MHD Transport Phenomena of Oscillatory Channel of Blood Flow with Hall Current

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Abstract: *In this paper, a model has been developed to the impact of Hall current in the transport phenomena of MHD oscillatory channel of blood flow in the presence of chemical reaction and an external magnetic field with porous medium. On the basis of certain simplifying assumptions, the fluid equations of continuity, momentum, energy and concentration are obtained. The partial differential equations governing the flow have been solved analytically and also the results are displayed graphically to illustrate the effect of various parameters on the dimensionless velocity, temperature and concentration profiles.*

Keywords: *Hall current, MHD, oscillatory flow, chemical reaction, porous medium.*

Nomenclature

B_o *intensity of the external magnetic field*
C *Concentration*
C_p *specific heat at constant pressure*
g *gravitational acceleration*
G_C *modified Grashof number*
G_r *Grashof number*
k *permeability factor*
K_c *chemical reaction parameter*
M *Hartmann number*
N *thermal radiation parameter*
N_u *Nusselt number*
p^* *pressure*
P_e *Peclet number*
q_r *Radiative heat flux*
R_e *Reynolds number*
β_0^* *Limiting viscosity*
Sh *Sherwood number*
S_c *Schmidt number*
T^* *time*
T *fluid temperature*
u^* *axial velocity*
(x^*, y^*) *space coordinates*
β_o *viscoelastic coefficient*
β *coefficient of volume expansion due to temperature*
λ *slip parameter*
μ *dynamic viscosity*
ν *kinematic viscosity*
σ *conductivity of the medium*
ω^* *angular frequency*
ρ *fluid density*
m *hall current effect*
β_C *coefficient of volume expansion due to concentrate*

I. INTRODUCTION

The different rates of biochemical reactions that are responsible for the contraction of muscles, secretion of different materials such as insulin, mucus and stomach acid by the glands and the transmission of massages by the nerves can be accelerated / decelerated by the action of drugs. The performance of kidney cells and the regulation of the volume of water / salts in the body is affected by rate of drugs. The application of drugs in which blood flows through arteries can also be enhanced / slowed down by the application of drugs. It may, however, be observed that the damaged structures / functions can only be repaired by drugs, but their restoration is not possible. When the clinician treat patients suffering from various types of degenerative / tissue-destroying diseases, they observe the multiple atherosclerosis (narrowing of arterial lumen due to deposition of different fatty substances, cholesterol, etc.), arthritis, Alzheimer disease, Parkinson disease, heart failure. When the damage takes place due to some infection some drugs (e.g. antibiotics) that help the body in the damage repair process. Several drugs (antacids, for example) produce effects, where the function of a cell remains unchanged and a receptor dose not have any cognition. Most of the antacids are functioning on bases that interact with stomach acid to neutralize it. Thus stomach acid is reduced simply through chemical reactions.

The porosity as well as permeability decides the characteristic feature of the porous medium, that is, the measure of flow conductivity in the porous medium. In studies related to the heat transfer in biological tissues, the theory of porous media is preferred by researchers, since the number of assumptions necessary for the development of a model is less than that in the case of other bioheat models. The role of porous media in different studies of the mechanics of different biological organs and tissues is enormous. Khaled and Vafai [1] in a review paper explained the role of porous media in studies related to the flow and heat transfer in biological tissues. The flow characteristics of a Casson fluid in a tube filled with a homogeneous porous medium were examined by Dash et al. [2].

Slip-flow is defined as the difference in the velocities between liquids and solids (or liquids and gases) in the vertical flow of two-phase mixtures through a tube, owing to the slip between the two phases. On the basis of an experimental study, Beavers and Joseph [3] proposed a theory according to which the effect of the boundary layer can be replaced by a slip velocity is proportional to the exterior velocity gradient.

