Multiple integral involving the S generalized Gauss's hypergeometric function ,a class of polynomials and multivariable Aleph-functions

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ABSTRACT
In the present paper we evaluate a generalized multiple integrals involving the product of rhe S generalized Gauss hypergeometric function, the multivariable Aleph-function, and general class of polynomials of several variables. The importance of the result established in this paper lies in the fact they involve the Aleph-function of several variables which is sufficiently general in nature and capable to yielding a large of results merely by specializating the parameters their in. We will study the case concerning the multivariable I-function defined by Sharma et al [2].

Keywords:Multivariable Aleph-function, general class of polynomials, multiple integral, S generalized hypergeometric function, multivariable I-function.

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1.Introduction and preliminaries.

The function Aleph of several variables is an extension the multivariable I-function recently studied by C.K. Sharma and Ahmad [2] , itself is a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integrals occurring in this paper will be referred to as the multivariable Aleph-function throughout our present study and will be defined and represented as follows.

We define : $\aleph(z_1,\cdots,z_r)=\aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array}\right|$

$[(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}]$, $[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]$:
$\cdots\cdots\cdots\cdots$, $[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{1,q_i}]$:

$$\left.\begin{array}{c} [(c_j^{(1)});\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;[(c_j^{(r)});\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ [(d_j^{(1)});\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;[(d_j^{(r)});\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}] \end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)y_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \tag{1.2}$$

and $$\theta_k(s_k)=\frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

Suppose , as usual , that the parameters

$a_j, j = 1, \cdots, p; b_j, j = 1, \cdots, q;$

$c_j^{(k)}, j = 1, \cdots, n_k; c_{ji^{(k)}}^{(k)}, j = n_k + 1, \cdots, p_{i^{(k)}};$

$d_j^{(k)}, j = 1, \cdots, m_k; d_{ji^{(k)}}^{(k)}, j = m_k + 1, \cdots, q_{i^{(k)}};$

with $k = 1 \cdots, r, i = 1, \cdots, R \ , i^{(k)} = 1, \cdots, R^{(k)}$

are complex numbers , and the $\alpha' s, \beta' s, \gamma' s$ and $\delta' s$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} + \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.4}$$

The reals numbers τ_i are positives for $i = 1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)} s_k)$ with $j = 1$ to m_k are separated from those of $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(k)} s_k)$ with $j = 1$ to n and $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)$ with $j = 1$ to n_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|arg z_k| < \frac{1}{2} A_i^{(k)} \pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k = 1 \cdots, r, i = 1, \cdots, R \ , i^{(k)} = 1, \cdots, R^{(k)} \tag{1.5}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \rightarrow 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \rightarrow \infty$

where $k = 1, \cdots, r : \alpha_k = min[Re(d_j^{(k)}/\delta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta_k = max[Re((c_j^{(k)} - 1)/\gamma_j^{(k)})], j = 1, \cdots, n_k$$

Serie representation of Aleph-function of several variables is given by

$$\aleph(y_1, \cdots, y_r) = \sum_{G_1, \cdots, G_r = 0}^{\infty} \sum_{g_1=0}^{m_1} \cdots \sum_{g_r=0}^{m_r} \frac{(-)^{G_1 + \cdots + G_r}}{\delta_{g_1} G_1! \cdots \delta_{g_r} G_r!} \psi(\eta_{G_1, g_1}, \cdots, \eta_{G_r, g_r})$$

$$\times\ \theta_1(\eta_{G_1,g_1})\cdots\theta_r(\eta_{G_r,g_r})y_1^{-\eta_{G_1,g_1}}\cdots y_r^{-\eta_{G_r,g_r}} \tag{1.6}$$

Where $\psi(.,\cdots,.)$, $\theta_i(.)$,$i=1,\cdots,r$ are given respectively in (1.2), (1.3) and

$$\eta_{G_1,g_1}=\frac{d_{g_1}^{(1)}+G_1}{\delta_{g_1}^{(1)}},\cdots,\ \eta_{G_r,g_r}=\frac{d_{g_r}^{(r)}+G_r}{\delta_{g_r}^{(r)}}$$

which is valid under the conditions $\delta_{g_i}^{(i)}[d_j^i+p_i]\neq\delta_j^{(i)}[d_{g_i}^i+G_i]$ (1.7)

for $j\neq m_i, m_i=1,\cdots\eta_{G_i,g_i};p_i,n_i=0,1,2,\cdots,;y_i\neq 0,i=1,\cdots,r$ (1.8)

