^j}{2}\right)(d_{12}^{j+1} + d_{12}^j)(e_{12}^{j+1} + e_{12}^j) - \frac{\alpha}{h^2}(d_{22}^{j+1} - d_{22}^j)$$
$$- \frac{6\gamma \Delta t}{32h^4}(d_{12}^{j+1} + d_{12}^j)^2(d_{22}^{j+1} + d_{22}^j) - \frac{\alpha R_e \Delta t}{2h^3}(d'^{j+1}_{32} + d'^{j}_{32}) + m^2\left(\frac{u_2^{j+1} + u_2^j}{2}\right)\Delta t$$

$$R_i \equiv (u_i^{j+1} - u_i^j) + \frac{R_e \Delta t}{4h}(d_{1i}^{j+1} + d_{1i}^j) - \frac{\Delta t}{2h^2} exp\left(-\delta \frac{\theta_i^{j+1} + \theta_i^j}{2}\right)(d_{2i}^{j+1} - d_{2i}^j)$$
$$+ \frac{\delta \Delta t}{16h^2} exp\left(-\delta \frac{\theta_i^{j+1} + \theta_i^j}{2}\right)(d_{1i}^{j+1} + d_{1i}^j)(e_{1i}^{j+1} + e_{1i}^j) - \frac{\alpha}{h^2}(d_{2i}^{j+1} - d_{2i}^j)$$
$$- \frac{6\gamma \Delta t}{32h^4}(d_{2i}^{j+1} + d_{2i}^j) - \frac{\alpha R_e \Delta t}{2h^3}(d_{1i}^{j+1} + d_{1i}^j)^2(d'^{j+1}_{3i} + d'^{j}_{3i}) + m^2\left(\frac{u_i^{j+1} + u_i^j}{2}\right)\Delta t$$

$$R_N \equiv (u_N^{j+1} - u_N^j) + \frac{R_e \Delta t}{4h}(-u_{N-1}^{j+1} - u_{N-1}^j) - \frac{\Delta t}{2h^2} exp\left(-\delta \frac{\theta_N^{j+1} + \theta_N^j}{2}\right)(-2u_N^{j+1} + u_{N-1}^{j+1} - 2u_N^j + u_{N-1}^j)$$
$$+ \frac{\delta \Delta t}{16h^2} exp\left(-\delta \frac{\theta_N^{j+1} + \theta_N^j}{2}\right)(-u_{N-1}^j - u_{N-1}^j)(e_{N-1}^{j+1} + e_{N-1}^j)$$
$$- \frac{\alpha}{h^2}(-2u_N^{j+1} + u_{N-1}^{j+1} - 2u_N^j + u_{N-1}^j)$$
$$- \frac{6\gamma \Delta t}{32h^4}(-u_{N-1}^{j+1} - u_{N-1}^j)^2(-2u_N^{j+1} + u_{N-1}^{j+1} - 2u_N^j + u_{N-1}^j) - \frac{\alpha R_e \Delta t}{2h^3}(d''^{j+1}_{31} + d''^{j}_{31})$$
$$+ m^2\left(\frac{u_N^{j+1} + u_N^j}{2}\right)\Delta t$$

Jacobians:

$$J(1,1) = \frac{\partial R_1}{\partial u_1^{j+1}} = 1 + \frac{\Delta t}{h^2} exp\left(-\delta \frac{\theta_1^{j+1} + \theta_1^j}{2}\right) + \frac{2\alpha}{h^2} + \frac{12\gamma \Delta t}{32h^4}(d_{11}^{j+1} + d_{11}^j)^2 + m^2\frac{\Delta t}{2} - \frac{10\alpha R_e \Delta t}{2h^3}$$

$$J(1,2) = \frac{\partial R_1}{\partial u_2^{j+1}} = \frac{R_e \Delta t}{4h} - \frac{\Delta t}{2h^2} exp\left(-\delta \frac{\theta_1^{j+1} + \theta_1^j}{2}\right) + \frac{\delta \Delta t}{16h^2} exp\left(-\delta \frac{\theta_1^{j+1} + \theta_1^j}{2}\right)(e_{11}^{j+1} + e_{11}^j) - \frac{\alpha}{h^2} - \frac{12\gamma \Delta t}{32h^4}(d_{11}^{j+1} + d_{11}^j)(d_{21}^{j+1} + d_{21}^j) - \frac{6\gamma \Delta t}{32h^4}(d_{11}^{j+1} + d_{11}^j)^2 + 12\frac{\alpha R_e \Delta t}{2h^3}$$

$$J(1,3) = \frac{\partial R_1}{\partial u_3^{j+1}} = -\frac{6\alpha R_e \Delta t}{2h^3}$$

$$J(1,4) = \frac{\partial R_1}{\partial u_4^{j+1}} = \frac{\alpha R_e \Delta t}{2h^3}$$

$$J(2,1) = \frac{\partial R_2}{\partial u_1^{j+1}} = -\frac{R_e \Delta t}{4h} - \frac{\Delta t}{2h^2} exp\left(-\delta \frac{\theta_2^{j+1} + \theta_2^j}{2}\right) + \frac{3\alpha R_e \Delta t}{2h^3} - \frac{\alpha}{h^2} + \frac{6\gamma \Delta t}{16h^4}(d_{12}^{j+1} + d_{12}^j)(d_{21}^{j+1} + d_{21}^j) - \frac{3\gamma \Delta t}{32h^4}(d_{11}^{j+1} + d_{11}^j)^2$$

$$J(2,2) = \frac{\partial R_2}{\partial u_2^{j+1}} = 1 + \frac{\Delta t}{h^2} exp\left(-\delta \frac{\theta_2^{j+1} + \theta_2^j}{2}\right) - \frac{10\alpha R_e \Delta t}{2h^3} + \frac{2\alpha}{h^2} + \frac{6\gamma \Delta t}{16h^4}(d_{12}^{j+1} + d_{12}^j)^2 + m^2\frac{\Delta t}{2}$$