When the strength of the magnetic field is strong, one cannot neglect the effect of Hall current. It is of considerable importance and interest to study how the results of the hydrodynamical problems get modified by the effect of Hall currents. Hall currents give rise to a cross flow making the flow three dimensional. Olajuwon et al. [4] obtained the effect of thermal radiation and hall current on heat and mass transfer of unsteady MHD flow of a viscoelastic micropolar fluid through a porous mediam. Nisat et al. [5] investigated the hall current effects on MHD fluid over an infinite rotating vertical porous plate embedded in unsteady laminar flow. Omokhuale and Onwuka [6] studied the effect of mass transfer and hall current on unsteady MHD flow of a viscoelastic fluid in a porous medium. Hannington et al. [7] presented the effect of hall current and rotation on MHD free convection flow past a vertical infinite plate under a variable transverse magnetic field. Makinde and Mhone [8] studied heat transfer to MHD oscillatory flow in a channel filled with porous median. Rita and Utpal [9] performed heat transfer to MHD oscillatory visco-elastic flow in a channel filled with porous mediam.

Devika et al. [10] analysed the MHD oscillatory flow of a viscous elastic fluid in a porous channel with chemical reaction. Kathyani et al. [11] studied the heat and mass transfer on unsteady MHD oscillatory flow of non-Newtonian fluid through porous medium in parallel plate channel. Misra and Adhikary [12] analysed MHD oscillatory channel flow, heat and mass transfer in physiological fluid in the presence of chemical reaction.

This study has been focussed toward examining the impact of Hall current in the transport phenomena of MHD oscillatory channel flow of blood in the presence of chemical reaction and an external magnetic field with porous medium. Velocity-slip of erythrocytes contained in blood has been taken into account. A magnetic field is considered to be applied externally in a direction transverse to that of blood flow. The radiative heat transfer and chemical reaction in the mass transfer on the flow field are of particular concern here. Analytical expressions are obtained for the velocity profiles, the temperature, the concentration, and rate of heat and mass transfer on the upper wall of the channel are obtained

II. Mathematical formulation

We consider an unsteady MHD oscillatory channel blood flow of fluid with chemical reaction, thermal radiation and heat source in the presence of transverse magnetic field. It is assumed that the fluid has small electrical conductivity and electromagnetic force produced is very small. We consider the flow of a fluid between two parallel walls at $y^* = 0$ and $y^* = h$, where the axis of x^* is taken parallel length of walls and y^* - axis along a direction perpendicular to the walls. A magnetic field of constant intensity B_0 is applied in the y^* direction. The physical sketch of the problem is shown in Figure 1.

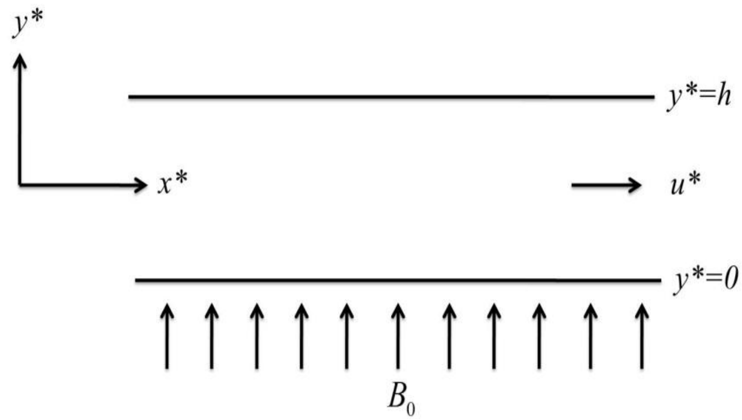

Figure 1. Geometrical Configuration

The momentum, heat and mass transfer equations considered for our model is in the form

$$\frac{\partial u^*}{\partial t^*} = -\frac{1}{\rho}\frac{\partial p^*}{\partial x^*} + \nu \frac{\partial^2 u^*}{\partial y^{*2}} + \beta_0^* \frac{\partial^3 u^*}{\partial y^{*2}\partial t^*} - \frac{\nu}{k^*}u^* - \frac{\sigma B_0^2}{\rho(1+m^2)}u^* + g\beta(T - T_0) + g\beta_c(C^* - C_0) \quad (1)$$

$$0 = -\frac{1}{\rho}\frac{\partial p^*}{\partial y^*} \quad (2)$$

$$\frac{\partial T}{\partial t^*} = \frac{K'}{\rho c_p}\frac{\partial^2 T}{\partial y^{*2}} - \frac{1}{\rho c_p}\frac{\partial q_r}{\partial y^*} \quad (3)$$

And

$$\frac{\partial C^*}{\partial t^*} = D\frac{\partial^2 C^*}{\partial y^{*2}} - K_c^{'}(C^* - C_0) \quad (4)$$

where the meanings of all the symbols appearing in the equations are listed in the **Nomenclature**.