Consider the Aleph-function of s variables

$$\aleph(z_1,\cdots,z_s)=\aleph_{P_i,Q_i,\iota_i;r':P_{i(1)},Q_{i(1)},\iota_{i(1)};r^{(1)};\cdots;P_{i(s)},Q_{i(s)};\iota_{i(s)};r^{(s)}}^{0,N:M_1,N_1,\cdots,M_s,N_s}\left(\begin{array}{c|c} \mathrm{z}_1 & \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_s & \end{array}\right.$$

$$\begin{array}{ll}[(\mathrm{u}_j;\mu_j^{(1)},\cdots,\mu_j^{(r')})_{1,N}] & ,[\iota_i(u_{ji};\mu_{ji}^{(1)},\cdots,\mu_{ji}^{(r')})_{N+1,P_i}]: \\ \ldots\ldots\ldots\ldots\ldots\ldots & ,[\iota_i(v_{ji};\upsilon_{ji}^{(1)},\cdots,\upsilon_{ji}^{(r')})_{1,Q_i}]:\end{array}$$

$$\left.\begin{array}{l}[(\mathrm{a}_j^{(1)});\alpha_j^{(1)})_{1,N_1}],[\iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)};\alpha_{ji^{(1)}}^{(1)})_{N_1+1,P_i^{(1)}}];\cdots;[(\mathrm{a}_j^{(s)});\alpha_j^{(s)})_{1,N_s}],[\iota_{i^{(s)}}(a_{ji^{(s)}}^{(s)};\alpha_{ji^{(s)}}^{(s)})_{N_s+1,P_i^{(s)}}] \\ [(\mathrm{b}_j^{(1)});\beta_j^{(1)})_{1,M_1}],[\iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)};\beta_{ji^{(1)}}^{(1)})_{M_1+1,Q_i^{(1)}}];\cdots;[(\mathrm{b}_j^{(s)});\beta_j^{(s)})_{1,M_s}],[\iota_{i^{(s)}}(b_{ji^{(s)}}^{(s)};\beta_{ji^{(s)}}^{(s)})_{M_s+1,Q_i^{(s)}}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^s}\int_{L_1'}\cdots\int_{L_s'}\zeta(t_1,\cdots,t_s)\prod_{k=1}^s\phi_k(t_k)z_k^{t_k}\,\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{1.9}$$

with $\omega=\sqrt{-1}$

$$\zeta(t_1,\cdots,t_s)=\frac{\prod_{j=1}^N\Gamma(1-u_j+\sum_{k=1}^s\mu_j^{(k)}t_k)}{\sum_{i=1}^{r'}[\iota_i\prod_{j=N+1}^{P_i}\Gamma(u_{ji}-\sum_{k=1}^s\mu_{ji}^{(k)}t_k)\prod_{j=1}^{Q_i}\Gamma(1-\upsilon_{ji}+\sum_{k=1}^s\upsilon_{ji}^{(k)}t_k)]} \tag{1.10}$$

and $$\phi_k(t_k)=\frac{\prod_{j=1}^{M_k}\Gamma(b_j^{(k)}-\beta_j^{(k)}t_k)\prod_{j=1}^{N_k}\Gamma(1-a_j^{(k)}+\alpha_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{r^{(k)}}[\iota_{i^{(k)}}\prod_{j=M_k+1}^{Q_{i^{(k)}}}\Gamma(1-b_{ji^{(k)}}^{(k)}+\beta_{ji^{(k)}}^{(k)}t_k)\prod_{j=N_k+1}^{P_{i^{(k)}}}\Gamma(a_{ji^{(k)}}^{(k)}-\alpha_{ji^{(k)}}^{(k)}s_k)]} \tag{1.11}$$

Suppose , as usual , that the parameters

$u_j,j=1,\cdots,P;v_j,j=1,\cdots,Q;$

$a_j^{(k)},j=1,\cdots,N_k;a_{ji^{(k)}}^{(k)},j=n_k+1,\cdots,P_{i^{(k)}};$

$b_{ji^{(k)}}^{(k)},j=m_k+1,\cdots,Q_{i^{(k)}};b_j^{(k)},j=1,\cdots,M_k;$

with $k=1\cdots,s,i=1,\cdots,r',i^{(k)}=1,\cdots,r^{(k)}$

are complex numbers , and the $\alpha's, \beta's, \gamma's$ and $\delta's$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \iota_i \sum_{j=N+1}^{P_i} \mu_{ji}^{(k)} - \iota_{i^{(k)}} \sum_{j=N_k+1}^{P_{i^{(k)}}} \alpha_{ji^{(k)}}^{(k)} + \iota_i \sum_{j=1}^{Q_i} \upsilon_{ji}^{(k)} - \iota_{i^{(k)}} \sum_{j=M_k+1}^{Q_{i^{(k)}}} \beta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.12}$$