$$J(2,3) = \frac{\partial R_2}{\partial u_3^{j+1}} = \frac{R_e \Delta t}{4h} - \frac{\Delta t}{2h^2} exp\left(-\delta \frac{\theta_2^{j+1} + \theta_2^j}{2}\right) + \frac{\delta \Delta t}{16h^2} exp\left(-\delta \frac{\theta_2^{j+1} + \theta_2^j}{2}\right)(e_{12}^{j+1} + e_{12}^j) + \frac{12\alpha R_e \Delta t}{2h^3} - \frac{\alpha}{h^2} - \frac{6\gamma \Delta t}{16h^4}(d_{12}^{j+1} + d_{12}^j)(d_{22}^{j+1} + d_{22}^j) - \frac{3\gamma \Delta t}{16h^4}(d_{12}^{j+1} + d_{12}^j)^2$$

$$J(2,4) = \frac{\partial R_2}{\partial u_4^{j+1}} = \frac{6\alpha R_e \Delta t}{2h^3}$$

$$J(2,5)=\frac{\partial R_2}{\partial u_5^{j+1}}=\frac{\alpha R_e \Delta t}{2h^3}$$

$$J(i,i-2)=\frac{\partial R_i}{\partial u_{i-2}^{j+1}}=\frac{\alpha R_e \Delta t}{2h^3}$$

$$J(i,i-1)=\frac{\partial R_i}{\partial u_{i-1}^{j+1}}=-\frac{R_e \Delta t}{4h}-\frac{\Delta t}{2h^2}exp\left(-\delta\frac{\theta_i^{j+1}+\theta_i^j}{2}\right)-\frac{\delta\Delta t}{16h^2}exp\left(-\delta\frac{\theta_i^{j+1}+\theta_i^j}{2}\right)\left(e_{1i}^{j+1}+e_{1i}^j\right)-\frac{2\alpha R_e\Delta t}{2h^3}-\frac{\alpha}{h^2}-\frac{3\gamma\Delta t}{16h^4}\left(d_{1i}^{j+1}+d_{1i}^j\right)^2$$

$$J(i,i)=\frac{\partial R_2}{\partial u_2^{j+1}}=1+\frac{\Delta t}{h^2}exp\left(-\delta\frac{\theta_i^{j+1}+\theta_i^j}{2}\right)+\frac{\alpha}{h^2}+\frac{6\gamma\Delta t}{16h^4}\left(d_{1i}^{j+1}+d_{1i}^j\right)^2+m^2\frac{\Delta t}{2}$$

$$J(i,i+1)=\frac{\partial R_i}{\partial u_{i+1}^{j+1}}=\frac{R_e \Delta t}{4h}-\frac{\Delta t}{2h^2}exp\left(-\delta\frac{\theta_i^{j+1}+\theta_i^j}{2}\right)+\frac{\delta\Delta t}{16h^2}exp\left(-\delta\frac{\theta_i^{j+1}+\theta_i^j}{2}\right)\left(e_{1i}^{j+1}+e_{1i}^j\right)+\frac{\alpha R_e\Delta t}{h^3}-\frac{\alpha}{h^2}-\frac{6\gamma\Delta t}{16h^4}\left(d_{1i}^{j+1}+d_{1i}^j\right)\left(d_{2i}^{j+1}+d_{2i}^j\right)-\frac{3\gamma\Delta t}{16h^4}\left(d_{1i}^{j+1}+d_{1i}^j\right)^2$$

$$J(i,i+2)=\frac{\partial R_i}{\partial u_{i+2}^{j+1}}=-\frac{\alpha R_e \Delta t}{2h^3}$$

$for\ i=3,4,\dots,N-1.$

$$J(N,N)=\frac{\partial R_N}{\partial u_N^{j+1}}=1+\frac{R_e \Delta t}{4h}+\frac{\Delta t}{h^2}exp\left(-\delta\frac{\theta_N^{j+1}+\theta_N^j}{2}\right)+\frac{\alpha}{h^2}+\frac{6\gamma\Delta t}{16h^4}\left(-u_{N-1}^{j+1}+-u_{N-1}^j\right)^2+\frac{\alpha R_e\Delta t}{h^3}+m^2\frac{\Delta t}{2}$$

$$J(N,N-1)=\frac{\partial R_N}{\partial u_{N-1}^{j+1}}=-\frac{R_e \Delta t}{4h}-\frac{\Delta t}{2h^2}exp\left(-\delta\frac{\theta_N^{j+1}+\theta_N^j}{2}\right)-\frac{\delta\Delta t}{16h^2}exp\left(-\delta\frac{\theta_N^{j+1}+\theta_N^j}{2}\right)\left(-\theta_{N-1}^{j+1}-\theta_{N-1}^j\right)-\frac{12\alpha R_e\Delta t}{2h^3}-\frac{\alpha}{h^2}-\frac{3\gamma\Delta t}{16h^4}\left(-u_{N-1}^{j+1}+-2u_{N-1}^j\right)^2-\frac{3\gamma\Delta t}{16h^4}\left[\left(-2u_N^{j+1}+u_{N-1}^{j+1}\right)+\left(-2u_N^j+u_{N-1}^j\right)\right]$$

$$J(N,N-2)=\frac{\partial R_N}{\partial u_{N-2}^{j+1}}=\frac{6\alpha R_e \Delta t}{2h^3}$$

$$J(N,N-2)=\frac{\partial R_N}{\partial u_{N-2}^{j+1}}=-\frac{\alpha R_e \Delta t}{2h^3}$$

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