The boundary conditions for the problem under consideration is reported by Brunn [13] andNubar [14] and are given by

$$u^* = \lambda\frac{\partial u^*}{\partial y^*}, \quad T = T_0 + (T_w - T_0)e^{iw^*t^*}, \quad C^* = C_0 + (C_w - C_0)e^{iw^*t^*}, \quad \text{at } y^* = h \quad (5)$$

And

$$u^* = \lambda\frac{\partial u^*}{\partial y^*}, T = T_0, \quad C^* = C_0 \text{ at } y^* = 0 \quad y^* = 0 \quad (6)$$

Using Rosseland approximation, the radiative transfer term q_r in equation (3) may be expressed as

$$q_r = -\frac{4\sigma^*}{3\alpha_r}\frac{\partial T^4}{\partial y^*} \quad (7)$$

We assume that the temperature differences within the flow are such that T^4 can be expressed as a linear function of the temperature T. This is accomplished by expanding T^4 in a Taylor series about T_0 (which is assumed to be independent of y*) and neglecting powers of T higher than the first. Thus we have

$$T^4 = 4T_0^3 T - 3T_0^4 \tag{8}$$

Then the heat transfer equation becomes

$$\frac{\partial T}{\partial t^*} = \frac{K'}{\rho c_p}\frac{\partial^2 T}{\partial y^{*2}} + \frac{16\sigma^* T_0^3}{3\rho c_p \alpha_r}\frac{\partial^2 T}{\partial y^{*2}} \tag{9}$$

We now use the following non-dimensional variables:

$$y = \frac{y^*}{h}, x = \frac{x^*}{h}, u = \frac{u^*}{U_0}, R_e = \frac{U_0 h}{\nu}, k = \frac{k^*}{h^2\rho}, t = \frac{t^* U_0}{h}, \quad p = \frac{hp^*}{\rho \nu U_0}, \theta = \frac{T - T_0}{T_W - T_0}, C = \frac{C^* - C_0}{C_W - C_0},$$

$$G_r = \frac{g\beta(T_W - T_0)h^2}{\nu U_0}, G_c = \frac{g\beta_c(C_W - C_0)h^2}{\nu U_0}, M = \frac{\sigma B_0^2 h^2}{\rho \nu}, P_e = \frac{U_0 h \rho c_p}{k}, \tag{10}$$

$$N = \frac{16\sigma^* T_0^3}{3\alpha_r K}, S_c = \frac{U_o h}{D}, K_c = K'_c\frac{h^2}{D}, \beta_0 = \frac{\beta_0^*}{\nu h}U_0, w = \frac{w^* h}{U_0}$$

Using equation (10) in equation (1) to (4), we have,

$$R_e\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \beta_0\frac{\partial^3 u}{\partial y^2 \partial t} - \{\frac{1}{k} + \frac{M}{1+m^2}\}u + G_r\theta + G_c C \tag{11}$$

$$0 = -\frac{\partial p}{\partial y} \tag{12}$$

$$P_e\frac{\partial \theta}{\partial t} = (1+N)\frac{\partial^2 \theta}{\partial y^2} \tag{13}$$

$$S_c\frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial y^2} - K_c C \tag{14}$$

While the non-dimensional boundary conditions become

$$u = \lambda\frac{\partial u}{\partial y}, \theta = e^{iwt}, C = e^{iwt}, \text{ at } y=1 \quad \text{and} \tag{15}$$

$$u = \lambda\frac{\partial u}{\partial y}, \theta = 0, C = 0, \text{ at } y=0 \tag{16}$$

From (11) and (12) it follows that $\frac{\partial p}{\partial x}$ is a function of t only. We consider it to be of the form

$$\frac{\partial p}{\partial x} = Be^{iwt} \tag{17}$$

Where B is a constant and w is the frequency of oscillation.