The reals numbers τ_i are positives for $i = 1, \cdots, r$, $\iota_{i^{(k)}}$ are positives for $i^{(k)} = 1 \cdots r^{(k)}$

The contour L_k is in the t_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(b_j^{(k)} - \beta_j^{(k)} t_k)$ with $j = 1$ to M_k are separated from those of $\Gamma(1 - u_j + \sum_{i=1}^{s} \mu_j^{(k)} t_k)$with $j = 1$ to N and $\Gamma(1 - a_j^{(k)} + \alpha_j^{(k)} t_k)$ with $j = 1$ to N_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

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$$+ \sum_{j=1}^{M_k} \beta_j^{(k)} - \iota_{i^{(k)}} \sum_{j=M_k+1}^{q_{i^{(k)}}} \beta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k = 1 \cdots, s, i = 1, \cdots, r \ , i^{(k)} = 1, \cdots, r^{(k)} \tag{1.13}$$

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$\aleph(z_1, \cdots, z_s) \ = 0(|z_1|^{\beta'_1}, \cdots, |z_s|^{\beta'_s}) , min(|z_1|, \cdots, |z_s|) \rightarrow \infty$

where, $k = 1, \cdots, z : \alpha'_k = min[Re(b_j^{(k)} / \beta_j^{(k)})], j = 1, \cdots, M_k$ and

$$\beta'_k = max[Re((a_j^{(k)} - 1)/\alpha_j^{(k)})], j = 1, \cdots, N_k \tag{1.14}$$

We will use these following notations in this paper

$$U = P_i, Q_i, \iota_i; r' ; \ V = M_1, N_1; \cdots ; M_s, N_s \tag{1.15}$$

$$\mathrm{W} = P_{i^{(1)}}, Q_{i^{(1)}}, \iota_{i^{(1)}}; r^{(1)}, \cdots, P_{i^{(r)}}, Q_{i^{(r)}}, \iota_{i^{(s)}}; r^{(s)} \tag{1.16}$$

$$A = \{(u_j; \mu_j^{(1)}, \cdots, \mu_j^{(s)})_{1,N}\}, \{\iota_i(u_{ji}; \mu_{ji}^{(1)}, \cdots, \mu_{ji}^{(s)})_{N+1,P_i}\} \tag{1.17}$$

$$B = \{\iota_i(v_{ji}; \upsilon_{ji}^{(1)}, \cdots, \upsilon_{ji}^{(s)})_{M+1,Q_i}\} \tag{1.18}$$

$$C = (a_j^{(1)}; \alpha_j^{(1)})_{1,N_1} , \iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)}; \alpha_{ji^{(1)}}^{(1)})_{N_1+1,P_{i^{(1)}}}, \cdots, (a_j^{(s)}; \alpha_j^{(s)})_{1,N_s} , \iota_{i^{(s)}}(a_{ji^{(s)}}^{(s)}; \alpha_{ji^{(s)}}^{(s)})_{N_s+1,P_{i^{(s)}}} \tag{1.19}$$

$$D = (b_j^{(1)}; \beta_j^{(1)})_{1,M_1}, \iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)}; \beta_{ji^{(1)}}^{(1)})_{M_1+1,Q_{i^{(1)}}}, \cdots, (b_j^{(s)}; \beta_j^{(s)})_{1,M_s}, \iota_{i^{(s)}}(\beta_{ji^{(s)}}^{(s)}; \beta_{ji^{(s)}}^{(s)})_{M_s+1,Q_{i^{(s)}}} \tag{1.20}$$

The multivariable Aleph-function writess :

$$\aleph(z_1,\cdots,z_s) = \aleph_{U:W}^{0,N:V}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_s \end{array}\middle| \begin{array}{c} A : C \\ \cdot\ \cdot\ \cdot \\ B : D \end{array}\right) \tag{1.21}$$

The generalized polynomials defined by Srivastava [5], is given in the following manner :

$$S_{N'_1,\cdots,N'_t}^{M'_1,\cdots,M'_t}[y_1,\cdots,y_t] = \sum_{K_1=0}^{[N'_1/M'_1]}\cdots\sum_{K_t=0}^{[N'_t/M'_t]}\frac{(-N'_1)_{M'_1K_1}}{K_1!}\cdots\frac{(-N'_t)_{M'_tK_t}}{K_t!}$$

$$A[N'_1,K_1;\cdots;N'_t,K_t]y_1^{K_1}\cdots y_t^{K_t} \tag{1.22}$$

Where $M'_1,\cdots,M'_t$ are arbitrary positive integers and the coefficients $A[N'_1,K_1;\cdots;N'_t,K_t]$ are arbitrary constants, real or complex. In the present paper, we use the following notation

$$a_1 = \frac{(-N'_1)_{M'_1K_1}}{K_1!}\cdots\frac{(-N'_t)_{M'_tK_t}}{K_t!}A[N'_1,K_1;\cdots;N'_t,K_t] \tag{1.23}$$