To solve equations (11) to (14) subject to the boundary conditions (15) and (16), for purely oscillatory flow, let

$$u(y,t) = u_f(y)e^{iwt}$$

$$\theta(y,t) = \theta_f(y)e^{iwt} \tag{18}$$

$$C(y,t) = C_f(y)e^{iwt}$$

Substituting these expressions in (11), (13) and (14) and comparing like terms we have the equations

$$(1+\beta_0 iw)\frac{\partial^2 u_f}{\partial y^2}-(R_e iw+\frac{1}{k}+\frac{M}{1+m^2})u_f = B - G_r\theta_f - G_c C_f \quad (19)$$

$$(1+N)\frac{\partial^2 \theta_f}{\partial y^2} - P_e iw\theta_f = 0 \quad (20)$$

$$\frac{\partial^2 C_f}{\partial y^2} - (S_c iw + K_c)C_f = 0 \quad (21)$$

Along with boundary conditions

$$u_f = \lambda\frac{\partial u_f}{\partial y}, \theta_f = 1, C_f = 1 \text{ at y=1} \quad (22)$$

$$u_f = \lambda\frac{\partial u_f}{\partial y}, \theta_f = 0, C_f = 0 \text{ at y=0} \quad (23)$$

Solving equations (19) to (21) subject to the conditions (22) and (23) and using the equation(18),
We have

$$\theta(y,t) = \frac{e^{iwt}}{e^{m_1}-e^{m2}}(e^{m_1 y}-e^{m_2 y}) \quad (24)$$

$$C(y,t) = \frac{e^{iwt}}{e^{m_3}-e^{m4}}(e^{m_3 y}-e^{m_4 y}) \quad (25)$$

$$u(y,t) = \left(E_1 e^{m_5 y} + E_2 e^{m_6 y} - \frac{B}{R_e iw + \frac{1}{k} + \frac{M}{1+m^2}} - \frac{G_r}{e^{m_1}-e^{m_2}}[\frac{e^{m_1 y}}{A_1} - \frac{e^{m_2 y}}{A_2}] - \frac{G_c}{e^{m_3}-e^{m_4}}[\frac{e^{m_3 y}}{A_3} - \frac{e^{m_4 y}}{A_4}] \right) e^{iwt}$$
(26)

Which respectively represent the temperature, concentration and velocity fields, where the expression for the constants m_i (i=1,2,3,..6), A_i (i=1,2,3,4), E_1 and E_2 are given in **Appendix.**

The rate of heat and mass transfer across the upper plate (wall) are calculated as

$$N_u = -\frac{\partial\theta}{\partial y}]_{\text{at y=1}} = -\frac{e^{iwt}}{e^{m_1}-e^{m_2}}(m_1 e^{m_1} - m_2 e^{m_2}) \quad (27)$$

$$S_h = -\frac{\partial C}{\partial y}]_{\text{at y=1}} = -\frac{e^{iwt}}{e^{m_3}-e^{m_4}}(m_3 e^{m_3} - m_4 e^{m_4}) \quad (28)$$

III. Results and Discussions

The motivation behind the study has been to analyze the effects of heat and mass transfer in the presence of chemical reaction in the flow of blood, by accounting for the velocity slip.

For the purpose of numerical computation, the following values of the different parameters involved in the analytical study have been used Misra and Adhikary [12].

$B = w = 1,\ G_r = G_c = 2,\ 0 \le P_e \le 3,\ 0 \le N \le 3,\ 0 \le R_e \le 6,\ 0 \le R_e \le 6,\ 0 \le \beta_0 \le 1,$

$0 \le M \le 5,\ 0.1 \le k \le 2,\ 0 \le \lambda \le 0.1,\ 0 \le K_c \le 2,\ 0 \le m \le 1,\ t = 0.$

These ranges of values of the physical quantities are mostly representative of blood flow, when a chemical reaction sets in. By using these values, the analytical expressions derived in the previous section have been computed by employing a suitable software, viz. MATHEMATICA.

From Figures 2 – 10, it is observed that the velocity increases gradually near the wall and decreases slowly away from the wall.

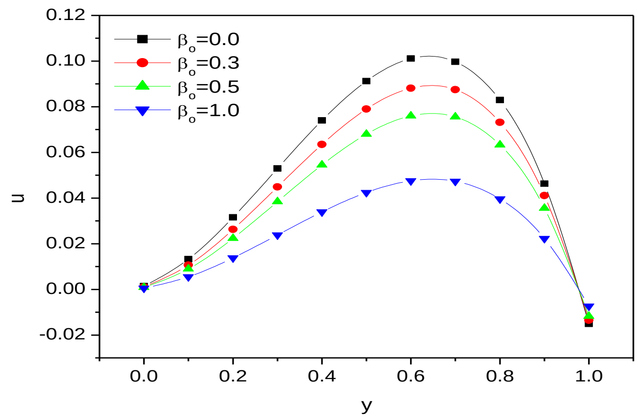

Figure 2. Velocity Profiles against y for various values of viscoelastic parameter β_0 .