In the document , we note :

$$G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r}) = \phi(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})\theta_1(\eta_{G_1,g_1})\cdots\theta_r(\eta_{G_r,g_r}) \tag{1.24}$$

where $\phi(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r}),\theta_1(\eta_{G_1,g_1}),\cdots,\theta_r(\eta_{G_r,g_r})$ are given respectively in (1.2) and (1.3)

2. S Generalized Gauss's hypergeometric function

The S generalized Gauss hypergeometric function $F_p^{(\alpha,\beta;\tau,\mu)}(a,b,c;z)$ introduced and defined by Srivastava et al [5, page 350 , Eq.(1.12)] is represented in the following manner :

$$F_p^{(\alpha,\beta;\tau,\mu)}(a,b,c;z) = \sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta;\tau,\mu)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!} \quad (|z|<1) \tag{2.1}$$

provided that $(Re(p)\geqslant 0, minRe(\alpha,\beta,\tau,\mu)>0; Re(c)>Re(b)>0)$

where the S generalized Beta function $B_p^{(\alpha,\beta;\tau,\mu)}(x,y)$ was introduced and defined by Srivastava et al [6, page 350, Eq(1.13)]

$$B_p^{(\alpha,\beta;\tau,\mu)}(x,y) = \int_1^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha;\beta;\frac{-p}{t^\tau(1-t)^\mu}\right)\mathrm{d}t \tag{2.2}$$

provided that $(Re(p)\geqslant 0, minRe(x,y,\alpha,\beta)>0; Min\{Re(\tau),Re(\mu\}>0)$

3. Required integral

We have the following multiple integrals transformation, see Marichev et al. ([1], 33.4 13 page 592).

let $\bar{ax} = a_1x_1 + \cdots + a_nx_n$

$$\int_{x_1\geqslant 0}\cdots\int_{x_n\geqslant 0}\frac{(1-x_1-\cdots-x_n)^{\upsilon_0-1}}{(\bar{ax}+b)^{\mu}}\prod_{i=1}^{n}x_i^{\upsilon_i-1}\mathrm{d}x_1\cdots\mathrm{d}x_n=\frac{\Gamma(\upsilon_0)\Gamma(\upsilon_1)\cdots\Gamma(\upsilon_n)}{\Gamma(\upsilon_1+\cdots+\upsilon_n-\mu)\Gamma(\mu)}b^{-\upsilon_0}$$

$$\times\int_0^1 x^{\mu-1}(1+bx)^{-\upsilon_0}\prod_{i=1}^{n}1+b+a_ix)^{-\upsilon_k}\,\mathrm{d}x \tag{3.1}$$

where $x_1+\cdots+x_n\leqslant 1, b>0, \upsilon_0, \mu, a_i>0, \upsilon_i>0, i=1,\cdots,n; \upsilon_0+\upsilon_1+\cdots+\upsilon_n>\mu$

4. Main integral

We note $X_{\upsilon_1,\cdots,\upsilon_n,\mu}=\frac{x_1^{\upsilon_1}\cdots x_n^{\upsilon_n}}{(\bar{ax}+b)^{\mu}}$, we have the following formula

Theorem

$$\int_{x_1\geqslant 0}\cdots\int_{x_n\geqslant 0}F_p^{(\alpha,\beta;\tau,\mu)}(a,b,c;yX_{\xi_1,\cdots,\xi_n;\xi})\,S_{N_1,\cdots,N_t}^{M_1',\cdots,M_t'}\left(\begin{array}{c}\mathrm{y}_1X_{\gamma_1^{(1)},\cdots,\gamma_1^{(n)},\gamma_1}\\ \cdots\\ \mathrm{y}_tX_{\gamma_t^{(1)}\cdots,\gamma_t^{(n)},\gamma_t}\end{array}\right)$$

$$\aleph_{u:w}^{0,\mathrm{n}:v}\left(\begin{array}{c}\mathrm{z}_1X_{\alpha_1^{(1)},\cdots\alpha_1^{(n)},\alpha_1}\\ \cdots\\ \mathrm{z}_rX_{\alpha_r^{(1)}\cdots\alpha_r^{(n)},\alpha_r}\end{array}\right)\aleph_{U:W}^{0,N:V}\left(\begin{array}{c}\mathrm{Z}_1X_{\eta_1^{(1)}\cdots\eta_1^{(n)},\eta_1}\\ \cdots\\ \mathrm{Z}_sX_{\eta_s^{(1)}\cdots\eta_s^{(n)},\eta_s}\end{array}\right)\frac{(1-x_1-\cdots-x_n)^{\upsilon_0-1}}{(\bar{ax}+b)^{\mu}}\prod_{i=1}^{n}x_i^{\upsilon_i-1}$$

$$\mathrm{d}x_1\cdots\mathrm{d}x_n=\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{m_1}\cdots\sum_{g_r=0}^{m_r}\sum_{K_1=0}^{[N_1/M_1']}\cdots\sum_{K_t=0}^{[N_t/M_t']}\sum_{n'=0}^{\infty}(a)_{n'}\frac{B_p^{(\alpha,\beta;\tau,\mu)}(b+n',c-b)}{B(b,c-b)}\Gamma(\upsilon_0)$$

$$\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}\,G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})a_1\frac{y^{n'}}{n'!}z_1^{\eta_{G_1,g_1}}\cdots z_r^{\eta_{G_r,g_r}}y_1^{K_1}\cdots y_t^{K_t}$$

$$\int_0^1\frac{x^{n'\xi+\sum_{j=1}^{t}K_j\gamma_j+\sum_{j=1}^{r}\eta_{G_j,g_j}\alpha_j-1}}{(1+bx)^{-\upsilon_0}\prod_{i=1}^{n}(1+b+a_ix)^{n'\xi_i+\sum_{j=1}^{t}K_j\gamma_j^{(i)}+\sum_{j=1}^{r}\eta_{G_j,g_j}\alpha_j^{(i)}}}\aleph_{U_{n,2}:W}^{0,N+n:V}\left(\begin{array}{c}\mathrm{Z}_1\frac{x^{\eta_1}}{\prod_{i=1}^{n}(1+b+a_ix)^{\eta_1^{(i)}}}\\ \cdots\\ \cdots\\ \mathrm{Z}_s\frac{x^{\eta_s}}{\prod_{i=1}^{n}(1+b+a_ix)^{\eta_s^{(i)}}}\end{array}\right|$$

$$\left.\begin{array}{c}\left[1-(\upsilon_i+n'\xi_i+\sum_{j=1}^{t}K_i\gamma_j^{(i)}+\sum_{j=1}^{r}\eta_{G_i,g_i}\alpha_j^{(i)});\eta_1^{(i)},\cdots,\eta_s^{(i)}\right]_{1,n},A:C\\ \cdots\\ (1\text{-}\mu-n'\xi-\sum_{j=1}^{t}K_j\gamma_j-\sum_{j=1}^{r}\eta_{G_j,g_j}\alpha_i;\eta_1,\cdots,\eta_s),B_1,B:D\end{array}\right)\mathrm{d}x \tag{4.1}$$

where $U_{n,2} = P_i + n; Q_i + 2; \iota_i; r'$

where : $$B_1 = \left\{ -\sum_{i=1}^{n} \left[\upsilon_i + n'\xi_i + \sum_{j=1}^{t} K_j \gamma_j^{(i)} + \sum_{j=1}^{r} \eta_{G_j,g_j} \alpha_j^{(i)} \right] + \right.$$

$$\left. \left[\mu + n'\xi + \sum_{j=1}^{t} K_j \gamma_j + \sum_{j=1}^{r} \eta_{G_j,g_j} \alpha_j \right]; \eta_1^{(1)} + \cdots + \eta_1^{(n)} - \eta_1, \cdots, \eta_s^{(1)} + \cdots + \eta_s^{(n)} - \eta_s \right\} \quad (4.2)$$

Provided that

a) $min\{\xi_i, \upsilon_i, \gamma_j^{(i)}, \alpha_k^{(i)}, \eta_l^{(i)}, \} > 0, i = 1, \cdots, n, j = 1, \cdots, t, k = 1, \cdots, r, l = 1, \cdots, s$

b) $min\{\xi, \gamma_j, \alpha_k, \eta_l, \} > 0, j = 1, \cdots, t, k = 1, \cdots, r, l = 1, \cdots, s$

c) $$Re(\upsilon_i + n'\xi_i) + \sum_{j=1}^{r} \alpha_j^{(i)} \min_{1 \leqslant k \leqslant m_j} Re\left(\frac{d_k^{(j)}}{\delta_k^{(j)}} \right) + \sum_{j=1}^{s} \eta_j^{(i)} \min_{1 \leqslant j \leqslant M_j} \left(\frac{b_k^{(j)}}{\beta_k^{(j)}} \right) > 0, i = 1, \cdots, n$$