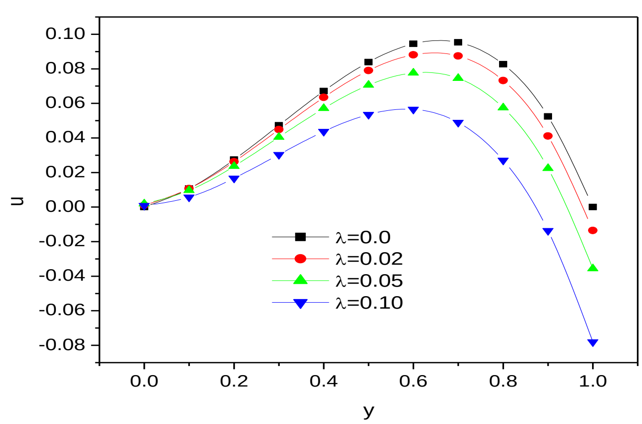

Figure 3. Velocity profiles against y for various values of slip parameter λ.

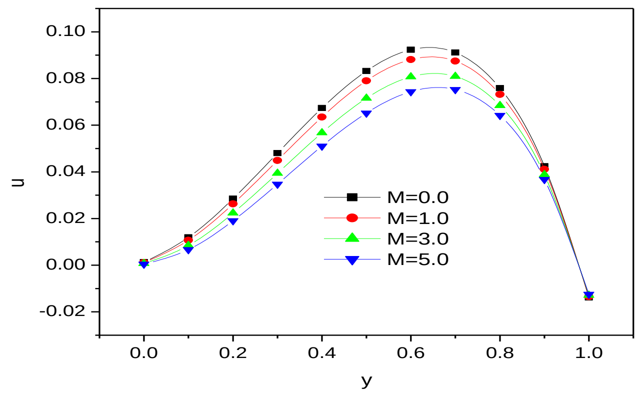

Figure 4. Velocity Profiles against y for various values of magnetic parameter M.

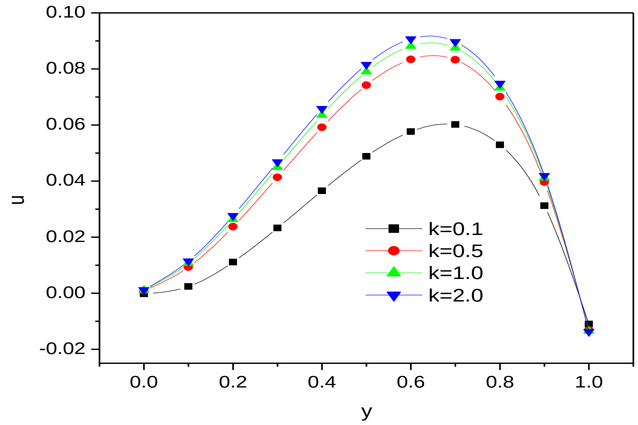

Figure 5. Velocity Profiles against y for various values of permeability parameter k.

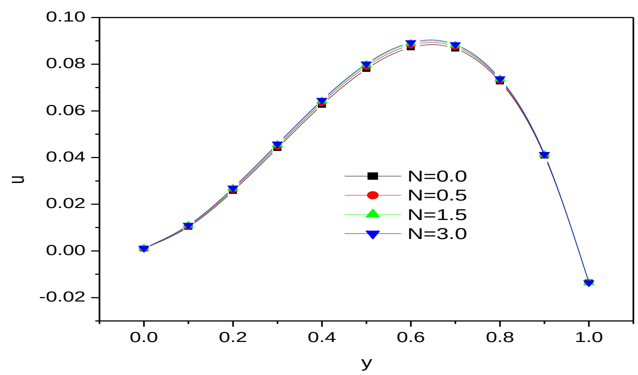

Figure 6. Velocity Profiles against y for various values of heat radiation parameter *N*.