d) $$\sum_{i=1}^{n} \left[Re(\upsilon_i + n'\xi_i) + \sum_{j=1}^{r} \alpha_j^{(i)} \min_{1 \leqslant k \leqslant m_j} Re\left(\frac{d_k^{(j)}}{\delta_k^{(j)}} \right) + \sum_{j=1}^{s} \eta_j^{(i)} \min_{1 \leqslant j \leqslant M_j} \left(\frac{b_k^{(j)}}{\beta_k^{(j)}} \right) \right] + 1 >$$

$$> Re(\mu + n'\xi) + \sum_{j=1}^{r} \alpha_j \min_{1 \leqslant k \leqslant m_j} \left(\frac{d_k^{(j)}}{\delta_k^{(j)}} \right) + \sum_{j=1}^{s} \eta_j \min_{1 \leqslant k \leqslant M_j} \left(\frac{b_k^{(j)}}{\beta_k^{(j)}} \right) > 0$$

d) $|argz_k| < \frac{1}{2} A_i^{(k)} \pi$, where $A_i^{(k)}$ is defined by (1.5) ; $i = 1, \cdots, r$

e) $|argZ_k| < \frac{1}{2} B_i^{(k)} \pi$, where $B_i^{(k)}$ is defined by (1.13) ; $i = 1, \cdots, s$

f) $a_0, \mu, a_i, b_i, \upsilon_i > 0, i = 1, \cdots, n$

g) The series occurring on the right-hand side of (4.1) is absolutely and uniformly convergent.

h) $(Re(p) \geqslant 0, minRe(\alpha, \beta, \tau, \mu) > 0; Re(c) > Re(b) > 0)$

The quantities U, V, W, A, B, C and D are defined by the equations (1.15) to (1;20)

Proof of (4.1) : Let $M\{\} = \frac{1}{(2\pi\omega)^s} \int_{L_1'} \cdots \int_{L_s'} \zeta(t_1, \cdots, t_s) \prod_{k=1}^{s} \phi_k(t_k) \{\}$. We have :

$$F_p^{(\alpha,\beta;\tau,\mu)}(a, b, c; yX_{\xi_1,\cdots,\xi_n;\xi})\, S_{N_1,\cdots,N_t}^{M_1',\cdots,M_t'} \left(\begin{array}{c} \mathrm{y}_1 X_{\gamma_1^{(1)},\cdots,\gamma_1^{(n)},\gamma_1} \\ \cdots \\ \mathrm{y}_t X_{\gamma_t^{(1)}\cdots,\gamma_t^{(n)},\gamma_t} \end{array} \right) \aleph_{u:w}^{0,\mathfrak{n}:v} \left(\begin{array}{c} \mathrm{z}_1 X_{\alpha_1^{(1)},\cdots\alpha_1^{(n)},\alpha_1} \\ \cdots \\ \mathrm{z}_r X_{\alpha_r^{(1)}\cdots\alpha_r^{(n)},\alpha_r} \end{array} \right)$$

$$\aleph_{U:W}^{0,N:V}\left(\begin{array}{c} Z_1 X_{\eta_1^{(1)}\cdots\eta_1^{(n)},\eta_1} \\ \cdots \\ Z_s X_{\eta_s^{(1)}\cdots\eta_s^{(n)},\eta_s} \end{array}\right) = \sum_{g_1=0}^{m_1}\cdots\sum_{g_r=0}^{m_r}\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1']}\cdots\sum_{K_t=0}^{[N_t/M_t']}$$

$$\sum_{n'=0}^{\infty}(a)_{n'}\frac{B_p^{(\alpha,\beta;\tau,\mu)}(b+n',c-b)}{B(b,c-b)}\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}\,G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})a_1\frac{y^{n'}}{n'!}z_1^{\eta_{G_1,g_1}}\cdots z_r^{\eta_{G_r,g_r}}y_1^{K_1}\cdots y_t^{K_t}$$

$$\prod_{j=1}^{t}X^{K_j}_{\gamma_j^{(1)},\cdots,\gamma_j^{(n)},\gamma_j}X^{n'}_{\xi_{(1)},\cdots,\xi_{(n)},\xi}\prod_{j=1}^{r}X^{\eta_{G_j,g_j}}_{\alpha_j^{(1)},\cdots,\alpha_j^{(n)},\alpha_j}\;M\left[\prod_{j=1}^{s}Z_j^{t_j}X^{t_j}_{\eta_j^{(1)},\cdots,\eta_j^{(n)},\eta_j}\right]\mathrm{d}t_1\cdots\mathrm{d}t_s \quad (4.2)$$