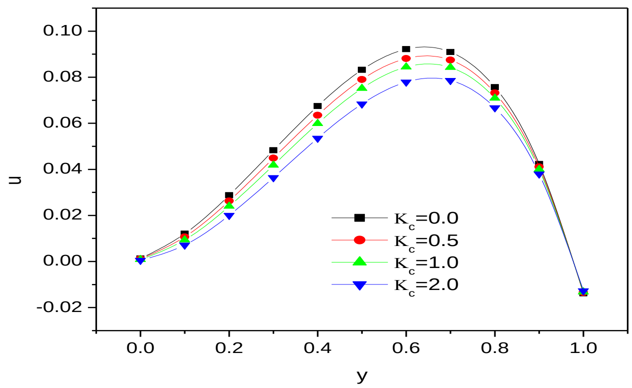

Figure 7. Velocity Profiles against y for various values of chemical reaction parameter K_c.

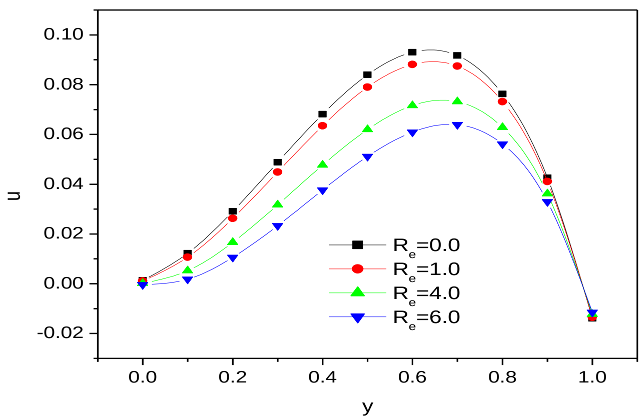

Figure 8. Velocity Profiles against y for various values of Reynolds number R_e .

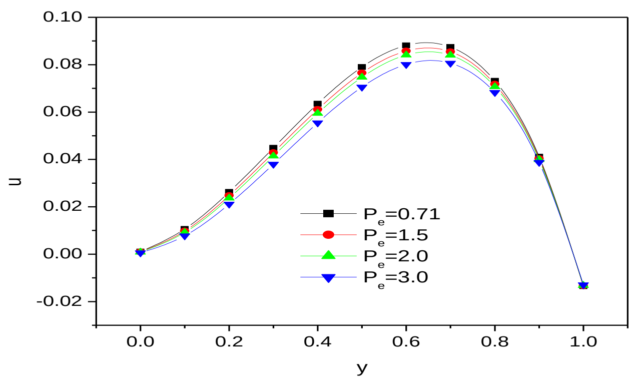

Figure 9. Velocity Profiles against y for various values of Peclet number P_e .

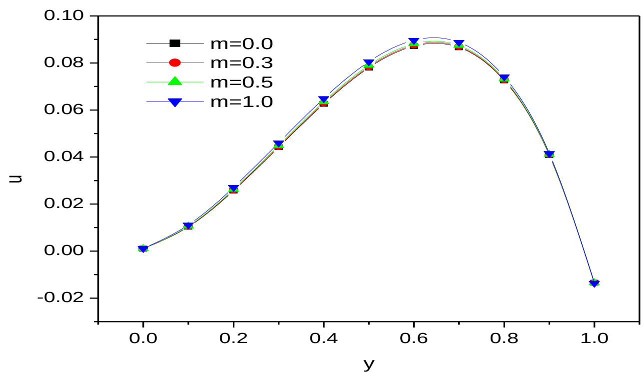

Figure 10. Velocity Profiles against y for various values of Hall current m.

Figure 2 shows the effect of the viscoelastic parameter (β_o) on velocity profiles. It is found that the velocity decreases as viscoelastic parameter increases. Figure 3 displays the effect of the Slip parameter (λ) on velocity profiles. The velocity decreases with increasing slip parameter. Figure 4 presents the effect of the Hartmann number (M) on velocity profiles and it is found that the increasing value of Hartmann number results in a decreasing velocity. This will happen for the effect of magnetic field gives raise to a force called the Lorentz force. Figure 5 presents the effect of the Permeability of porous medium parameter (k) on velocity profiles. It is obvious that the velocity increases as permeability parameter increases.