Multiplying both sides of (4.3) by $\dfrac{(1-x_1-\cdots-x_n)^{\upsilon_0-1}}{(\bar{a}x+b)^{\mu}}\prod_{i=1}^{n}x_i^{\upsilon_i-1}$ and integrating with respect to $x_1,\cdots,x_s$ verifying the conditions f), changing the order of integrations and summations (which is easily seen to be justified due to the absolute convergence of the integrals and the summations involved in the process), we obtain :

$$\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{M_1}\cdots\sum_{g_r=0}^{M_r}\sum_{K_1=0}^{[N_1'/M_1']}\cdots\sum_{K_t=0}^{[N_t'/M_t']}\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})z^{R}a_1$$

$$\sum_{n'=0}^{\infty}(a)_{n'}\frac{B_p^{(\alpha,\beta;\tau,\mu)}(b+n',c-b)}{B(b,c-b)}z_1^{\eta_{G_1,g_1}}\cdots z_r^{\eta_{G_r,g_r}}y_1^{K_1}\cdots y_t^{K_t}\int_{x_1\geqslant 0}\cdots\int_{x_n\geqslant 0}X^{n'}_{\xi_1,\cdots,\xi_n}\prod_{j=1}^{t}X^{K_j}_{\gamma_j^{(1)},\cdots,\gamma_j^{(n)},\gamma_j}$$

$$\frac{y^{n'}}{n'!}\prod_{j=1}^{r}X^{\eta_{G_j,g_j}}_{\alpha_j^{(1)},\cdots,\alpha_j^{(n)},\alpha_j}\left\{M\left[\prod_{j=1}^{s}Z_j^{t_j}X^{t_j}_{\eta_j^{(1)},\cdots,\eta_j^{(n)},\eta_j}\right]\mathrm{d}t_1\cdots\mathrm{d}t_s\right\}\frac{(1-x_1-\cdots-x_n)^{\upsilon_0-1}}{(\bar{a}x+b)^{\mu}}\prod_{i=1}^{n}x_i^{\upsilon_i-1}$$

$$\mathrm{d}x_1\cdots\mathrm{d}x_n \quad (4.3)$$

Change the order of the $(x_1,\cdots,x_n)$-integrals and $(t_1,\cdots,t_s)$-integrals ,we get :

$$\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{M_1}\cdots\sum_{g_r=0}^{M_r}\sum_{K_1=0}^{[N_1'/M_1']}\cdots\sum_{K_t=0}^{[N_t'/M_t']}\sum_{n'=0}^{\infty}(a)_{n'}\frac{B_p^{(\alpha,\beta;\tau,\mu)}(b+n',c-b)}{B(b,c-b)}\frac{(-)^{G_1+\cdots+G_r}\;y^{n'}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!n'!}a_1$$

$$\Gamma(\upsilon_0)G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})z^{R}z_1^{\eta_{G_1,g_1}}\cdots z_r^{\eta_{G_r,g_r}}\;y_1^{K_1}\cdots y_t^{K_t}\;M\left\{\prod_{j=1}^{s}Z_j^{t_j}\int_{x_1\geqslant 0}\cdots\int_{x_n\geqslant 0}\right.$$

$$\prod_{i=1}^{n}x^{n'\xi_i+\sum_{j=1}^{t}K_j\gamma_j^{(i)}+\sum_{j=1}^{r}\eta_{G_j,g_j}\alpha_j^{(i)}+\sum_{j=1}^{s}t_j\eta_j^{(i)}+\upsilon_i-1}\frac{(1-x_1-\cdots-x_n)^{\upsilon_0-1}}{(\bar{a}x+b)^{\mu+n'\xi+\sum_{j=1}^{t}K_j\gamma_j+\sum_{j=1}^{r}\eta_{G_j;g_j}\alpha_j}}$$

$$\mathrm{d}x_1\cdots\mathrm{d}x_n\Big]\mathrm{d}t_1\cdots\mathrm{d}t_s\Big\} \tag{4.4}$$

Now, we transform the inner $(x_1,\cdots,x_n)$-integrals by using the equation (3.1) and interpreting the result thus obtained in the Mellin-barnes contour integrals (1.9), we arrive at the desired result.

5. Multivariable I-function

If $\iota_i,\iota_{i^{(1)}},\cdots,\iota_{i^{(s)}}\rightarrow 1$, the Aleph-function of several variables reduces in the I-function of several variables. The multiple integrals transformation have been derived in this section for multivariable I-functions defined by Sharma et Ahmad [2].