Figure 6 displays the effect of the heat radiation parameter (N) with velocity profiles. The effect of increasing heat radiation parameter is to increase (slightly) the velocity. Figure 7 displays the effect of the chemical reaction parameter (K_c) on velocity profiles. It is noticed that when the chemical reaction parameter increases, the velocity profile also decreases. Figure 8 shows the effect of the Reynolds number (R_e) on velocity profiles. The velocity decreases as Reynolds number increases. Figure 9 shows the effect of the Peclet number (P_e) on velocity profiles. It is shown that the increase of peclet number leads to the decrease of velocity. Figure 10 depicts the effect of the Hall current effect (m) on velocity profiles and it is shown that as hall current parameter increases, the velocity increases.

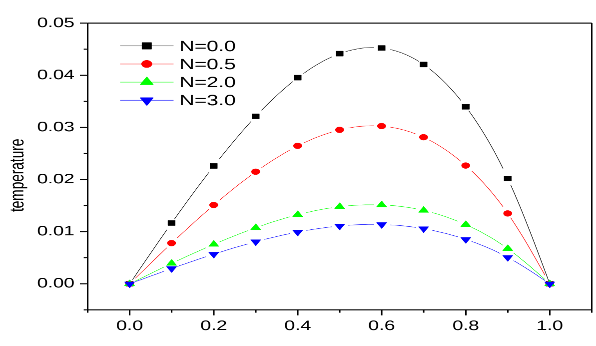

Figure 11. Temperature Profiles against y for various values of Heat radiation parameter N.

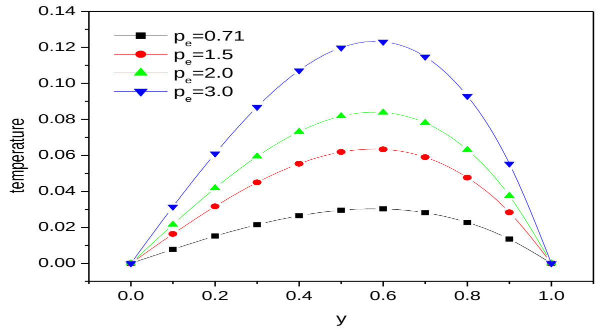

Figure 12. Temperature Profiles against y for various values of Peclet number P_e .

Figure 11 presents the effect of the heat radiation parameter (N) on temperature profiles. It can be noted that the temperature profile is parabolic. The temperature decreases as the heat radiation parameter increases. Figure 12 illustrates the effect of the Peclet number (P_e) on temperature profiles. It is noticed that as the Peclet number increases, the temperature also increases.

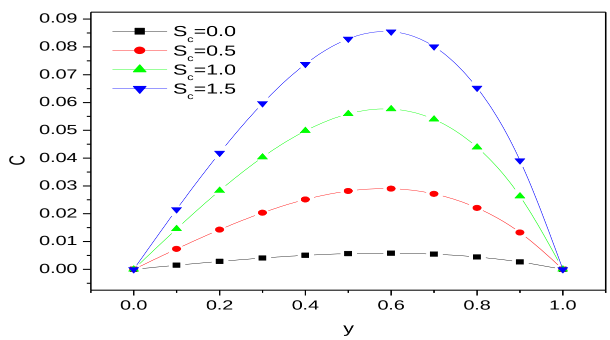

Figure 13 Concentration Profiles against y for various values of Schmidt number S_c .

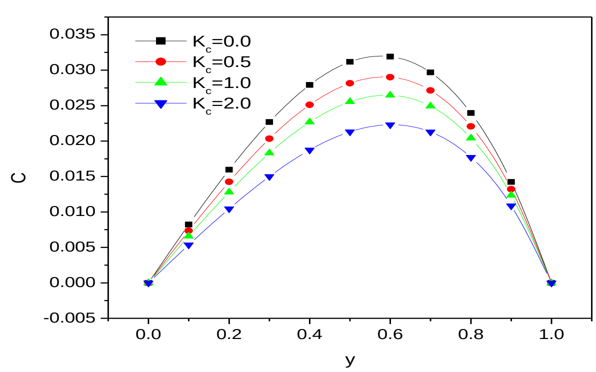

Figure 14 Concentration Profiles against y for various values of chemical reaction parameter K_c.