Corollary

$$\int_{x_1\geqslant 0}\cdots\int_{x_n\geqslant 0} F_p^{(\alpha,\beta;\tau,\mu)}(a,b,c;yX_{\xi_1,\cdots,\xi_n;\xi})\, S^{M'_1,\cdots,M'_t}_{N_1,\cdots,N_t}\begin{pmatrix} \mathrm{y}_1X_{\gamma_1^{(1)},\cdots,\gamma_1^{(n)},\gamma_1}\\ \cdots\\ \mathrm{y}_tX_{\gamma_t^{(1)}\cdots,\gamma_t^{(n)},\gamma_t}\end{pmatrix}$$

$$\aleph^{0,\mathfrak{n}:v}_{u:w}\begin{pmatrix} \mathrm{z}_1X_{\alpha_1^{(1)},\cdots\alpha_1^{(n)},\alpha_1}\\ \cdots\\ \mathrm{z}_rX_{\alpha_r^{(1)}\cdots\alpha_r^{(n)},\alpha_r}\end{pmatrix} I^{0,N:V}_{U:W}\begin{pmatrix} \mathrm{Z}_1X_{\eta_1^{(1)}\cdots\eta_1^{(n)},\eta_1}\\ \cdots\\ \mathrm{Z}_sX_{\eta_s^{(1)}\cdots\eta_s^{(n)},\eta_s}\end{pmatrix}\frac{(1-x_1-\cdots-x_n)^{\upsilon_0-1}}{(a\bar{x}+b)^{\mu}}\prod_{i=1}^n x_i^{\upsilon_i-1}$$

$$\mathrm{d}x_1\cdots\mathrm{d}x_n=\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{m_1}\cdots\sum_{g_r=0}^{m_r}\sum_{K_1=0}^{[N_1/M'_1]}\cdots\sum_{K_t=0}^{[N_t/M'_t]}\sum_{n'=0}^{\infty}(a)_{n'}\frac{B_p^{(\alpha,\beta;\tau,\mu)}(b+n',c-b)}{B(b,c-b)}\Gamma(\upsilon_0)$$

$$\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}\,G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})a_1\frac{y^{n'}}{n'!}z_1^{\eta_{G_1,g_1}}\cdots z_r^{\eta_{G_r,g_r}}y_1^{K_1}\cdots y_t^{K_t}$$

$$\int_0^1\frac{x^{n'\xi+\sum_{j=1}^tK_j\gamma_j+\sum_{j=1}^r\eta_{G_j,g_j}\alpha_j-1}}{(1+bx)^{-\upsilon_0}\prod_{i=1}^n(1+b+a_ix)^{n'\xi_i+\sum_{j=1}^tK_j\gamma_j^{(i)}+\sum_{j=1}^r\eta_{G_j,g_j}\alpha_j^{(i)}}}I^{0,N+n:V}_{U_{n,2}:W}\left(\begin{array}{c|c} \mathrm{Z}_1\frac{x^{\eta_1}}{\prod_{i=1}^n(1+b+a_ix)^{\eta_1^{(i)}}} & \\ \cdots & \\ \cdots & \\ \mathrm{Z}_s\frac{x^{\eta_s}}{\prod_{i=1}^n(1+b+a_ix)^{\eta_s^{(i)}}} & \end{array}\right.$$

$$\left.\begin{array}{c}\left[1-(\upsilon_i+n'\xi_i+\sum_{j=1}^tK_j\gamma_j^{(i)}+\sum_{j=1}^r\eta_{G_i,g_i}\alpha_j^{(i)});\eta_1^{(i)},\cdots,\eta_s^{(i)}\right]_{1,n},A:C\\ \cdots\\ (1\text{-}\mu-n'\xi-\sum_{j=1}^tK_j\gamma_j-\sum_{j=1}^r\eta_{G_j,g_j}\alpha_i;\eta_1,\cdots,\eta_s),B_1,B:D\end{array}\right)\mathrm{d}x \tag{5.1}$$

under the same conditions and notations that (4.1) with $\iota_i,\iota_{i^{(1)}},\cdots,\iota_{i^{(s)}}\rightarrow 1$

Remarks :

If s= 2 , we obtain the similar relation with the Aleph-function of two variables defined by Sharma [4]

If s= 2 and $\iota_i, \iota_i', \iota_i'' \rightarrow 1$, the multivariable Aleph-function reduces to the I-function of two variables defined by Sharma and Mishra [3].

6. Conclusion

In this paper we have evaluated multiple integrals transformation involving the multivariable Aleph-functions, class of polynomials of several variables and S generalized Gauss's hypergeometric function. The integral established in this paper is of very general nature as it contains multivariable Aleph-function, which is a general function of several variables studied so far. Thus, the integral established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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