Figure 13 displays the effect of Schmidt number (S_c) on concentration profiles. It can be noted from the figure it is parabolic and as the concentration of the blood increaseas, the Schmidt number increases. Figure 14 shows the effect of chemical reaction parameter (K_c) on concentration profiles. The concentration profile decreases as the chemical reaction parameter increases. The numerical results for the rate of heat transfer (N_u, Nusselt number) and the rate of mass transfer (*Sh* , Sherwood number), are shown in Table 1 and 2.

Table 1. Effect of *Pe* and *N* on *Nu*

P_e	N	N_u
0.71	0.5	0.157554
2.0	0.5	0.439516
3.0	0.5	0.650393
0.71	1.0	0.118239
0.71	2.0	0.078861
0.71	3.0	0.059155

Table 2. Effect of S_c and K_c on *Sh*

S_c	K_c	*sh*
0.5	0.5	0.156082
1.0	0.5	0.310873
1.5	0.5	0.463139
1.0	1.0	0.293052
1.0	1.5	0.277349
1.0	2.0	0.263429

Table 1 shows the effect of Peclet number and heat radiation on the Nusselt number. The Nusselt number increases as Peclet number increases and decreases as heat radiation increases. Table 2 shows the effect of Schmidt number and Chemical reaction parameter on Sherwood number. The Sherwood number increases as Schmidt number increases and decreases as Chemical reaction parameter increases

. IV. Conclusion

The study of MHD transport phenomena of oscillatory channel of blood flow with chemical reaction through a porous medium has been developed. The results are discussed through graphs and tables for different values of parameters. The following conclusions have been drawn from the results obtained:

(a) The velocity increases as permeability factor and heat radiation parameter increased.
(b) Increase in hall current parameter increases the velocity.
(c) The temperature increases as Peclet number increases.
(d) The concentration of the blood increases as Schmidt number increases, whereas decreases when chemical reaction is increased.
(e) The Nusselt number increases as the Peclet number increased and decreases as heat Radiation parameter is increased.
(f) The Sherwood number increases as the Schmidt number increased.

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Appendix

$$m_1, m_2 = +\sqrt{\frac{P_e iw}{1+N}}, -\sqrt{\frac{P_e iw}{1+N}}; \quad m_3, m_4 = +\sqrt{(S_c iw + K_c)}, -\sqrt{(S_c iw + K_c)};$$

$$m_5, m_6 = +\sqrt{\frac{R_e iw + \frac{1}{k} + \frac{M}{1+m^2}}{1+\beta_0 iw}}, -\sqrt{\frac{R_e iw + \frac{1}{k} + \frac{M}{1+m^2}}{1+\beta_0 iw}}; \; A_s = (1+\beta_0 iw) m_s^2 - \left(R_e iw + \frac{1}{k} + \frac{M}{1+m^2}\right), (s = 1,2,3,4)$$

$$X_1 = \frac{B}{R_e iw + \frac{1}{k} + \frac{M}{1+m^2}} + \frac{G_r}{e^{m_1} - e^{m_2}}\left[\frac{1-\lambda m_1}{A_1} - \frac{1-\lambda m_2}{A_2}\right] + \frac{G_r}{e^{m_3} - e^{m_4}}\left[\frac{1-\lambda m_3}{A_3} - \frac{1-\lambda m_4}{A_4}\right]$$

$$X_2 = \frac{B}{R_e iw + \frac{1}{k} + \frac{M}{1+m^2}} + \frac{G_r}{e^{m_1} - e^{m_2}}\left[\frac{(1-\lambda m_1)e^{m_1}}{A_1} - \frac{(1-\lambda m_2)e^{m_2}}{A_2}\right] + \frac{G_r}{e^{m_3} - e^{m_4}}\left[\frac{(1-\lambda m_3)e^{m_3}}{A_3} - \frac{(1-\lambda m_4)e^{m_4}}{A_4}\right]$$

$$E_1 = \frac{X_2 - X_1 e^{m_6}}{(e^{m_5} - e^{m_6})(1-\lambda m_5)}; \qquad E_2 = -\frac{X_2 - X_1 e^{m_5}}{(e^{m_5} - e^{m_6})(1-\lambda m_6)}$$